Time compact difference methods for wave propagation in discontinuous media

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Abstract

In [2] a fourth order one-step method was constructed for the solution of wave propagation problems. The method is based on the first order system form of the PDE, and uses a staggered grid both in space and time. The method was also applied with good results to a problem with discontinuous coefficients without using any special procedure across the discontinuity. In this paper we will analyze a model problem from acoustics, and demonstrate the theoretical foundation for this behavior. Furthermore, we shall present more detailed numerical experiments which confirm the theoretical results.

1 Introduction

In [2] a fourth order one-step method was constructed for the solution of wave propagation problems, see also [4]. The method is based on the first order system form of the PDE, and uses a staggered grid both in space and time. We refer to the introduction of that paper for a discussion of other papers (included in the bibliography) and techniques on this matter. It was shown that the method is norm conserving, which is a good property for long time integrations. The method was also applied to a problem with discontinuous coefficients without using any special procedure across the discontinuity. The numerical experiments showed that the solution was much improved when going from the standard second order method to the fourth order method, despite the fact that it is formally only first order accurate for discontinuous coefficients.

In this paper we will analyze a model problem from acoustics, and demonstrate the theoretical foundation for this behavior. Furthermore, we shall present more detailed numerical experiments which confirm the theoretical results.

2 The model problem

In this section we shall define the model problem for the acoustic equations

$$\begin{aligned} \frac{\partial p}{\partial t} &= -\rho c^2 \frac{\partial u}{\partial x}\,, \\ \frac{\partial u}{\partial t} &= -\frac{1}{\rho}\frac{\partial p}{\partial x} \end{aligned} \tag{2.1}$$

in the interval $-a \leq x \leq a$. It is assumed that the density $\rho(x)$ and the speed of sound $c(x)$ are piecewise constant with a discontinuity at $x = 0$:

$$\rho(x) = \begin{cases} \rho_L\,, & x \leq 0 \\ \rho_R\,, & x > 0 \end{cases}$$

$$c(x) = \begin{cases} c_L\,, & x \leq 0 \\ c_R\,, & x > 0\,. \end{cases}$$

We consider time harmonic solutions proportional to $e^{i\omega t}$. Since the coefficients ρ, c are independent of time, the analysis can be done in Fourier space, and for convenience we assume $\omega > 0$. With the ansatz

$$
\begin{aligned}
p(x,t) &= \hat{p}(x,\omega)e^{i\omega t}\,, \\
u(x,t) &= \hat{u}(x,\omega)e^{i\omega t}\,,
\end{aligned}
$$

we define the solution in Fourier space on each side of the interface as

$$
\begin{aligned}
\hat{p}(x,\omega) &= \begin{cases} \hat{p}_L(x,\omega)\,, & x \le 0 \\ \hat{p}_R(x,\omega)\,, & x > 0 \end{cases} \\
\hat{u}(x,\omega) &= \begin{cases} \hat{u}_L(x,\omega)\,, & x \le 0 \\ \hat{u}_R(x,\omega)\,, & x > 0\,. \end{cases}
\end{aligned}
$$

The solution is

$$
\begin{aligned}
\begin{bmatrix} \hat{p}_L \\ \hat{u}_L \end{bmatrix} &= \sigma_{L1} \begin{bmatrix} -\rho_L c_L \\ 1 \end{bmatrix} e^{i\omega x/c_L} + \sigma_{L2} \begin{bmatrix} \rho_L c_L \\ 1 \end{bmatrix} e^{-i\omega x/c_L}\,, \\
\begin{bmatrix} \hat{p}_R \\ \hat{u}_R \end{bmatrix} &= \sigma_{R1} \begin{bmatrix} -\rho_R c_R \\ 1 \end{bmatrix} e^{i\omega x/c_R} + \sigma_{R2} \begin{bmatrix} \rho_R c_R \\ 1 \end{bmatrix} e^{-i\omega x/c_R}\,,
\end{aligned}
\tag{2.2}
$$

where $\sigma_{L\nu}$, $\sigma_{R\nu}$ are coefficients to be determined by boundary conditions and by continuity conditions across the interface at $x = 0$. In our model problem we assume that u is specified at $x = \pm a$, i.e., $\hat{u}_L(-a)$ and $\hat{u}_R(a)$ are known, but we could as well have used specification of any other combination of p and u except the outgoing characteristic variable. For later use, we write down the equations for the coefficients $\sigma_{L\nu}$, $\sigma_{R\nu}$:

$$
\begin{aligned}
e^{-i\omega a/c_L}\sigma_{L1} + e^{i\omega a/c_L}\sigma_{L2} &= \hat{u}_L(-a)\,, \\
-\rho_L c_L \sigma_{L1} + \rho_L c_L \sigma_{L2} + \rho_R c_R \sigma_{R1} - \rho_R c_R \sigma_{R2} &= 0\,, \\
\sigma_{L1} + \sigma_{L2} - \sigma_{R1} - \sigma_{R2} &= 0\,, \\
e^{i\omega a/c_R}\sigma_{R1} + e^{-i\omega a/c_R}\sigma_{R2} &= \hat{u}_R(a)\,.
\end{aligned}
\tag{2.3}
$$

The coefficient matrix is

$$
A = \begin{bmatrix}
e^{-i\omega a/c_L} & e^{i\omega a/c_L} & 0 & 0 \\
-\rho_L c_L & \rho_L c_L & \rho_R c_R & -\rho_R c_R \\
1 & 1 & -1 & -1 \\
0 & 0 & e^{i\omega a/c_R} & e^{-i\omega a/c_R}
\end{bmatrix}
\tag{2.4}
$$

with the determinant

$$
\mathrm{Det}(A) = 4i[\rho_L c_L \sin(\frac{\omega a}{c_R})\cos(\frac{\omega a}{c_L}) + \rho_R c_R \sin(\frac{\omega a}{c_L})\cos(\frac{\omega a}{c_R})]\,.
$$

This determinant may be zero for some combinations of the parameters allowing for non-unique solutions. However, in this investigation we are interested only in the effect of the interface. Since the speed of propagation is finite, we can without restriction place the outer boundaries at any points $\pm a$ with sufficiently large a and such that $\mathrm{Det}(A) \neq 0$.

In the numerical experiments, we will consider the case $\sigma_{R1} = 0$, which means that a rightgoing wavetrain is passing through the interface causing a reflected leftgoing wave in the left halfspace. The boundary values $\hat{u}_L(-a)$, $\hat{u}_R(a)$ are chosen such that

$$\begin{aligned}
\sigma_{L1} &= \frac{\rho_L c_L - \rho_R c_R}{\rho_L c_L}\,, \\
\sigma_{L2} &= \frac{\rho_L c_L + \rho_R c_R}{\rho_L c_L}\,, \\
\sigma_{R1} &= 0\,, \\
\sigma_{R2} &= 2\,,
\end{aligned} \tag{2.5}$$

which gives the solution in physical space

$\boldsymbol{x \leq 0}$ **:**

$$\begin{aligned}
p(x,t) &= e^{i\omega(t-x/c_L)} - \frac{\rho_L c_L - \rho_R c_R}{\rho_L c_L + \rho_R c_R} e^{i\omega(t+x/c_L)}\,, \\
u(x,t) &= \frac{1}{\rho_L c_L}[e^{i\omega(t-x/c_L)} + \frac{\rho_L c_L - \rho_R c_R}{\rho_L c_L + \rho_R c_R} e^{i\omega(t+x/c_L)}]\,,
\end{aligned}$$

$\boldsymbol{x > 0}$ **:**

$$\begin{aligned}
p(x,t) &= \frac{2\rho_R c_R}{\rho_L c_L + \rho_R c_R} e^{i\omega(t-x/c_R)}\,, \\
u(x,t) &= \frac{2}{\rho_L c_L + \rho_R c_R} e^{i\omega(t-x/c_R)}\,.
\end{aligned}$$

The solution is continuous at $x = 0$, but the space derivative is discontinuous. The real world problems we have in mind are defined over a large domain both in space and time compared to the typical wavelengths. In Fourier space this means that the required accuracy must be retained for large values of ω. Furthermore, we intend to use Cartesian grids in several space dimensions, and we don't expect the interfaces to always pass through gridpoints. In order to simulate this situation in one space dimension, we let gridpoints be slightly shifted, such that $x = 0$ is not a gridpoint, see Figure 2.1.

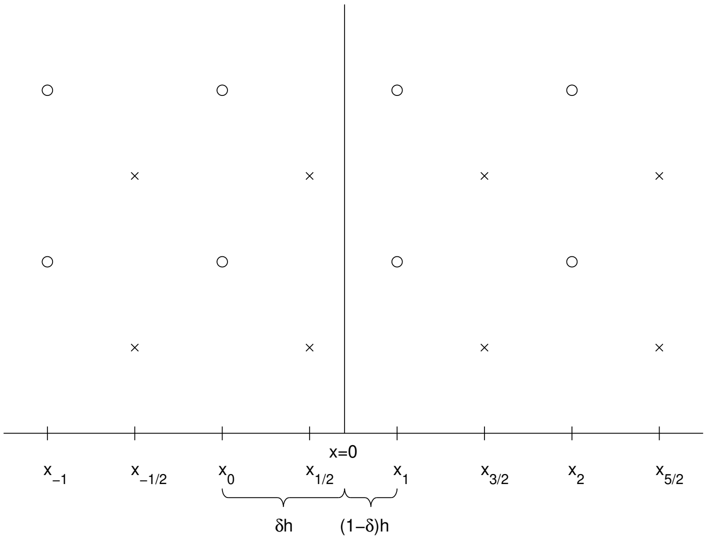

Figure 2.1: Staggered grid around $x = 0$. p stored at $(\times)$, u stored at $(\circ)$.

