Second Order ODEs are Sufficient for Modeling of Many Periodic Signals

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Abstract

Which is the minimum order an autonomous nonlinear ordinary differential equation (ODE) needs to have to be able to model a periodic signal? This question is motivated by recent research on periodic signal analysis, where nonlinear ODEs are used as models. The results presented here show that an order of two of the ODE is sufficient for a large class of periodic signals. More precisely, conditions on a periodic signal are established that imply the existence of an ODE that has the periodic signal as a solution. A criterion that characterizes the above class of periodic signals by means of the overtone contents of the signals is also presented. The reason why higher order ODEs are sometimes needed is illustrated with geometric arguments. Extensions of the theoretical analysis to cases with orders higher than two are developed using this insight.

Keywords: Identification, Nonlinear systems, Periodic motion, Phase plane.

I. INTRODUCTION

Periodic signals arise in many applications. Power network supervision, auto-tuning [1] of PID regulators, instability phenomena in nonlinear feedback systems [2] and the measurement of linearity of electronic power amplifiers with sinusoidal input are only a few examples where the analysis of periodic signals is central.

As a result, a wide variety of methods has been proposed for periodic signal analysis, see e.g. [3]. The periodogram combined with the fast Fourier transform (FFT) techniques forms the baseline for performance comparisons. More advanced methods in use include various parametric approaches [4], comb filters [5] as well as high resolution methods like MUSIC and ESPRIT from the sensor array processing field. See [3] for algorithmic details and a performance analysis.

The problem discussed in the present paper originates from the idea to use nonlinear ODEs as models for periodic signal generation. Many such systems have been documented, e.g. nonlinear pendulum systems, tunnel diode circuits as well as the predator-prey equations [2]. As described in detail in [2] an extensive theory around periodic orbit phenomena can be built up for second order nonlinear ODEs. Well known results include e.g. the Poincare map and the Bendixon theorem that treat the existence and the stability of periodic orbits. The reason why the majority of these result are only valid for second order ODEs is that closed orbits in R^2 that do not intersect themselves divide the state space into one part interior to the orbit and another part exterior to the orbit (the Jordan curve theorem). Because of this theoretical background, an ODE of order two was used for the algorithm development in [6] and [7]. The right hand side of the ODE model was parameterized with a bi-polynomial in the two state variables, with the parameters being the polynomial coefficients. Recursive algorithms for estimating the parameters were then derived in [6], based on the Kalman filter and the extended Kalman filter (EKF). An off-line least squares algorithm was presented in [7]. The parameterization allows for identification of arbitrary right hand side functions of the ODE and the performance of the methods seems to be good. The question regarding the generality of the restriction to second order ODEs remains though, and is dealt with here.

The first contribution of this paper is a characterization of the class of periodic signals that can be generated by a nonlinear second order ODE that fulfills normal regularity conditions. The analysis is performed in the phase plane. This allows the discussion to be supported by straightforward and intuitive geometric interpretations, thereby making the result available to a wide audience. The geometric interpretations has the additional benefit of making it clear precisely when an order of two is not sufficient, this being the second contribution of the paper. This observation is used to extend the theoretical results for second order ODEs to arbitrary order. The construction of a criterion that implies that a second order ODE is sufficient constitutes the third contribution. This criterion is shown to be related to the overtone contents of the periodic signal. It is proved that the criterion is unaffected by linear coordinate transformations.

The paper is organized as follows. Main results are presented in section II. The frequency domain sufficiency criterion is discussed in section III. Numerical examples appear in section IV, and the conclusions are summarized in section V. Conditions required in the discussion are denoted C1), C2),…, and they are introduced when needed in the development that follows.

II. MAIN RESULTS

Some assumptions of general validity are first introduced to set the framework for the discussion. The measured signal is denoted by $x(t)$ where t denotes continuous time. The signal is periodic, i.e. it fulfills

C1) $x(t+T)=x(t)$, $\forall t \in R$, $0<T<\infty$, where T denotes the period.

Throughout the paper it is assumed that no noise affects the signals. This is in line with the purpose of the analysis since a periodic signal with noise added is no longer periodic. This assumption is expressed as

C2) The signal $x(t)$ is not corrupted by any disturbances.