The definition of the grid is

$$\begin{aligned} &x_j = (j - \delta)h\ , \quad j = 0, \pm 1/2, \pm 1,\ \dots\ , \quad Nh = a\,, \\ &t_n = nk\ , \quad n = 0, 1/2, 1,\ \dots\ , \end{aligned}$$

with the gridvalues defined by

$$\begin{aligned} p_{j+1/2}^{n+1/2} &= p(x_{j+1/2}, t_{n+1/2})\ , \\ u_j^n &= u(x_j, t_n)\ , \quad j = 0, \pm 1, \pm 2,\ \dots\ , \quad n = 0, 1, 2,\ \dots\ . \end{aligned}$$

Here it is assumed that $1/2 \le \delta < 1$, such that the variables ($p_{j+1/2}^{n+1/2}$ and u_j^n) are always on the same side of the interface $x = 0$ for all j. If instead $0 \le \delta < 1/2$, then we would keep $(p_{j-1/2}^{n+1/2}, u_j^n)$ together in the equations.

3 The second order scheme

The second order approximation (called the Yee scheme in electromagnetics) is

$$\begin{aligned} p_{j+1/2}^{n+1/2} - p_{j+1/2}^{n-1/2} &= -\lambda \rho_{j+1/2} c_{j+1/2}^2 (u_{j+1}^n - u_j^n)\,, \\ u_j^{n+1} - u_j^n &= -\frac{\lambda}{\rho_j}(p_{j+1/2}^{n+1/2} - p_{j-1/2}^{n+1/2})\,, \end{aligned} \tag{3.1}$$

where $\lambda = k/h$. This scheme is to be applied at all gridpoints, even at the ones around the interface. In [2] it was shown that a sufficient condition for stability is

$$\frac{\lambda}{2}[\max_j (c_{j+1/2}\sqrt{\rho_{j+1/2}/\rho_j}) + \max_j (c_{j-1/2}\sqrt{\rho_{j-1/2}/\rho_j})] < 1\,, \tag{3.2}$$

For the Fourier analysis we make the ansatz

$$p_{j+1/2}^{n+1/2} = \hat{p}_{j+1/2} e^{i\omega t_{n+1/2}}\,,$$
$$u_j^n = \hat{u}_j e^{i\omega t_n}\,,$$

and with $\xi = \omega k$, we get the transformed equations

$$\begin{aligned}(e^{i\xi/2} - e^{-i\xi/2})\hat{p}_{j+1/2} &= -\lambda \rho_{j+1/2} c_{j+1/2}^2 (\hat{u}_{j+1} - \hat{u}_j)\,,\\ (e^{i\xi/2} - e^{-i\xi/2})\hat{u}_j &= -\frac{\lambda}{\rho_j}(\hat{p}_{j+1/2} - \hat{p}_{j-1/2})\,.\end{aligned} \tag{3.3}$$

For the model problem we consider each halfline $x \le 0\,,\ x \ge 0$ separately, and each one has constant but different coefficients $\rho\,,\ c$. Hence, we look for solutions of the type

$$\begin{bmatrix} \hat{p}_j \\ \hat{u}_j \end{bmatrix} = \underline{\hat{v}}\kappa^j\,,$$

where $\underline{\hat{v}}$ is a constant vector, and κ is a complex number. The system is

$$\begin{bmatrix} 2i\sin(\xi/2)\kappa^{1/2} & \lambda\rho c^2(\kappa - 1) \\ \frac{\lambda}{\rho}(\kappa^{1/2} - \kappa^{-1/2}) & 2i\sin(\xi/2) \end{bmatrix} \underline{\hat{v}} = 0\,,$$

and nontrivial solutions exist if

$$-4\sin^2(\xi/2)\kappa^{1/2} - \lambda^2 c^2 \kappa^{-1/2}(\kappa - 1)^2 = 0\,.$$

With the notation $s = \sin(\xi/2)/(\lambda c)$, the solutions are

$$\kappa_1 = 1 - 2s^2 + 2s\sqrt{s^2 - 1}\,,$$
$$\kappa_2 = 1 - 2s^2 - 2s\sqrt{s^2 - 1}\,,$$

where the square-root is defined such that $\sqrt{-1} = i$. Since we are concerned about accuracy, we consider small values of ξ. Assuming $0 < s < 1$, the corresponding eigenvectors are

$$\underline{\hat{v}}_\nu = \begin{bmatrix} \frac{2i\rho\sin(\xi/2)}{\lambda(\kappa_\nu^{-1/2} - \kappa_\nu^{1/2})} \\ 1 \end{bmatrix}\,, \quad \nu = 1, 2\,.$$

Taylor expansions show that

$$\begin{aligned}\kappa_1 &= e^{i\xi/(\lambda c)} + \mathcal{O}(\xi^3) = e^{i\omega h/c} + \mathcal{O}(\omega^3 h^3)\,,\\ \kappa_2 &= e^{-i\xi/(\Lambda c)} + \mathcal{O}(\xi^3) = e^{-i\omega h/c} + \mathcal{O}(\omega^3 h^3)\,,\end{aligned} \tag{3.4}$$

leading to

$$\frac{2i\sin(\xi/2)}{\lambda(\kappa_1^{-1/2} - \kappa_1^{1/2})} = -\tilde{c}\,,$$
$$\frac{2i\sin(\xi/2)}{\lambda(\kappa_2^{-1/2} - \kappa_2^{1/2})} = \tilde{c}\,,$$

where

$$\tilde{c} = c + \mathcal{O}(\omega^2 h^2)\,. \tag{3.6}$$

The solutions are different on the left and right halflines, and we use the subscript L and R for reference to the two domains.

The solution in Fourier space is

$\boldsymbol{j \leq 0}$ **:**

$$\begin{bmatrix} \hat{p}_{L(j+1/2)} \\ \hat{u}_{Lj} \end{bmatrix} = \tilde{\sigma}_{L1} \begin{bmatrix} -\rho_L \tilde{c}_L \kappa_{L1}^{1/2} \\ 1 \end{bmatrix} \kappa_{L1}^{j} + \tilde{\sigma}_{L2} \begin{bmatrix} \rho_L \tilde{c}_L \kappa_{L2}^{1/2} \\ 1 \end{bmatrix} \kappa_{L2}^{j}\,, \tag{3.7}$$

$\boldsymbol{j > 1}$ **:**

$$\begin{bmatrix} \hat{p}_{R(j+1/2)} \\ \hat{u}_{Rj} \end{bmatrix} = \tilde{\sigma}_{R1} \begin{bmatrix} -\rho_R \tilde{c}_R \kappa_{R1}^{1/2} \\ 1 \end{bmatrix} \kappa_{R1}^{j} + \tilde{\sigma}_{R2} \begin{bmatrix} \rho_R \tilde{c}_R \kappa_{R2}^{1/2} \\ 1 \end{bmatrix} \kappa_{R2}^{j}\,, \tag{3.8}$$

where the constants σ_{L1}, σ_{L2}, σ_{R1}, σ_{R2} are to be determined by the interface conditions and by the boundary conditions given at the outer boundaries. The purpose of the investigation in this paper is to demonstrate the effect of the internal interface, therefore we assume that the outer boundary conditions are represented exactly. This means that we prescribe u at the boundaries $x = \pm a - \delta h$, which in Fourier space corresponds to

$$\begin{aligned} \hat{u}_{L(-N)} &= \sigma_{L1} e^{i\omega(-N-\delta)h/c_L} + \sigma_{L2} e^{-i\omega(-N-\delta)h/c_L}\,, \\ \hat{u}_{RN} &= \sigma_{R1} e^{i\omega(N-\delta)h/c_R} + \sigma_{R2} e^{-i\omega(N-\delta)h/c_R}\,, \end{aligned} \tag{3.9}$$

with the coefficients σ_{L1}, σ_{L2}, σ_{R1}, σ_{R2} given in (2.5). The interface conditions are

$$\begin{aligned} 2i\sin(\xi/2)\hat{p}_{L(1/2)} &= -\lambda \rho_L c_L^2 (\hat{u}_{R1} - \hat{u}_{L0})\,, \\ 2i\sin(\xi/2)\hat{u}_{R1} &= -\frac{\lambda}{\rho_R}(\hat{p}_{R(3/2)} - \hat{p}_{L(1/2)})\,. \end{aligned}$$

These are clearly satisfied by the general form of the solution if the definition of $\hat{u}_{Lj}$ in (3.7) and $\hat{p}_{R(j+1/2)}$ in (3.8) are extended such that

$$\begin{aligned} \hat{u}_{L1} &= \hat{u}_{R1}\,, \\ \hat{p}_{R(1/2)} &= \hat{p}_{L(1/2)}\,. \end{aligned}$$

The complete system determining the coefficients $\tilde{\sigma}_{L\nu}$, $\tilde{\sigma}_{R\nu}$ is

$$\begin{aligned} \kappa_{L1}^{-N}\tilde{\sigma}_{L1} + \kappa_{L2}^{-N}\tilde{\sigma}_{L2} &= \hat{u}_{L(-N)}\,, \\ -\rho_L\tilde{c}_L\kappa_{L1}^{1/2}\tilde{\sigma}_{L1} + \rho_L\tilde{c}_L\kappa_{L2}^{1/2}\tilde{\sigma}_{L2} + \rho_R\tilde{c}_R\kappa_{R1}^{1/2}\tilde{\sigma}_{R1} - \rho_R\tilde{c}_R\kappa_{R2}^{1/2}\tilde{\sigma}_{R2} &= 0\,, \\ \kappa_{L1}\tilde{\sigma}_{L1} + \kappa_{L2}\tilde{\sigma}_{L2} - \kappa_{R1}\tilde{\sigma}_{R1} - \kappa_{R2}\tilde{\sigma}_{R2} &= 0\,, \\ \kappa_{R1}^{N}\tilde{\sigma}_{R1} + \kappa_{R2}^{N}\tilde{\sigma}_{R2} &= \hat{u}_{RN}\,. \end{aligned} \tag{3.10}$$

We now make the ansatz

$$\begin{aligned} \tilde{\sigma}_{L\nu} &= \sigma_{L\nu} + \tau_{L\nu}\ , \quad \nu = 1,2\,, \\ \tilde{\sigma}_{R\nu} &= \sigma_{R\nu} + \tau_{R\nu}\ , \quad \nu = 1,2\,, \end{aligned}$$

where the σ-coefficients are the ones representing the true solution. By using the relations (3.4), (3.6) and $e^{\mathcal{O}(\xi)} = 1 + \mathcal{O}(\xi)$, we obtain

$$
\begin{aligned}
&[e^{-i\omega Nh/c_L} + \mathcal{O}(\omega^3 h^2)](\sigma_{L1} + \tau_{L1}) + [e^{i\omega Nh/c_L} + \mathcal{O}(\omega^3 h^2)](\sigma_{L2} + \tau_{L2}) \\
&\quad = \hat{u}_L(-a - \delta h)\,, \\
&-\rho_L[c_L + \mathcal{O}(\omega^2 h^2)](\sigma_{L1} + \tau_{L1}) + \rho_L[c_L + \mathcal{O}(\omega^2 h^2)](\sigma_{L2} + \tau_{L2}) \\
&\quad +\rho_R[c_R + \mathcal{O}(\omega^2 h^2)](\sigma_{R1} + \tau_{R1}) - \rho_R[c_R + \mathcal{O}(\omega^2 h^2)](\sigma_{R2} + \tau_{R2}) = 0\,, \\
&[1 + \mathcal{O}(\omega h)](\sigma_{L1} + \tau_{L1}) + [1 + \mathcal{O}(\omega h)](\sigma_{L2} + \tau_{L2}) \\
&\quad -[1 + \mathcal{O}(\omega h)](\sigma_{R1} + \tau_{R1}) - [1 + \mathcal{O}(\omega h)](\sigma_{R2} + \tau_{R2}) = 0\,, \\
&[e^{i\omega Nh/c_R} + \mathcal{O}(\omega^3 h^2)](\sigma_{R1} + \tau_{R1}) + [e^{-i\omega Nh/c_R} + \mathcal{O}(\omega^3 h^2)](\sigma_{R2} + \tau_{R2}) \\
&\quad = \hat{u}_R(a - \delta h)\,.
\end{aligned}
$$

By defining the vectors

$$
\underline{\sigma} = \begin{bmatrix} \sigma_{L1} \\ \sigma_{L2} \\ \sigma_{R1} \\ \sigma_{R2} \end{bmatrix}, \quad
\underline{\tau} = \begin{bmatrix} \tau_{L1} \\ \tau_{L2} \\ \tau_{R1} \\ \tau_{R2} \end{bmatrix}, \quad
\underline{g} = \begin{bmatrix} \hat{u}_L(-a - \delta h) \\ 0 \\ 0 \\ \hat{u}_R(a - \delta h) \end{bmatrix},
$$

the system can be written in the form

$$
A\underline{\sigma} + \tilde{A}\underline{\tau} = \underline{g} + \mathcal{O}(\omega h + \omega^3 h^2)\underline{\sigma}\,,
$$

where A is defined in (2.4), and $\tilde{A} = A + \mathcal{O}(\omega h + \omega^3 h^2)$. Note that we have included the terms of order $\omega^2 h^2$ in $\mathcal{O}(\omega^3 h^2)$. Obviously, $\underline{\sigma}$ satisfies $A\underline{\sigma} = \underline{g}$, and the solution is $\mathcal{O}(1)$. Therefore, the remaining system has the form

$$
\tilde{A}\underline{\tau} = \mathcal{O}(\omega h + \omega^3 h^2)\,,
$$

and by assumption $|\mathrm{Det}(A)| \geq d > 0$, where d is independent of h, i.e., $\mathrm{Det}(\tilde{A}) \neq 0$ for all h small enough. Hence, there is a solution with $\underline{\tau} = \mathcal{O}(\omega h + \omega^3 h^2)$, and we have

$\boldsymbol{j \leq 0}$:

$$
\begin{aligned}
\hat{p}_{L(j+1/2)} &= -\rho_L c_L \sigma_{L1} e^{i\omega(j+1/2-\delta)h/c_L} + \rho_L c_L \sigma_{L2} e^{-i\omega(j+1/2-\delta)h/c_L} \\
&\quad + \mathcal{O}(\omega h + \omega^3 h^2)\,, \\
\hat{u}_{Lj} &= \sigma_{L1} e^{i\omega(j-\delta)h/c_L} + \sigma_{L2} e^{-i\omega(j-\delta)h/c_L} + \mathcal{O}(\omega h + \omega^3 h^2)\,,
\end{aligned} \tag{3.11}
$$

$\boldsymbol{j > 1}$:

$$
\begin{aligned}
\hat{p}_{R(j+1/2)} &= -\rho_R c_R \sigma_{R1} e^{i\omega(j+1/2-\delta)h/c_R} + \rho_R c_R \sigma_{R2} e^{-i\omega(j+1/2-\delta)/c_R} \\
&\quad + \mathcal{O}(\omega h + \omega^3 h^2)\,, \\
\hat{u}_{Rj} &= \sigma_{R1} e^{i\omega(j-\delta)h/c_R} + \sigma_{R2} e^{-i\omega(j-\delta)h/c_R} + \mathcal{O}(\omega h + \omega^3 h^2)\,.
\end{aligned} \tag{3.12}
$$

Comparing to the true solution, we see that there are two types of errors. The first type is the usual second order $\mathcal{O}(\omega^3 h^2)$ error obtained even for problems with continuous coefficients. If we allow an error ε, then we need

$$
\frac{\omega^3 a^2}{N^2} \leq \gamma_1 \varepsilon\,,
$$

where γ_1 is a constant, cf. [3]. The number of points N_ω per wavelength is proportional to N/ω, and we need

$$N_\omega \geq \gamma_2 \sqrt{\omega/\varepsilon}\,,$$

where γ_2 is a constant. For any given error level ε, the number of gridpoints per wavelength must be increased with increasing frequencies ω. This is a well known effect, that makes second order methods insufficient for large problems.

The second type of error in our solution is the phase shift $e^{\pm i\omega\delta h/c_{L,R}}$ and the $\mathcal{O}(\omega h)$ error in the amplitude. This is a first order error, but there is an important difference compared to the first type. If we require

$$\omega h \leq \gamma_1 \varepsilon\,,$$

then

$$N_\omega \geq \gamma_3/\varepsilon\,,$$

where γ_3 is a constant independent of ω and ε. Hence, the number of gridpoints per wavelength is independent of the frequency ω. The important conclusion is that this type of error is unaffected by the size of the problem, or expressed differently, the solution can be computed over any length of time without increasing this part of the error. This is why it makes sense to increase the order of accuracy for the basic approximation even if the interface is dislocated by $\mathcal{O}(h)$. As we will prove in the next section, the first type of error will be diminished, while the first order error remains of the same type as above.

4 The time compact fourth order scheme

We define the forward and backward difference operators by

$$D_+ v_j = (v_{j+1} - v_j)/h\,,\ D_- v_j = (v_j - v_{j-1})/h\,.$$

The fourth order scheme derived in [2] is

$$\begin{aligned} p_{j+1/2}^{n+1/2} &= p_{j+1/2}^{n-1/2} + k a_{j+1/2} D_+ u_j^n \\ &\quad + \frac{k}{24} a_{j+1/2} (k^2 D_+ b_j D_- a_{j+1/2} D_+ - h^2 D_+^2 D_-) u_j^n\,, \\ u_j^{n+1} &= u_j^n + k b_j D_- p_{j+1/2}^{n+1/2} \\ &\quad + \frac{k}{24} b_j (k^2 D_- a_{j+1/2} D_+ b_j D_- - h^2 D_+ D_-^2) p_{j+1/2}^{n+1/2}\,, \end{aligned} \tag{4.1}$$

where

$$\begin{aligned} a_{j+1/2} &= -\rho_{j+1/2} c_{j+1/2}^2\,, \\ b_j &= -1/\rho_j\,. \end{aligned}$$

On each halfspace we consider these equations with constant coefficients ρ, c :

$$\begin{aligned} p_{j+1/2}^{n+1/2} - p_{j+1/2}^{n-1/2} &= -\lambda\rho c^2 [u_{j+1}^n - u_j^n + \frac{c^2\lambda^2-1}{24}(u_{j+2}^n - 3u_{j+1}^n + 3u_j^n - u_{j-1}^n)]\,, \\ u_j^{n+1} - u_j^n &= -\frac{\lambda}{\rho}[p_{j+1/2}^{n+1/2} - p_{j-1/2}^{n+1/2} + \frac{c^2\lambda^2-1}{24}(p_{j+3/2}^{n+1/2} - 3p_{j+1/2}^{n+1/2} \\ &\quad + 3p_{j-1/2}^{n+1/2} - p_{j-3/2}^{n+1/2})]\,. \end{aligned} \tag{4.2}$$

After Fourier transformation we get the system

$$\begin{bmatrix} 2i\sin(\xi/2)\kappa^{5/2} & \lambda\rho c^2[\kappa^3 - \kappa^2 + \frac{c^2\lambda^2-1}{24}\kappa(\kappa-1)^3] \\ \frac{\lambda}{\rho}[\kappa^{5/2} - \kappa^{3/2} + \frac{c^2\lambda^2-1}{24}\kappa^{1/2}(\kappa-1)^3] & 2i\sin(\xi/2)\kappa^2 \end{bmatrix} \underline{\hat{v}} = 0\,,$$

which has nontrivial solutions if

$$4\sin^2(\xi/2)\kappa^3 + \lambda^2 c^2(\kappa-1)^2[\kappa + \frac{c^2\lambda^2-1}{24}(\kappa-1)^2]^2 = 0\,.$$

There are 6 solutions κ_ν, and for small values of ξ we have

$$\begin{aligned} \kappa_1 &= e^{i\xi/(\lambda c)} + \mathcal{O}(\xi^5) = e^{i\omega h/c} + \mathcal{O}(\omega^5 h^5)\,, \\ \kappa_2 &= e^{-i\xi/(\Lambda c)} + \mathcal{O}(\xi^5) = e^{-i\omega h/c} + \mathcal{O}(\omega^5 h^5)\,, \\ \kappa_{3,4} &= 1 + \frac{12}{1-c^2\lambda^2}(1 + \sqrt{1 + \frac{1-c^2\lambda^2}{6}}) + \mathcal{O}(\xi)\,, \\ \kappa_{5,6} &= 1 + \frac{12}{1-c^2\lambda^2}(1 - \sqrt{1 + \frac{1-c^2\lambda^2}{6}}) + \mathcal{O}(\xi)\,. \end{aligned} \tag{4.3}$$