A. When Order Two Is Enough

A brief outline of the development of the result is as follows. First $x(t)$ and $\dot{x}(t)$ are used to introduce a (tentative) state space of second order. This allows the periodic signal to be represented by a closed curve in the state space. A condition can then be formulated that allows the curve to be uniquely described without reference to any quantities other than points in the state space. By the introduction of smoothness assumptions on the closed curve a formal treatment is used to construct a second order ODE that has the closed curve, and hence the periodic signal $x(t)$, as a solution. Finally, the uniqueness of the solution of this ODE is assessed.

When referring to signals the notation $x(t)$ and $\dot{x}(t)$ is used, while the corresponding states of the ODE are referred to as $x_1(t)$ and $x_2(t)$. For the 2-dimensional state space the most suitable of the notations is used.

To proceed the following assumption is introduced

C3) The signal $x(t)$ is twice continuously differentiable.

The state space spanned by $\mathbf{x}=\begin{pmatrix} x & \dot{x} \end{pmatrix}^T$ is then well defined. Since $\dot{x}(t+T)=\dot{x}(t)$, $\forall t \in R$ by C3) it follows that the periodic signal $x(t)$ can be represented as the first coordinate of the *closed* curve $\mathbf{x}(t)=\begin{pmatrix} x(t) & \dot{x}(t) \end{pmatrix} \in R^2$.

The objective is now to construct an ODE whose solution generates $\mathbf{x}(t)$. In order for this to be possible the state vector $\begin{pmatrix} x_1 & x_2 \end{pmatrix}^T$ of the ODE (if it exists) must contain *all* information of the signal. No use of any additional parameters (like t) is allowed. The following example illustrates this fact further.

Example 1: Fig. 1 is useful to gain insight into what can happen. The signal of Fig. 1a is generated as $x(t) = y_1(t) + 1/2\, y_2(t)$, where $y_1(t) = \cos(t)$ and $y_2(t) = \cos(2t)$ are solutions of $\ddot{y}_1 + y_1 = 0$ and $\ddot{y}_2 + 4y_2 = 0$, respectively. Note that it is straightforward to write the signal $x(t)$ as a solution to the following fourth order linear ODE, obtained by a combination of the two ODEs above

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 1/2 \\ -1 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -4 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}, \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} y_1 + 1/2\, y_2 \\ \dot{y}_1 \\ y_2 \\ \dot{y}_2 \end{pmatrix}, \quad x = x_1 . \qquad (1)$$

(a) (b)

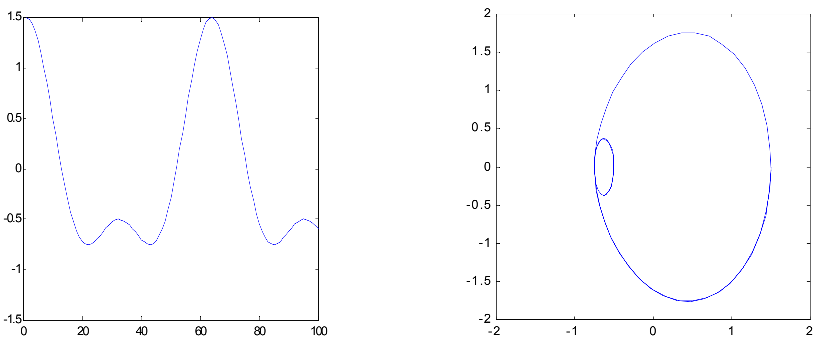

Figure 1: A periodic signal generated by a fourth order linear ODE and the corresponding model phase plane plot.

Fig. 1b shows the curve $\mathbf{x}(t) = \begin{pmatrix} x(t) & \dot{x}(t) \end{pmatrix}^T$, i.e. the attempted model corresponds to a tentative second order ODE. Since the curve $\mathbf{x}(t)$ makes an additional loop, encircling points on the negative horizontal axis, it follows that $\mathbf{x}(t)$ intersects itself at least once. This means that there is a point $\mathbf{x}_I$ in the "model state space" and on the curve $\mathbf{x}(t)$ where the further evolution of $\mathbf{x}(t)$ depends on the evolution of $\mathbf{x}(t)$ before reaching $\mathbf{x}_I$. The consequence is that the further evolution of $\mathbf{x}(t)$ cannot be determined only from one point in the model of the state space, i.e. the state of a second order ODE *cannot* contain all information needed to generate $\mathbf{x}(t)$.