The roots $\kappa_{3,4,5,6}$ give rise to parasitic solutions, but since

$$|\kappa_{3,4}| >> 1\ , \quad |\kappa_{5,6}| << 1\ ,$$

the effect is very local. The eigenvectors $\underline{\hat{v}}_{1,2}$ have the same form as in the second order case, and with obvious notation the complete solution is

$\mathbf{j \le 0}$:

$$\begin{aligned} \begin{bmatrix} \hat{p}_{L(j+1/2)} \\ \hat{u}_{Lj} \end{bmatrix} &= \tilde{\sigma}_{L1}\begin{bmatrix} -\rho_L\tilde{c}_L\kappa_{L1}^{1/2} \\ 1 \end{bmatrix}\kappa_{L1}^j + \tilde{\sigma}_{L2}\begin{bmatrix} \rho_L\tilde{c}_L\kappa_{L2}^{1/2} \\ 1 \end{bmatrix}\kappa_{L2}^j \\ &\quad + \sum_{\nu=3}^{4}\tilde{\sigma}_{L\nu}\begin{bmatrix} v_{L\nu}^{(1)} \\ v_{L\nu}^{(2)} \end{bmatrix}\kappa_{L\nu}^j + \sum_{\nu=5}^{6}\tilde{\sigma}_{L\nu}\begin{bmatrix} v_{L\nu}^{(1)} \\ v_{L\nu}^{(2)} \end{bmatrix}\kappa_{L\nu}^{j+N}\ , \end{aligned} \tag{4.4}$$

$\mathbf{j > 1}$:

$$\begin{aligned} \begin{bmatrix} \hat{p}_{R(j+1/2)} \\ \hat{u}_{Rj} \end{bmatrix} &= \tilde{\sigma}_{R1}\begin{bmatrix} -\rho_R\tilde{c}_R\kappa_{R1}^{1/2} \\ 1 \end{bmatrix}\kappa_{R1}^j + \tilde{\sigma}_{R2}\begin{bmatrix} \rho_R\tilde{c}_R\kappa_{R2}^{1/2} \\ 1 \end{bmatrix}\kappa_{R2}^j \\ &\quad + \sum_{\nu=3}^{4}\tilde{\sigma}_{R\nu}\begin{bmatrix} v_{R\nu}^{(1)} \\ v_{R\nu}^{(2)} \end{bmatrix}\kappa_{R\nu}^{j-N} + \sum_{\nu=5}^{6}\tilde{\sigma}_{R\nu}\begin{bmatrix} v_{R\nu}^{(1)} \\ v_{R\nu}^{(2)} \end{bmatrix}\kappa_{R\nu}^j\ , \end{aligned} \tag{4.5}$$

where

$$\begin{aligned}\tilde{c}_L &= c_L + \mathcal{O}(\omega^4 h^4)\,, \\ \tilde{c}_R &= c_R + \mathcal{O}(\omega^4 h^4)\,.\end{aligned} \tag{4.6}$$

Note the similarity with the form (3.7), (3.8) of the solution to the second order scheme. The first two terms is the principal part corresponding to the right- and leftgoing waves, while the remaining terms are parasitic solutions. In the following we shall prove that the total error is $\mathcal{O}(\omega h + \omega^5 h^4)$, where the second part will dominate for long time integrations. However, this part is much smaller than the $\mathcal{O}(\omega^3 h^2)$-term that dominates for the second order scheme.

In contrast to the second order case, there are now points in the neighborhood of the interface where the constant coefficient equations (4.2) are not equivalent to the original equations (4.1). For instance, the second equation in (4.1) for $j = 1$ contains $\rho(x_0) = \rho(x_{1/2}) = \rho_L$ and $c(x_{1/2}) = c_L$ as well as ρ_R, c_R.

On each halfspace there are 6 parameters $\tilde{\sigma}_\nu$ to be determined. As for the second order case, we disregard the influence of any numerical type of boundary conditions at the outer boundaries, and for convenience we simply prescribe the correct values for u at the three outer points x_{N-2}, x_{N-1}, x_N at the right boundary, and similarly at the left boundary. (For an actual implementation, it would be simpler to prescribe u at x_{N-1}, x_N, and p at $x_{N-1/2}$.) There are then 3 remaining parameters on each side to be determined by 6 interface conditions.

These conditions are in Fourier space

$$
\begin{aligned}
2i\sin(\xi/2)\hat{p}_{L(-1/2)} &= -\lambda\rho_L c_L^2[\hat{u}_{L0} - \hat{u}_{L(-1)} + \frac{c_L^2\lambda^2 - 1}{24}(\hat{u}_{R1} - 3\hat{u}_{L0} \\
&\quad + 3\hat{u}_{L(-1)} - \hat{u}_{L(-2)})]\,, \\
2i\sin(\xi/2)\hat{p}_{L(1/2)} &= -\lambda\rho_L c_L^2[\hat{u}_{R1} - \hat{u}_{L0} + \frac{\lambda^2}{24\rho_R}\{\rho_R c_R^2(\hat{u}_{R2} - \hat{u}_{R1}) \\
&\quad - \rho_L c_L^2(\hat{u}_{R1} - \hat{u}_{L0})\} \\
&\quad - \frac{\lambda^2}{24\rho_L}\{\rho_L c_L^2(\hat{u}_{R1} - \hat{u}_{L0}) - \rho_L c_L^2(\hat{u}_{L0} - \hat{u}_{L(-1)})\} \\
&\quad - \frac{1}{24}(\hat{u}_{R2} - 3\hat{u}_{R1} + 3\hat{u}_{L0} - \hat{u}_{L(-1)})]\,, \\
2i\sin(\xi/2)\hat{p}_{R(3/2)} &= -\lambda\rho_R c_R^2[\hat{u}_{R2} - \hat{u}_{R1} + \frac{\lambda^2}{24\rho_R}\{\rho_R c_R^2(\hat{u}_{R3} - \hat{u}_{R2}) \\
&\quad - \rho_R c_R^2(\hat{u}_{R2} - \hat{u}_{R1})\} \\
&\quad - \frac{\lambda^2}{24\rho_R}\{\rho_R c_R^2(\hat{u}_{R2} - \hat{u}_{R1}) - \rho_L c_L^2(\hat{u}_{R1} - \hat{u}_{L0})\} \\
&\quad - \frac{1}{24}(\hat{u}_{R3} - 3\hat{u}_{R2} + 3\hat{u}_{R1} - \hat{u}_{L0})]\,, \\
2i\sin(\xi/2)\hat{u}_{L0} &= -\frac{\lambda}{\rho_L}[\hat{p}_{L(1/2)} - \hat{p}_{L(-1/2)} + \frac{\lambda^2}{24}\rho_L c_L^2\{\frac{1}{\rho_R}(\hat{p}_{R(3/2)} - \hat{p}_{L(1/2)}) \\
&\quad - \frac{1}{\rho_L}(\hat{p}_{L(1/2)} - \hat{p}_{L(-1/2)})\} - \frac{\lambda^2}{24}\rho_L c_L^2\{\frac{1}{\rho_L}(\hat{p}_{L(1/2)} - \hat{p}_{L(-1/2)}) \\
&\quad - \frac{1}{\rho_L}(\hat{p}_{L(-1/2)} - \hat{p}_{L(-3/2)})\} - \frac{1}{24}(\hat{p}_{R(3/2)} - 3\hat{p}_{L(1/2)} + 3\hat{p}_{L(-1/2)} - \hat{p}_{L(-3/2)})]\,, \\
2i\sin(\xi/2)\hat{u}_{R1} &= -\frac{\lambda}{\rho_R}[\hat{p}_{R(3/2)} - \hat{p}_{L(1/2)} + \frac{\lambda^2}{24}\rho_R c_R^2\{\frac{1}{\rho_R}(\hat{p}_{R(5/2)} - \hat{p}_{R(3/2)}) \\
&\quad - \frac{1}{\rho_R}(\hat{p}_{R(3/2)} - \hat{p}_{L(1/2)})\} - \frac{\lambda^2}{24}\rho_L c_L^2\{\frac{1}{\rho_R}(\hat{p}_{R(3/2)} - \hat{p}_{L(1/2)}) \\
&\quad - \frac{1}{\rho_L}(\hat{p}_{L(1/2)} - \hat{p}_{L(-1/2)})\} - \frac{1}{24}(\hat{p}_{R(5/2)} - 3\hat{p}_{R(3/2)} + 3\hat{p}_{L(1/2)} - \hat{p}_{L(-1/2)})]\,, \\
2i\sin(\xi/2)\hat{u}_{R2} &= -\frac{\lambda}{\rho_R}[\hat{p}_{R(5/2)} - \hat{p}_{R(3/2)} + \frac{c_R^2\lambda^2 - 1}{24}(\hat{p}_{R(7/2)} - 3\hat{p}_{R(5/2)} \\
&\quad + 3\hat{p}_{R(3/2)} - \hat{p}_{L(1/2)})]\,,
\end{aligned}
$$

Let

$$
\begin{aligned}
q_p(\hat{p}_{C(j+1/2)}) &= 2i\sin(\xi/2)\hat{p}_{C(j+1/2)} + \lambda\rho_C c_C^2[\hat{u}_{C(j+1)} - \hat{u}_{Cj} \\
&\quad + \frac{c_C^2\lambda^2 - 1}{24}(\hat{u}_{C(j+2)} - 3\hat{u}_{C(j+1)} + 3\hat{u}_{Cj} - \hat{u}_{C(j-1)})]
\end{aligned}
$$

and

$$
\begin{aligned}
q_u(\hat{u}_{Cj}) &= 2i\sin(\xi/2)\hat{u}_{Cj} + \frac{1}{\rho_C}[\hat{p}_{C(j+1/2)} - \hat{p}_{C(j-1/2)} \\
&\quad + \frac{c_C^2\lambda^2 - 1}{24}(\hat{p}_{C(j+3/2)} - 3\hat{p}_{C(j+1/2)} + 3\hat{p}_{C(j-1/2)} - \hat{p}_{C(j-3/2)})]
\end{aligned}
$$

for $C = L,\ R$. For j well separated from zero, the difference scheme is

$$
\begin{aligned}
q_p(\hat{p}_{L(j+1/2)}) &= 0\,, \\
q_u(\hat{u}_{Lj}) &= 0
\end{aligned}
$$

on the left side of the interface, and

$$
\begin{aligned}
q_p(\hat{p}_{R(j+1/2)}) &= 0\,, \\
q_u(\hat{u}_{Rj}) &= 0
\end{aligned}
$$

on the right side. Close to the interface, these equations are perturbed. Except for the first and the last of the six interface conditions above, the coefficients ρ_L, c_L, ρ_R, c_R are all occurring in each equation. For example, in the third equation, which is centered at the right hand side of the interface, there is a term $\rho_L c_L^2(\hat{u}_{R1} - \hat{u}_{L0})$, which we would like to replace by $\rho_R c_R^2(\hat{u}_{R1} - \hat{u}_{R0})$, where we have extended the definition of u_R to hold also to the left of the interface. Therefore we write

$$
\rho_L c_L^2(\hat{u}_{R1} - \hat{u}_{L0}) = \rho_R c_R^2(\hat{u}_{R1} - \hat{u}_{R0}) + \rho_L c_L^2(\hat{u}_{R0} - \hat{u}_{L0}) + (\rho_L c_L^2 - \rho_R c_R^2)(\hat{u}_{R1} - \hat{u}_{R0})\,.
$$

There is also the last term, which we write as

$$
\frac{1}{24}\hat{u}_{L0} = \frac{1}{24}\hat{u}_{R0} - \frac{1}{24}(\hat{u}_{R0} - \hat{u}_{L0})\,.
$$

The whole equation can be written in the form

$$
q(\hat{p}_{R(3/2)}) = \alpha(\hat{u}_{R0} - \hat{u}_{L0}) + \beta(\hat{u}_{R1} - \hat{u}_{R0})\,,
$$

where α and β are constants of order one. With similar rewriting in the other equations, we get

$$
\begin{aligned}
q(\hat{p}_{L(-1/2)}) &= \alpha_1(\hat{u}_{R1} - \hat{u}_{L1})\,, \\
q(\hat{p}_{L(1/2)}) &= \alpha_2(\hat{u}_{R2} - \hat{u}_{L2}) + \alpha_3(\hat{u}_{R1} - \hat{u}_{L1}) + \alpha_4(\hat{u}_{R0} - \hat{u}_{L0}) \\
&\quad + \beta_1(\hat{u}_{R2} - \hat{u}_{R1}) + \beta_2(\hat{u}_{R1} - \hat{u}_{R0})\,, \\
q(\hat{p}_{R(3/2)}) &= \alpha_5(\hat{u}_{R0} - \hat{u}_{L0}) + \beta_3(\hat{u}_{R1} - \hat{u}_{R0})\,, \\
q(\hat{u}_{L0}) &= \alpha_6(\hat{p}_{R(3/2)} - \hat{p}_{L(3/2)}) + \beta_4(\hat{p}_{R(3/2)} - \hat{p}_{R(1/2)})\,, \\
q(\hat{u}_{R1}) &= \alpha_7(\hat{p}_{R(1/2)} - \hat{p}_{L(1/2)}) + \alpha_8(\hat{p}_{R(-1/2)} - \hat{p}_{L(-1/2)}) + \alpha_9(\hat{p}_{R(3/2)} - \hat{p}_{L(3/2)}) \\
&\quad + \beta_5(\hat{p}_{R(3/2)} - \hat{p}_{R(1/2)}) + \beta_6(\hat{p}_{R(1/2)} - \hat{p}_{R(-1/2)})\,, \\
q(\hat{u}_{R2}) &= \alpha_{10}(\hat{p}_{R(1/2)} - \hat{p}_{L(1/2)})\,,
\end{aligned}
$$

where α_ν, β_ν are nonzero constants. If we can find the constants $\tilde{\sigma}_{L\nu}$, $\tilde{\sigma}_{R\nu}$ making the right hand sides of these equations zero, and at the same time satisfying the outer boundary conditions at

$$x_j\,,\ j=-N,-N+1,-N+2,N-2,N-1,N\,,$$

then the solution is determined, and has the form (4.4), (4.5). When adding the outer boundary conditions, the complete set of equations becomes

$$\begin{aligned}
\hat{u}_{L(-N)} &= \hat{u}_L(-a-\delta h)\,,\\
\hat{u}_{L(-N+1)} &= \hat{u}_L(-a+(1-\delta)h)\,,\\
\hat{u}_{L(-N+2)} &= \hat{u}_L(-a+(2-\delta)h)\,,\\
\hat{u}_{L0}-\hat{u}_{R0} &= \gamma_1(\hat{u}_{R1}-\hat{u}_{R0})\,,\\
\hat{u}_{L1}-\hat{u}_{R1} &= 0\,,\\
\hat{u}_{L2}-\hat{u}_{R2} &= \gamma_2(\hat{u}_{R2}-\hat{u}_{R1})+\gamma_3(\hat{u}_{R1}-\hat{u}_{R0})\,,\\
\hat{p}_{L(-1/2)}-\hat{p}_{R(-1/2)} &= \gamma_4(\hat{p}_{R(3/2)}-\hat{p}_{R(1/2)})+\gamma_5(\hat{p}_{R(1/2)}-\hat{p}_{R(-1/2)})\,,\\
\hat{p}_{L(1/2)}-\hat{p}_{R(1/2)} &= 0\,,\\
\hat{p}_{L(3/2)}-\hat{p}_{R(3/2)} &= \gamma_6(\hat{p}_{R(3/2)}-\hat{p}_{R(1/2)})\,,\\
\hat{u}_{R(N-2)} &= \hat{u}_R(a-(2+\delta)h)\,,\\
\hat{u}_{R(N-1)} &= \hat{u}_R(a-(1+\delta)h)\,,\\
\hat{u}_{RN} &= \hat{u}_R(a-\delta h)\,,
\end{aligned}$$

where the constants γ_ν are determined by the constants α_ν and β_ν . The corresponding system for the $\tilde{\sigma}$-coefficients is

$$\begin{aligned}
&\kappa_{L1}^{-N+\nu}\tilde{\sigma}_{L1}+\kappa_{L2}^{-N+\nu}\tilde{\sigma}_{L2}+\kappa_{L5}^{\nu}v_{L5}^{(2)}\tilde{\sigma}_{L5}+\kappa_{L6}^{\nu}v_{L6}^{(2)}\tilde{\sigma}_{L6}\\
&\quad=\hat{u}_L(-a+(\nu-\delta)h)\,,\ \nu=0,1,2\,,\\
&\kappa_{L1}^{\nu}\tilde{\sigma}_{L1}+\kappa_{L2}^{\nu}\tilde{\sigma}_{L2}+\kappa_{L3}^{\nu}v_{L3}^{(2)}\tilde{\sigma}_{L3}+\kappa_{L4}^{\nu}v_{L4}^{(2)}\tilde{\sigma}_{L4}\\
&\quad-\kappa_{R1}^{\nu}\tilde{\sigma}_{R1}-\kappa_{R2}^{\nu}\tilde{\sigma}_{R2}-\kappa_{R5}^{\nu}v_{R5}^{(2)}\tilde{\sigma}_{R5}-\kappa_{R6}^{\nu}v_{R6}^{(2)}\tilde{\sigma}_{R6}=g_\nu\,,\ \nu=0,1,2\,,\\
&-\rho_L\tilde{c}_L\kappa_{L1}^{\nu-1/2}\tilde{\sigma}_{L1}+\rho_L\tilde{c}_L\kappa_{L2}^{\nu-1/2}\tilde{\sigma}_{L2}+\kappa_{L3}^{\nu-1}v_{L3}^{(1)}\tilde{\sigma}_{L3}+\kappa_{L4}^{\nu-1}v_{L4}^{(1)}\tilde{\sigma}_{L4}\\
&\quad+\rho_R\tilde{c}_R\kappa_{R1}^{\nu-1/2}\tilde{\sigma}_{R1}-\rho_R\tilde{c}_R\kappa_{R2}^{\nu-1/2}\tilde{\sigma}_{R2}-\kappa_{R5}^{\nu-1}v_{R5}^{(1)}\tilde{\sigma}_{R5}-\kappa_{R6}^{\nu-1}v_{R6}^{(1)}\tilde{\sigma}_{R6}\\
&\quad=g_{\nu+3}\,,\ \nu=0,1,2\,,\\
&\kappa_{R1}^{N-\nu}\tilde{\sigma}_{R1}+\kappa_{R2}^{N-\nu}\tilde{\sigma}_{R2}+\kappa_{R3}^{-\nu}v_{R3}^{(2)}\tilde{\sigma}_{R3}+\kappa_{R4}^{-\nu}v_{R4}^{(2)}\tilde{\sigma}_{R4}\\
&\quad=\hat{u}_R(a-(\nu+\delta)h)\,,\ \nu=0,1,2\,,
\end{aligned}\tag{4.7}$$

where

$$\begin{aligned}
g_0 &= \gamma_1(\hat{u}_{R1} - \hat{u}_{R0})\,,\\
g_1 &= 0\,,\\
g_2 &= \gamma_2(\hat{u}_{R2} - \hat{u}_{R1}) + \gamma_3(\hat{u}_{R1} - \hat{u}_{R0})\,,\\
g_3 &= \gamma_4(\hat{p}_{R(3/2)} - \hat{p}_{R(1/2)}) + \gamma_5(\hat{p}_{R(1/2)} - \hat{p}_{R(-1/2)})\,,\\
g_4 &= 0\,,\\
g_5 &= \gamma_6(\hat{p}_{R(3/2)} - \hat{p}_{R(1/2)})\,.
\end{aligned}$$