A condition that excludes situations where $\mathbf{x}(t)$ intersects itself hence needs to be imposed. That condition must also exclude other degenerate cases that can be thought of as limiting cases of example 1. Cusps, corners and stops are examples of such limiting cases. The condition needed can be heuristically formulated as "the curve $\mathbf{x}(t) = \begin{pmatrix} x(t) & \dot{x}(t) \end{pmatrix}^T$ is smooth and does not intersect itself." The smoothness assumption excludes the degenerate cases discussed above. In order to formulate the heuristic condition mathematically, the set S of all points in the model of the state space that fall on $\mathbf{x}(t)$ is introduced

$$S = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in R^2 | \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x(t) \\ \dot{x}(t) \end{pmatrix}, t \in R \right\}. \qquad (2)$$

The mathematical formulation of the condition that is needed then becomes

C4) $\forall \begin{pmatrix} x \\ \dot{x} \end{pmatrix} \in S \subset R^2$:

1) $\begin{pmatrix} x(t_1) \\ \dot{x}(t_1) \end{pmatrix} = \begin{pmatrix} x(t_2) \\ \dot{x}(t_2) \end{pmatrix} \Rightarrow t_1 = t_2 + kT, k \in Z.$

2) $\exists \delta, L_1, L_2,\ \delta > 0,\ 0 < L_1 \le L_2 < \infty \ \mid\ |t_1 - t_2| < \delta \ \Rightarrow\ L_1|t_1 - t_2| \le \left\| \begin{pmatrix} x(t_1) \\ \dot{x}(t_1) \end{pmatrix} - \begin{pmatrix} x(t_2) \\ \dot{x}(t_2) \end{pmatrix} \right\|_2 \le L_2 |t_1 - t_2|.$

Z denotes the set of all integers. Part 2) of C4) is a statement of smoothness. The following lemma can now be proved

Lemma 1: Let $\dot{x} = m(x)$ denote the multi-valued mapping that results when $\dot{x}$ is considered as a function of x for $\begin{pmatrix} x & \dot{x} \end{pmatrix}^T \in S$. If C3) and C4) holds, then $\dot{x} = m(x)$ is continuously differentiable in the interior of each separate single-valued branch. Further, the speed $\sqrt{\dot{\mathbf{x}}^T(t)\dot{\mathbf{x}}(t)}$ of the curve $\mathbf{x}(t)$ is strictly positive for all t.

Remark 1: $\dot{x} = m(x)$ cannot be single valued since $\mathbf{x}(t)$ is closed.

Proof: C4) implies that

$$0 < L_1 \leq \left\| \frac{\mathbf{x}(t_1) - \mathbf{x}(t_2)}{t_1 - t_2} \right\|_2 \leq L_2 < \infty. \tag{3}$$

Using C3) to evaluate (3) in the limit where $t_1 \to t_2$ then gives the following bound for the speed $\sqrt{\dot{\mathbf{x}}^T(t)\dot{\mathbf{x}}(t)}$ of the curve $\mathbf{x}(t)$

$$0 < L_1 \leq \sqrt{\dot{\mathbf{x}}^T(t)\dot{\mathbf{x}}(t)} \leq L_2 < \infty. \tag{4}$$

Since the curve $\mathbf{x}(t)$ is continuously differentiable in t by C3), and since the speed of the curve is always strictly greater than zero, it follows that the nonzero velocity vector $\dot{\mathbf{x}}(t)$ (tangent to $\mathbf{x}(t)$) varies smoothly along $\mathbf{x}(t)$. The result of Lemma 1 follows.

The next step of the construction of the second order ODE is to *select* a tentative second order state space model as

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x(t) \\ \dot{x}(t) \end{pmatrix}. \tag{5}$$

The main objective of the development of this section is now to prove that this choice represents a state. This fact will be proved by explicit construction of right hand side functions of an ODE that has the right hand side of (5) as a solution. A differentiation of (5) first gives

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ \ddot{x}(t) \end{pmatrix}. \tag{6}$$

Equation (6) will be the key to the construction of an ODE with the sought properties. The construction of the ODE requires that the right hand side of the second state equation of (6) is expressed in terms of the states x_1

and x_2, instead of in terms of $\ddot{x}(t)$. This can be accomplished by a solution of (5) with respect to t, so that t is expressed in terms of the states. In order to address this problem, the following lemma is needed.