Here we have disregarded terms of order $\mathcal{O}(\kappa_{C\nu}^{-N})$ and $\mathcal{O}(\kappa_{C(\nu+2)}^{N})$, $C = L, R$, $\nu = 3, 4$, since their size is on rounding error level. As in the second order case we make the ansatz

$$\begin{aligned}
\tilde{\sigma}_{L\nu} &= \sigma_{L\nu} + \tau_{L\nu}\ , \quad \nu = 1, 2, \ldots, 6\,,\\
\tilde{\sigma}_{R\nu} &= \sigma_{R\nu} + \tau_{R\nu}\ , \quad \nu = 1, 2, \ldots, 6\,,
\end{aligned}$$

where σ_{L1}, σ_{L2}, σ_{R1}, σ_{R2} are defined by the true solution, and where

$$\sigma_{L\nu} = \sigma_{R\nu} = 0\ , \quad \nu = 3, 4, 5, 6\,.$$

By using the relations (4.3), (4.6), we obtain with $\mu = \omega^5 h^4$

$$\begin{aligned}
&[e^{-i\omega(N-\nu)h/c_L} + \mathcal{O}(\mu)](\sigma_{L1} + \tau_{L1}) + [e^{i\omega(N-\nu)h/c_L} + \mathcal{O}(\mu)](\sigma_{L2} + \tau_{L2})\\
&\quad + \kappa_{L5}^{\nu} v_{L5}^{(2)} \tau_{L5} + \kappa_{L6}^{\nu} v_{L6}^{(2)} \tau_{L6} = \hat{u}_L(-a + (\nu - \delta)h)\ , \quad \nu = 0, 1, 2\,,\\[2ex]
&[e^{i\omega\nu h/c_L} + \mathcal{O}(\omega^5 h^5)](\sigma_{L1} + \tau_{L1}) + [e^{-i\omega\nu h/c_L} + \mathcal{O}(\omega^5 h^5)](\sigma_{L2} + \tau_{L2})\\
&\quad + \kappa_{L3}^{\nu} v_{L3}^{(2)} \tau_{L3} + \kappa_{L4}^{\nu} v_{L4}^{(2)} \tau_{L4}\\
&\quad - [e^{i\omega\nu h/c_R} + \mathcal{O}(\omega^5 h^5)](\sigma_{R1} + \tau_{R1}) - [e^{-i\omega\nu h/c_R} + \mathcal{O}(\omega^5 h^5)](\sigma_{R2} + \tau_{R2})\\
&\quad - \kappa_{R5}^{\nu} v_{R5}^{(2)} \tau_{R5} - \kappa_{R6}^{\nu} v_{R6}^{(2)} \tau_{R6}\\
&\quad = \mathcal{O}(\xi)(\sigma_{R1} + \sigma_{R2} + \tau_{R1} + \tau_{R2}) + \tilde{g}_\nu\ , \quad \nu = 0, 1, 2\,,\\[2ex]
&- \rho_L[c_L + \mathcal{O}(\mu)][e^{\omega\nu h/c_L} + \mathcal{O}(\omega^5 h^5)](\sigma_{L1} + \tau_{L1}) + \rho_L[c_L + \mathcal{O}(\mu)]\\
&\quad [e^{-i\omega\nu h/c_L} + \mathcal{O}(\omega^5 h^5)](\sigma_{L2} + \tau_{L2}) + \kappa_{L3}^{\nu-1} v_{L3}^{(1)} \tau_{L3} + \kappa_{L4}^{\nu-1} v_{L4}^{(1)} \tau_{L4}\\
&\quad + \rho_R[c_R + \mathcal{O}(\mu)][e^{i\omega\nu h/c_R} + \mathcal{O}(\omega^5 h^5)](\sigma_{R1} + \tau_{R1}) - \rho_R[c_R + \mathcal{O}(\omega^5 h^5)]\\
&\quad [e^{-i\omega\nu h/c_R} + \mathcal{O}(\omega^5 h^5)](\sigma_{R2} + \tau_{R2}) - \kappa_{R5}^{\nu-1} v_{R5}^{(1)} \tau_{R5} - \kappa_{R6}^{\nu-1} v_{R6}^{(1)} \tau_{R6}\\
&\quad = \mathcal{O}(\xi)(\sigma_{R1} + \sigma_{R2} + \tau_{R1} + \tau_{R2}) + \tilde{g}_{\nu+3}\ , \quad \nu = 0, 1, 2\,,\\[2ex]
&[e^{i\omega(N-\nu)h/c_R} + \mathcal{O}(\mu)](\sigma_{R1} + \tau_{R1}) + [e^{-i\omega(N-\nu)h/c_R} + \mathcal{O}(\mu)](\sigma_{R2} + \tau_{R2})\\
&\quad + \kappa_{R3}^{-\nu} v_{R3}^{(2)} \tau_{R3} + \kappa_{R4}^{-\nu} v_{R4}^{(2)} \tau_{R4} = \hat{u}_R(a - (\nu + \delta)h)\ , \quad \nu = 0, 1, 2\,,
\end{aligned} \tag{4.8}$$

where

$$
\begin{aligned}
\tilde{g}_0 &= \gamma_1[(\kappa_{R5}-1)v_{R5}^{(2)}\tau_{R5} + (\kappa_{R6}-1)v_{R6}^{(2)}\tau_{R6}]\,, \\
\tilde{g}_1 &= 0\,, \\
\tilde{g}_2 &= \gamma_2[\kappa_{R5}(\kappa_{R5}-1)v_{R5}^{(2)}\tau_{R5} + \kappa_{R6}(\kappa_{R6}-1)v_{R6}^{(2)}\tau_{R6}] \\
&\quad + \gamma_3[(\kappa_{R5}-1)v_{R5}^{(2)}\tau_{R5} + (\kappa_{R6}-1)v_{R6}^{(2)}\tau_{R6}]\,, \\
\tilde{g}_3 &= \gamma_4[\kappa_{R5}^{1/2}(\kappa_{R5}-1)v_{R5}^{(1)}\tau_{R5} + \kappa_{R6}^{1/2}(\kappa_{R6}-1)v_{R6}^{(1)}\tau_{R6}] \\
&\quad + \gamma_5[\kappa_{R5}^{-1/2}(\kappa_{R5}-1)v_{R5}^{(1)}\tau_{R5} + \kappa_{R6}^{-1/2}(\kappa_{R6}-1)v_{R6}^{(1)}\tau_{R6}]\,, \\
\tilde{g}_4 &= 0\,, \\
\tilde{g}_5 &= \gamma_6[\kappa R5^{1/2}(\kappa_{R5}-1)v_{R5}^{(1)}\tau_{R5} + \kappa_{R6}^{1/2}(\kappa_{R6}-1)v_{R6}^{(1)}\tau_{R6}]\,.
\end{aligned}
$$

Consider first the first three equations. By construction the σ-terms cancel except for those multiplied by $\mathcal{O}(\xi+\mu)$. These remaining terms are put on the right hand side in the system for the τ-coefficients, and we get

$$
\begin{aligned}
&[e^{-i\omega Nh/c_L} + \mathcal{O}(\xi+\mu)]\tau_{L1} + [e^{i\omega Nh/c_L} + \mathcal{O}(\xi+\mu)]\tau_{L2} + v_{L5}^{(2)}\tau_{L5} + v_{L6}^{(2)}\tau_{L6} \\
&\quad = \mathcal{O}(\xi+\mu)\,, \\
&[e^{-i\omega Nh/c_L} + \mathcal{O}(\xi+\mu)]\tau_{L1} + [e^{i\omega Nh/c_L} + \mathcal{O}(\xi+\mu)]\tau_{L2} + \kappa_{L5}v_{L5}^{(2)}\tau_{L5} + \kappa_{L6}v_{L6}^{(2)}\tau_{L6} \\
&\quad = \mathcal{O}(\xi+\mu)\,, \\
&[e^{-i\omega Nh/c_L} + \mathcal{O}(\xi+\mu)]\tau_{L1} + [e^{i\omega Nh/c_L} + \mathcal{O}(\xi+\mu)]\tau_{L2} + \kappa_{L5}^2v_{L5}^{(2)}\tau_{L5} + \kappa_{L6}^2v_{L6}^{(2)}\tau_{L6} \\
&\quad = \mathcal{O}(\xi+\mu)\,.
\end{aligned}
$$

By subtracting the second and the third equation from the first one, we get a system for τ_{L5} and τ_{L6} with a determinant that is bounded from zero independent of ξ and μ. We get

$$
\begin{aligned}
\tau_{L5} &= \mathcal{O}(\xi+\mu)(\tau_{L1}+\tau_{L2}) + \mathcal{O}(\xi+\mu)\,, && (4.9)\\
\tau_{L6} &= \mathcal{O}(\xi+\mu)(\tau_{L1}+\tau_{L2}) + \mathcal{O}(\xi+\mu)\,, && (4.10)\\
[e^{-i\omega Nh/c_L} &+ \mathcal{O}(\xi+\mu)]\tau_{L1} + [e^{i\omega Nh/c_L} + \mathcal{O}(\xi+\mu)]\tau_{L2} = \mathcal{O}(\xi+\mu)\,. && (4.11)
\end{aligned}
$$

In the same way we get from the last three equations

$$
\begin{aligned}
\tau_{R3} &= \mathcal{O}(\xi+\mu)(\tau_{R1}+\tau_{R2}) + \mathcal{O}(\xi+\mu)\,, && (4.12)\\
\tau_{R4} &= \mathcal{O}(\xi+\mu)(\tau_{R1}+\tau_{R2}) + \mathcal{O}(\xi+\mu)\,, && (4.13)\\
[e^{i\omega Nh/c_R} &+ \mathcal{O}(\xi+\mu)]\tau_{R1} + [e^{-i\omega Nh/c_R} + \mathcal{O}(\xi+\mu)]\tau_{R2} = \mathcal{O}(\xi+\mu)\,. && (4.14)
\end{aligned}
$$