Lemma 2 [The implicit function theorem, [2]] : Assume that $f: R^n \times R^m \to R^n$ is continuously differentiable at each point (x,y) of an open set $P \subset R^n \times R^m$. Let (x_0,y_0) be a point in P for which $f(x_0,y_0)=0$ and for which the Jacobian matrix $[\partial f/\partial x](x_0,y_0)$ is nonsingular. Then there exist neighbourhoods $U \subset R^n$ of x_0 and $V \subset R^m$ of y_0 such that for each $y \in V$ the equation $f(x,y)=0$ has a unique solution $x \in U$. Moreover, this solution can be given as $x = g(y)$, where g is continuously differentiable at $y = y_0$.

Then consider the solution of the state equations of (5) with respect to t. Towards this end, consider Fig. 2 which is generated in the same way as Fig. 1b, but with $x(t)=\cos(t)+0.1\cos(5t)$. A point $\mathbf{x}_0$ on the curve $\mathbf{x}(t)$ is given and the intention is to solve for t. The point $\mathbf{x}_0$ is marked with an 'x' in Fig. 2.

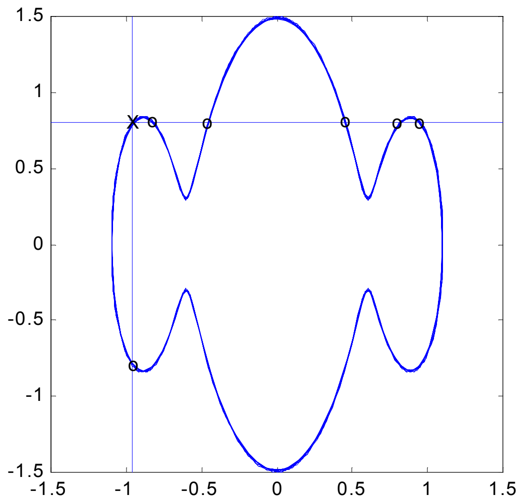

Figure 2: Solving for the time in a phase plane plot. The 'x' corresponds to the point of interest while the 'o' corresponds to alternative solution points. The horizontal line illustrates solution in terms of the second state variable while the vertical line illustrates solution in terms of the first state variable.

Generalizing the situation of Fig. 2, the state equations of (5) are written as

$$\begin{pmatrix} f_1(x_1, t) \\ f_2(x_2, t) \end{pmatrix} = \begin{pmatrix} x_1 - x(t) \\ x_2 - \dot{x}(t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} . \tag{7}$$

When the first equation of (7) is solved in the point $(x_{1,0} \quad x_{2,0})$ the times $t_{1,i}$, $i = 1,\ldots,I$ results. Referring to Fig. 2, there is one time for the point $\mathbf{x}_0$ (marked with 'x' in Fig. 2) and one time for each of the other intersections with the vertical line through $\mathbf{x}_0$ (marked with 'o' in Fig. 2). Now, because of C3) and Lemma 1, the regularity conditions of Lemma 2 are fulfilled. Further, in all points $t_{1,i}$ where

$$\frac{\partial f_1(x_{1,0}, t_{1,i})}{\partial t} = \dot{x}(t_{1,i}) \neq 0 \tag{8}$$

the remaining conditions of Lemma 2 are fulfilled for $f_1(x_{1,0}, t_{1,i})$. Therefore it can be concluded that for all $t_{1,i}$ fulfilling (8) there exist continuously differentiable functions $h_{1,i}$, $i = 1,\ldots,I$, and corresponding neighbourhoods $U_{1,i}$ of $t_{1,i}$ and $V_{1,i}$ of $x_{1,0}$, such that

$$t = h_{1,i}(x_1), \quad t \in U_{1,i}, \; x_1 \in V_{1,i}, \; i = 1,\ldots,I . \tag{9}$$

Analogously, when the second equation of (7) is solved the times $t_{2,j}$, $j = 1,\ldots,J$ result. These times correspond to the 'x' and the 'o' along the horizontal line through $\mathbf{x}_0$ of Fig. 2. Following the approach leading to (9), it is clear that in all points $t_{2,j}$ where

$$\frac{\partial f_2(x_{2,0}, t_{2,j})}{\partial t} = \ddot{x}(t_{2,j}) \neq 0 \tag{10}$$

there exist continuously differentiable functions $h_{2,j}$, $j = 1,\ldots,J$, and corresponding neighbourhoods $U_{2,j}$ of $t_{2,j}$ and $V_{2,j}$ of $x_{2,0}$, such that

$$t = h_{2,j}(x_2), \; t \in U_{2,j}, \; x_1 \in V_{2,j}, \; j = 1,\ldots,J . \tag{11}$$

The $I+J$ times $t_{1,i}$, $i=1,\ldots,I$, $t_{2,j}$, $j=1,\ldots,J$, are all candidates to the joint solution of both state equations of (7) (or (5)). By construction there is at least one i_1 and one j_1 such that $t_{1,i_1} = t_{2,j_1}$ ($\left(x_{1,0} \quad x_{2,0}\right)^T$ is a point (in the model state space) on the curve $\mathbf{x}(t)$, see (5) and Fig. 2). Furthermore, there cannot be any $t_{1,i_2} = t_{2,j_2} \neq t_{1,i_1} = t_{2,j_1}$ within the same period, since $t_{1,i_2} = t_{2,j_2}$ must then, by C4), correspond to a different point of the state space. Hence, there is exactly one time $t_0 = t_{1,i_1} = t_{2,j_1}$ corresponding to $\left(x_{1,0} \quad x_{2,0}\right)^T$. The time t in a neighborhood around t_0 can be expressed in terms of the state variables, either by (9) or (11). In cases where (8) but not (10) holds, (9) is used to solve in terms of x_1. In cases where (10) but not (8) holds, (11) is used to solve in terms of x_2. In cases where both (8) and (10) hold, any one of (9) or (11) can be used. Note that by Lemma 1

$$\sqrt{\dot{\mathbf{x}}^T(t)\dot{\mathbf{x}}(t)} = \sqrt{\dot{x}^2(t) + \ddot{x}^2(t)} \geq L_1 > 0 . \qquad (12)$$

Hence at least one of (8) or (10) always holds.

The idea is now to select a number of points $\left\{\left(x_{1,k} \quad x_{2,k}\right)^T\right\}_{k=1}^K$ in the model state space, ordered clockwise or counterclockwise around the curve $\mathbf{x}(t)$. The times t_k corresponding to the points $\left\{\left(x_{1,k} \quad x_{2,k}\right)^T\right\}_{k=1}^K$ are then computed. The points $\left\{\left(x_{1,k} \quad x_{2,k}\right)^T\right\}_{k=1}^K$ should be selected in such a way so that the neighborhoods (in which the sought functions exist) that result around each solution point t_k, *covers one complete period* T, cf. condition C6) just below this paragraph. The resulting neighborhoods are denoted $\left\{U_{\delta(k),k}\right\}_{k=1}^K$ and $\left\{V_{\delta(k),k}\right\}_{k=1}^K$, where $\delta(k) = 1,2$ indicates if (9) ($\delta(k)=1$) or if (11) ($\delta(k)=2$) is used. It should be noted that the domains of definition $\left\{V_{\delta(k),k}\right\}_{k=1}^K$ are consistent with the corresponding range spaces $\left\{U_{\delta(k),k}\right\}_{k=1}^K$. This is clear from the formulation and use of Lemma 2.

In order to conclude the construction it needs to be assumed that the (open) intervals $\left\{U_{\delta(k),k}\right\}_{k=1}^K$ overlap and cover one complete period. These assumptions are formulated as

C5) $U_{\delta(k),k} \cap U_{\delta(k+1),k+1} \neq \varnothing$, $k = 1,\ldots,K-1$ and $U_{\delta(K),K} \cap U_{\delta(1),1} \neq \varnothing$.

C6) $\bigcup_{k=1}^{K} U_{\delta(k),k} \supset [\bar{t}, \bar{t}+T]$ for an appropriately selected $\bar{t}$.

Remark 2: Note that the ordering of points is crucial in the formulation of C5) and C6). The use of $\bar{t}$ is introduced to allow an arbitrary position in time of the obtained intervals. One general situation where C5) and C6) can be expected to hold is when the signal is analytic. In such a case the techniques of analytic continuation [8] could be used to prove that C5) and C6) hold. It is expected that there are several different possible conditions that imply C5) and C6), which is one reason why these conditions are formulated in an explicit form.