In the 4th to 6th equation of (4.8) the σ-terms cancel except for $\mathcal{O}(\xi+\mu)$-terms, since they satisfy the continuity equations. The remaining part can be written

$$
\begin{aligned}
&(1+\mathcal{O}(\xi+\mu))\tau_{L1}+(1+\mathcal{O}(\xi+\mu))\tau_{L2}+v_{L3}^{(2)}\tau_{L3}+v_{L4}^{(2)}\tau_{L4} \\
&\quad -[1+\gamma_1(\kappa_{R5}-1)]v_{R5}^{(2)}\tau_{R5}-[1+\gamma_1(\kappa_{R6}-1)]v_{R6}^{(2)}\tau_{R6}=\mathcal{O}(\xi+\mu)\,, \qquad (4.15)\\
&(1+\mathcal{O}(\xi+\mu))\tau_{L1}+(1+\mathcal{O}(\xi+\mu))\tau_{L2}+\kappa_{L3}v_{L3}^{(2)}\tau_{L3}+\kappa_{L4}v_{L4}^{(2)}\tau_{L4} \\
&\quad -\kappa_{R5}v_{R5}^{(2)}\tau_{R5}-\kappa_{R6}v_{R6}^{(2)}\tau_{R6}=0\,, \qquad (4.16)\\
&(1+\mathcal{O}(\xi+\mu))\tau_{L1}+(1+\mathcal{O}(\xi+\mu))\tau_{L2}+\kappa_{L3}^2v_{L3}^{(2)}\tau_{L3}+\kappa_{L4}^2v_{L4}^{(2)}\tau_{L4} \\
&\quad -[\kappa_{R5}^2+(\gamma_3+\kappa_{R5}\gamma_2)(\kappa_{R5}-1)]v_{R5}^{(2)}\tau_{R5} \\
&\quad -[\kappa_{R6}^2+(\gamma_3+\kappa_{R6}\gamma_2)(\kappa_{R6}-1)]v_{R6}^{(2)}\tau_{R6}=\mathcal{O}(\xi+\mu)\,. \qquad (4.17)
\end{aligned}
$$

By subtracting (4.16) and (4.17) respectively from (4.15), we get two equations containing only τ_{L3}, τ_{L4}, τ_{R5}, τ_{R6} and $\mathcal{O}(\xi+\mu)(\tau_{L1}+\tau_{L2})$. Similarly, from the 7th to the 9th equation of (4.8) we can eliminate τ_{R1}, τ_{R2} and get two more equations of the same type. These four equations are solved and yield

$$
\begin{bmatrix} \tau_{L3} \\ \tau_{L4} \\ \tau_{R5} \\ \tau_{R6} \end{bmatrix} = \mathcal{O}(\xi+\mu) \begin{bmatrix} \tau_{L1} \\ \tau_{L2} \\ \tau_{R1} \\ \tau_{R2} \end{bmatrix} + \mathcal{O}(\xi+\mu)\,. \qquad (4.18)
$$

Finally, we collect the remaining four equations for τ_{L3}, τ_{L4}, τ_{R5}, τ_{R6}. Three of these equations are (4.11), (4.14) and (4.15), the fourth one is derived from the 7th equation of (4.8), but has not been written out here. The resulting system is

$$
(A+\mathcal{O}(\xi+\mu)) \begin{bmatrix} \tau_{L1} \\ \tau_{L2} \\ \tau_{R1} \\ \tau_{R2} \end{bmatrix} = \mathcal{O}(\xi+\mu)\,, \qquad (4.19)
$$

where the non-singular matrix A is defined in (2.4). Hence, there is a solution of order $\xi+\mu$. Consequently, the remaining coefficients $(\tau_{L\nu}, \tau_{R\nu})$, $\nu=3,4,5,6$ are also of order $\xi+\mu$.

Hence, we have proven that the solution has an error $\mathcal{O}(\omega h+\omega^5h^4)$. In analogy with the second order case, we prescribe an error tolerance ε, and require first

$$
\frac{\omega^5a^4}{N^4} \le \varepsilon\,.
$$

The number of points per wavelength N_ω must satisfy

$$
N_\omega \ge \gamma_1(\omega/\varepsilon)^{1/4}\,,
$$

i.e., it is still increasing with ω but can be chosen smaller than in the second order case. The first order error implies, just as for the second order scheme,

$$
N_\omega \ge \gamma_2/\varepsilon\,,
$$

which is independent of ω.

5 Numerical experiments

In the numerical experiments we are solving the problem (2.1) in the domain $0 \le x \le 2\pi$, $0 \le t \le T$. The discontinuity is located at $x = \pi/2$, and for convenience we are considering solutions that are periodic also in space. For that purpose, we introduce a second discontinuity at $x = 3\pi/2$ such that

$$(\rho(x),\, c(x)) = \begin{cases} (\rho_L,\, c_L)\ , & \{0 \le x \le \pi/2,\ 3\pi/2 \le x \le 2\pi\} \\ (\rho_R,\, c_R)\ , & \pi/2 < x < 3\pi/2\,. \end{cases}$$

We use the parameters $\rho_R = c_R = 1$ and $\rho_L < 1$, $c_L < 1$, and substitute $e^{i\omega t}$ used in the analysis above by $\sin(\omega t)$. As long as $0 \le t < \pi/2$, the solution in the x-interval $[0,\, \pi)$ is independent of the discontinuity at $x = 3\pi/2$, and we have

$\mathbf{0 \le x \le \pi/2}$ **:**

$$\begin{aligned} p(x,t) &= \sin(t - \frac{x-\pi/2}{c_L}) - \frac{\rho_L c_L - \rho_R c_R}{\rho_L c_L + \rho_R c_R}\sin(t + \frac{x-\pi/2}{c_L})\,, \\ u(x,t) &= \frac{1}{\rho_L c_L}[\sin(t - \frac{x-\pi/2}{c_L}) + \frac{\rho_L c_L - \rho_R c_R}{\rho_L c_L + \rho_R c_R}\sin(t + \frac{x-\pi/2}{c_L})]\,, \end{aligned}$$

$\boldsymbol{\pi/2 < x < \pi}$ **:**

$$\begin{aligned} p(x,t) &= \frac{2\rho_R c_R}{\rho_L c_L + \rho_R c_R}\sin(t - \frac{x-\pi/2}{c_R})\,, \\ u(x,t) &= \frac{2}{\rho_L c_L + \rho_R c_R}\sin(t - \frac{x-\pi/2}{c_R})\,. \end{aligned}$$

We want to simulate the case where many waves are hitting the interface $x = \pi/2$. Instead of running the scheme over long time intervals, we stay with $0 \le t < \pi/2$ and choose instead large ω-values. In order to keep periodicity in x, we use $\omega = \omega_0 c_L$,where ω_0 is an integer. Since the wavelength in time is $2\pi/\omega$, we are running $\omega/4$ wavelengths in time, and during that time, $\omega_0/4$ waves are hitting the interface $x = \pi/2$.

In the analysis above, we have discussed the necessary number of points N_ω per wavelength, which is proportional to N/ω. More precisely for our case, the shortest wavelength is in the x-interval $[0,\, \pi/2]$, and on a uniform grid we are interested in the shortest wavelength, which is represented by the fewest number of points. In our case, the shortest wavelength in space is $2\pi c_L/\omega = 2\pi/\omega_0$, and one wavelength is represented by $N_\omega = N/\omega_0$ points. In the figure captions we are giving the parameters ω_0 and N_ω.

In [2] we presented some preliminary experiments. Here we give a more detailed experimental analysis of the same case, which confirms the theoretical results above. The parameters are

$$\begin{aligned} \rho_L &= 0.55556\,, \quad \rho_R = 1.00000\,, \\ c_L &= 0.87879\,, \quad c_R = 1.00000\,, \end{aligned}$$

corresponding to sediment and water respectively. The Courant number is $\lambda = k/h = 0.8$.

In the following, we denote the two schemes by Q_2 and Q_4. We first present a relatively short run with $T = 0.12\pi$ corresponding to 7 wavelengths in time. Figures 5.1 and 5.2 show the error

$$\varepsilon^n = \max_j |p(x_{j+1/2}, t_{n+1/2}) - p_{j+1/2}^{n+1/2}|$$

as a function of time for Q_2 and Q_4 for $N = 960$ and $N = 1920$ respectively.

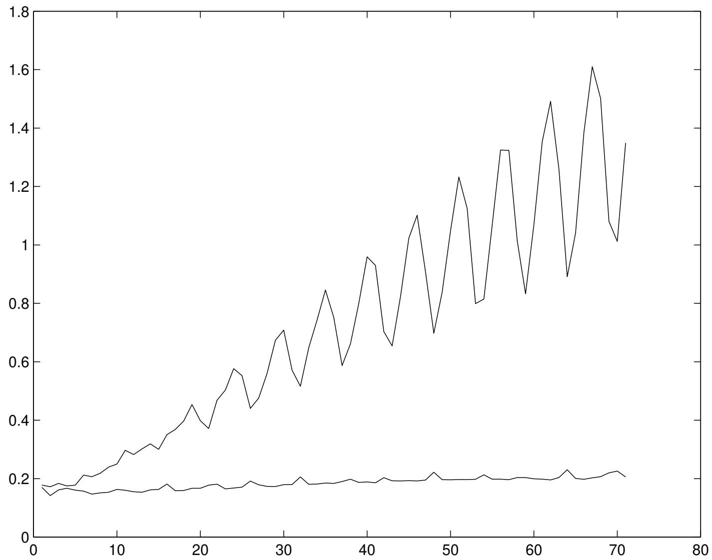

Figure 5.1: The error ε^n as a function of time for Q_2 and Q_4 (lower curve), $T = 0.12\pi$, $\omega_0 = 128$, $N_\omega = 7.5$.

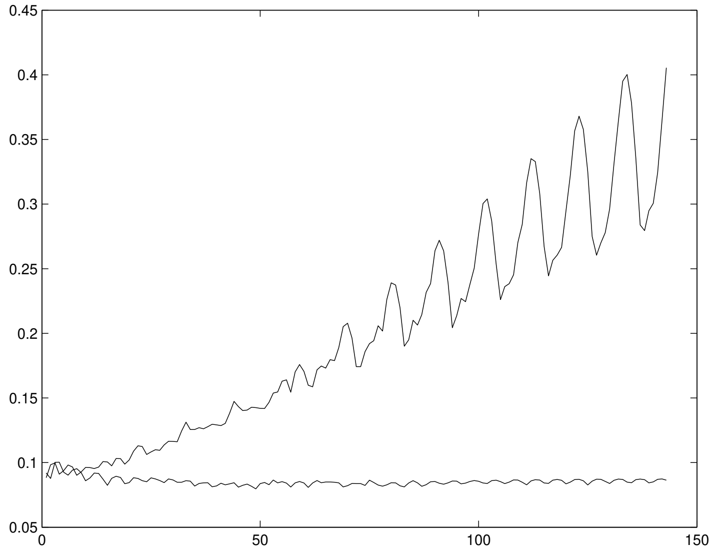

Figure 5.2: The error ε^n as a function of time for Q_2 and Q_4 (lower curve), $T = 0.12\pi$, $\omega_0 = 128$, $N_\omega = 15$.