Using C5), C6), (9), (10) and the treatment above it can be concluded that there exist closed intervals $\bar{U}_{\delta(k),k} \subset U_{\delta(k),k}$ and corresponding *minimal* domains of definition $\bar{V}_{\delta(k),k} \subseteq V_{\delta(k),k}$ such that

$$\bar{U}_{\delta(k),k} \cap \bar{U}_{\delta(k+1),k+1} = \varnothing,\ k = 1,\ldots,K-1,\ \bar{U}_{\delta(K),K} \cap \bar{U}_{\delta(1),1} = \varnothing$$

$$\bigcup_{k=1}^{K} \bar{U}_{\delta(k)} = [\bar{t}, \bar{t}+T] \qquad (13)$$

$$t \equiv h_{Period}(x_1,x_2) = \begin{cases} h^{k}_{1,i_1(k)}(x_1), & t \in \bar{U}_{1,k}, x_1 \in \bar{V}_{1,k}, \delta(k) = 1 \\ h^{k}_{2,j_1(k)}(x_2), & t \in \bar{U}_{2,k}, x_2 \in \bar{V}_{2,k}, \delta(k) = 2 \end{cases}, \quad k = 1,\ldots,K.$$

Inserting the expression for t of (13) in (6), the following second order ODE results

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ \ddot{x}(h_{Period}(x_1,x_2)) \end{pmatrix}. \qquad (14)$$

To prove that $\mathbf{x}(t)$ is indeed a solution, the components of the curve $\mathbf{x}(t)$ are inserted in (14). This results in

$$LHS = \begin{pmatrix} x_2 \\ \ddot{x}(h_{Period}(x_1,x_2)) \end{pmatrix} = \begin{pmatrix} \dot{x}(t) \\ \ddot{x}\left(h_{Period}\left(x(t),\dot{x}(t)\right)\right) \end{pmatrix} = \begin{pmatrix} \dot{x}(t) \\ \ddot{x}(t) \end{pmatrix} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = RHS. \qquad (15)$$

The following theorem has now been proved.

Theorem 1 [Loop criterion]: Consider the periodic signal $x(t)$. Assume that C1), C2), C3), C4), C5) and C6) hold. Then there exists a function $h_{Period}(x_1,x_2)$ and a *second order* ordinary differential equation

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ \ddot{x}\left(h_{Period}(x_1,x_2)\right) \end{pmatrix}$$

that has a solution given by $(x_1 \quad x_2)^T = \left(x(t) \quad \dot{x}(t)\right)^T$.

Remark 3: The interpretation of Theorem 1 is that it gives conditions on the signal that are sufficient to ensure the existence of right hand side *functions* of a second order ODE. Obviously, this is only a prerequisite for identification. The further steps of parameterization of these functions as well as the development of identification algorithms are described in [6] and [7] . Note also that there is no need for more than one nontrivial right hand side function. This function enters in the second state equation of (14). The result is denoted the "loop criterion" to reflect the geometrical intuition behind condition C4).

Remark 4: It should be noted that the above theorem does not state anything about the stability of the solution. The orbit may or may not be stable. In order to secure a stable orbit further constraints may have to be imposed, see e.g. [2] for available results for stability checks applicable to periodic orbit solutions. The topic of construction of an ODE with periodic orbit solutions that are guaranteed to be stable is outside the scope of the present paper and is left for future research.

The solution to the second order ODE of Theorem 1 may not be unique. Uniqueness also requires that $h_{Period}(x_1,x_2)$ is Lipschitz. Noting that the derived ODE is autonomous, the following two corollaries that guarantee also uniqueness to Theorem 1 then follows from [2], Theorem 2.3 and [2], Theorem 2.4, respectively:

Corollary 1: Assume that the conditions of Theorem 1 hold. Assume in addition that there are constants L_3 and L_4 such that

$$\left|\ddot{x}\left(h_{Period}(x_1,x_2)\right) - \ddot{x}\left(h_{Period}(y_1,y_2)\right)\right| \leq L_3 \left\| \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right\|$$

$$\left| \ddot{x}\left(h_{Period}\left(x_1^0, x_2^0\right)\right) \right| \le L_4$$

$\forall \ (x_1 \quad x_2)^T, (y_1 \quad y_2)^T \in R^2$, $\forall \ t \in [t_0, t_1]$. Then there is a second order ODE

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ \ddot{x}\left(h_{Period}(x_1, x_2)\right) \end{pmatrix}, \ \begin{pmatrix} x_1(t_0) \\ x_2(t_0) \end{pmatrix} = \begin{pmatrix} x_1^0 \\ x_2^0 \end{pmatrix} \in S$$

that has the unique solution $(x_1 \quad x_2)^T = \left(x(t) \quad \dot{x}(t) \right)^T$, $t \in [t_0, t_1]$.