The error for Q_2 grows quickly due to the $\omega^3 h^2$-term in the truncation error. On the contrary, Q_4 shows an almost constant error, which is due to the fact that the ωh-term in the truncation error dominates. Indeed, the following table shows the error

$$\varepsilon = \max_{0 \le t_n \le 0.12\pi} \varepsilon^n$$

for the same cases.

N_ω	Q_2	Q_4
7.5	1.61	0.23
15	0.41	0.10

Table 5.1: Max error for $0 \le t \le 0.12\pi$.

The second order (but larger) error is clearly demonstrated for Q_2, while the first order (but smaller) error is exhibited by Q_4 .

We next present a longer run with $T = 0.48\pi$ corresponding to 28 wavelengths in time, see Figures 5.3 and 5.4.

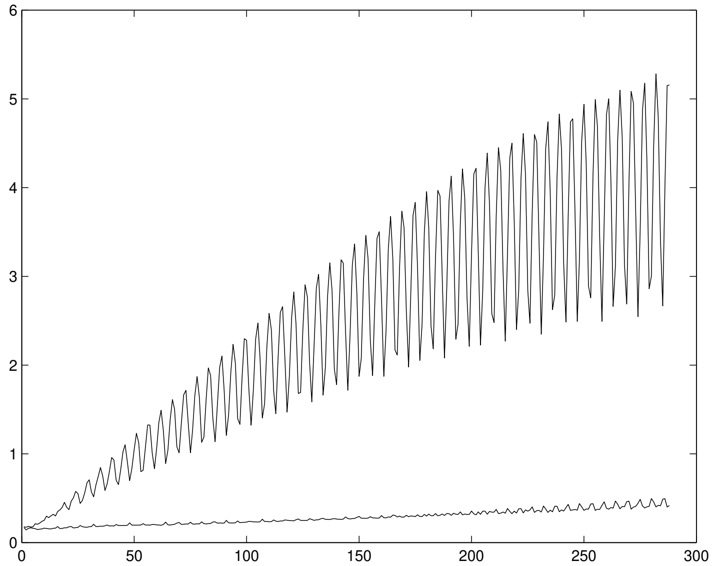

Figure 5.3: The error ε^n as a function of time for Q_2 and Q_4 (lower curve), $T = 0.48\pi$, $\omega_0 = 128$, $N_\omega = 7.5$.

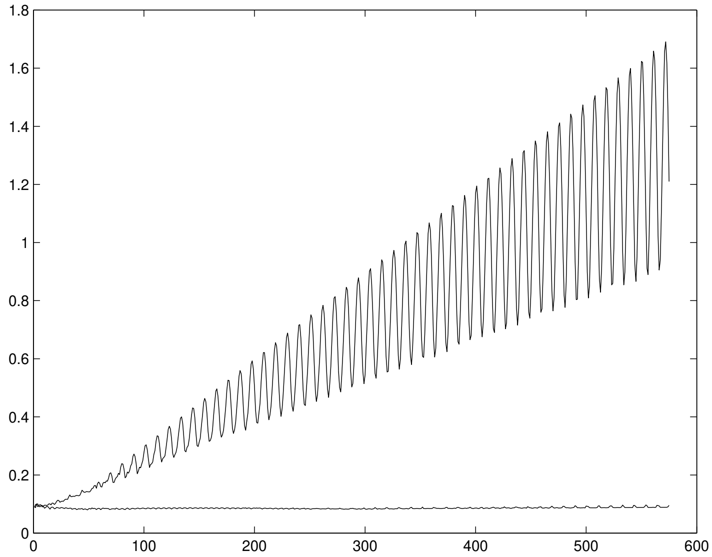

Figure 5.4: The error ε^n as a function of time for Q_2 and Q_4 (lower curve), $T = 0.48\pi$, $\omega_0 = 128$, $N_\omega = 15$.

Q_4 is again superior, but shows now a slight increase of the error at the end of the time interval. This is an effect of the growing influence of the $\omega^5 h^4$-term in the truncation error. The error

$$\varepsilon = \max_{0 \le t_n \le 0.48\pi} \varepsilon^n$$

is shown in Table 5.2.

N_ω	Q_2	Q_4
7.5	5.28	0.50
15	1.69	0.10

Table 5.2: Max error for $0 \le t \le 0.48\pi$.

The convergence rate is now somewhere between 1st and 4th order for Q_4 .

After even longer time, the error will behave like a normal 4th order scheme for smooth solutions. This type of behavior as a function of time is described in more detail in [1].

Next we show the solution as a function of x at $t = 0.48\pi$ for the higher frequency $\omega_0 = 256$ for the two schemes in Figures 5.5 and 5.6 respectively. The solution is shown in an interval of length $\pi/20$ around the interface $\pi/4$ (located at the point no. 81 in the figure).

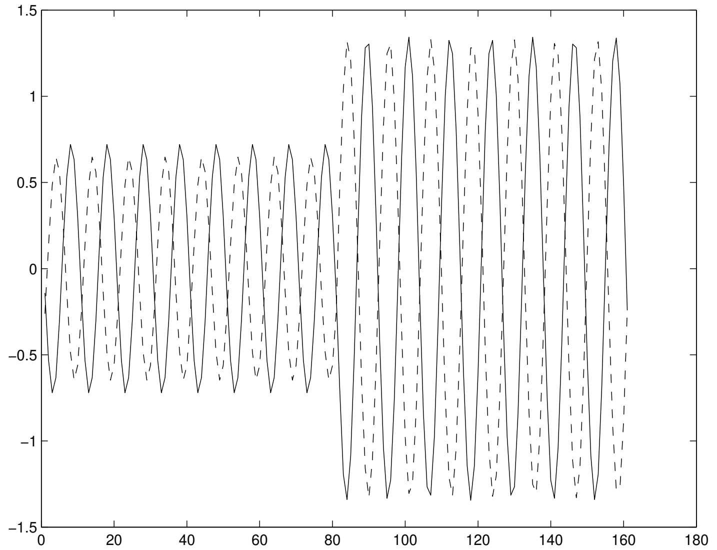

Figure 5.5: p as a function of x , $\omega_0 = 256$, $N_\omega = 10$, $N = 2560$. Exact solution (—), Q_2 approximation $(- \; - \; -)$

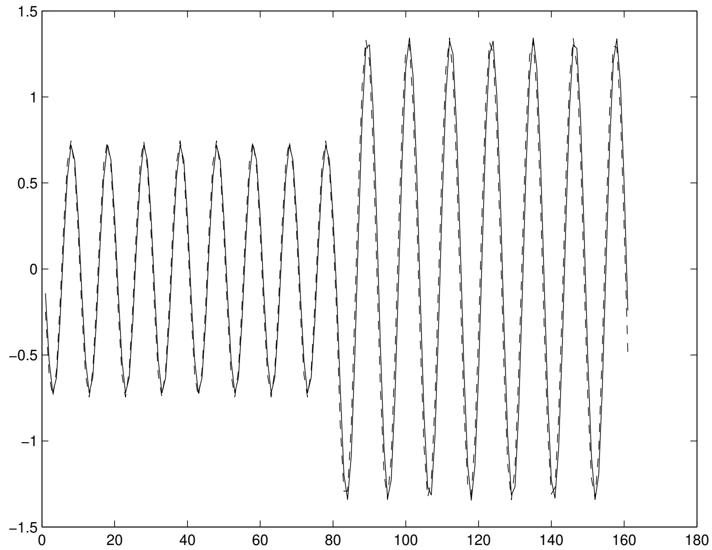

Figure 5.6: p as a function of x , $\omega_0 = 256$, $N_\omega = 10$, $N = 2560$. Exact solution (—), Q_4 approximation $(- \; - \; -)$.

The Q_2-scheme is completely out of phase, while the Q_4-scheme is almost exact.

Finally we present a case with an odd number of gridpoints, such that the interface falls in between a gridpoint x_j and its neighbor $x_{j+1/2}$. Figure 5.7 shows the solution for $N = 2561$ to be compared with the previous case $N = 2560$. As expected from the analysis, the accuracy is the same.

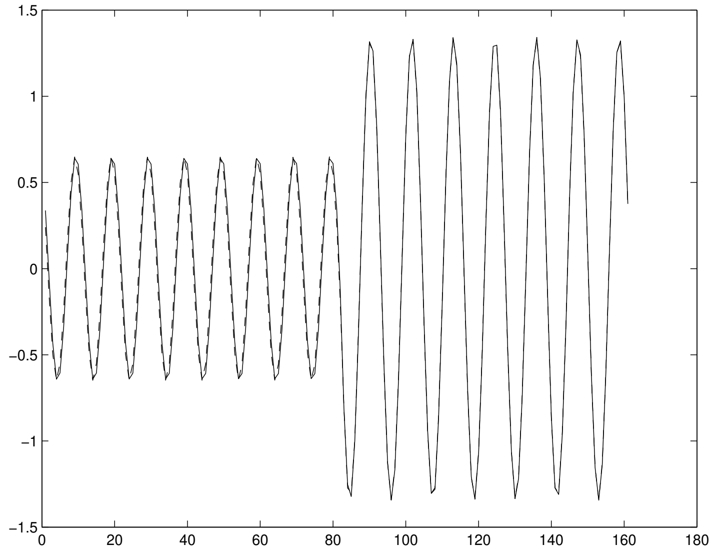

Figure 5.7: p as a function of x , $\omega_0 = 256$, $N_\omega \approx 10$, $N = 2561$. Exact solution (—), Q_4 approximation (− − −).

6 Conclusions

We have analyzed the behavior of the one-step difference schemes Q_2 and Q_4 developed in [2] when applied to acoustic wave propagation in discontinuous media. The schemes are applied without any special procedure across the interface. In this case the schemes are formally only first order accurate, even if they have 2nd and 4th order accuracy, respectively, for smooth solutions. The analysis explains why the Q_4-scheme still gives much more accurate results regardless of the location of the grid relative to the interface. In other words, when integrating over long time intervals, the necessary number of points per wavelength is determined by the properties of the particular scheme for smooth solutions.

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