Corollary 2: Assume that the conditions of Theorem 1 hold. Assume in addition that there is a constant L_5 such that

$$\left| \ddot{x}\left(h_{Period}(x_1, x_2)\right) - \ddot{x}\left(h_{Period}(y_1, y_2)\right) \right| \le L_5 \left\| \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right\|$$

$\forall \ (x_1 \quad x_2)^T, (y_1 \quad y_2)^T \in D \subset R^2$, $\forall \ t \ge t_0$. Let W be a compact subset of the domain D, $\left(x_1^0 \quad x_2^0\right) \in W$, and suppose that it is known that every solution of

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ \ddot{x}\left(h_{Period}(x_1, x_2)\right) \end{pmatrix}, \ \begin{pmatrix} x_1(t_0) \\ x_2(t_0) \end{pmatrix} = \begin{pmatrix} x_1^0 \\ x_2^0 \end{pmatrix} \in S$$

(which exists) lies entirely in W. Then this second order ODE has the unique solution $(x_1 \quad x_2)^T = \left(x(t) \quad \dot{x}(t) \right)^T$, $t \in [t_0, t_1]$.

The above construction of a second order ODE with $\mathbf{x}(t)$ as a solution is somewhat technical. In order to put further light on the details the following example goes through the details for the signal $x(t) = \cos(t)$ and it is shown that the result is the well known linear ODE describing the harmonic oscillator. A special emphasis is given to the selection of intervals when solving for t by the implicit function theorem (Lemma 2). The example also illustrates that the coverage of a complete period with intervals, as assumed by C6), is most often not a problem in applications.

Example 2 [Construction of the harmonic oscillator]: In this example the above construction procedure is illustrated. The signal $x(t) = \cos(t)$ is given. It is straightforward to see that C1)-C4) are valid. Differentiation first gives the following equations corresponding to (7) in a point $\begin{pmatrix} x_{1,0} & x_{2,0} \end{pmatrix}^T$ on the unit circle

$$\begin{pmatrix} x_{1,0} - \cos(t) \\ x_{2,0} - \sin(t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{16}$$

Solving the component equations of (16) in a point where (8) and (10) holds results in $t_{1,1} = \cos^{-1}(x_{1,1})$, $t_{1,2} = -\cos^{-1}(x_{1,0})$, $t_{2,1} = -\sin^{-1}(x_{2,0})$, $t_{2,2} = \pi + \sin^{-1}(x_{2,0})$. Select one point $\begin{pmatrix} x_{1,1} & x_{2,1} \end{pmatrix}^T$ on the unit circle in the interior of the first quadrant and one point $\begin{pmatrix} x_{1,2} & x_{2,2} \end{pmatrix}^T$ on the unit circle in the interior of the third quadrant. Then, the existence of the following one to one functions and neighborhoods that correspond to (9) and (11) follow.

$$\begin{aligned} t &= \cos^{-1}(x_1), \quad t \in (0,\pi), \quad x_1 \in (-1,1),\ x_2 > 0 \\ t &= -\cos^{-1}(x_1), \quad t \in (-\pi,0), \quad x_1 \in (-1,1)\ ,\ x_2 < 0 \\ t &= -\sin^{-1}(x_2), \quad t \in (-\pi/2,\pi/2), \quad x_2 \in (-1,1), \quad x_1 > 0 \\ t &= \pi + \sin^{-1}(x_2), \quad t \in (\pi/2,3\pi/2), \quad x_2 \in (-1,1), \quad x_1 > 0\,. \end{aligned} \tag{17}$$

It follows from (17) that also C5) and C6) hold. To proceed, select $\varepsilon > 0$ sufficiently small and select the intervals $\overline{U}_{1,1} = [\varepsilon,\pi-\varepsilon]$, $\overline{U}_{2,2} = [\pi-\varepsilon,\pi+\varepsilon]$, $\overline{U}_{1,3} = [-\pi+\varepsilon,-\varepsilon]$, $\overline{U}_{2,4} = [-\varepsilon,\varepsilon]$ and the corresponding domains of definition according to (17). Then it follows that

$$h_{Period}(x_1,x_2) = \begin{cases} \cos^{-1}(x_1), & t \in \overline{U}_{1,1} \\ \pi + \sin^{-1}(x_2) & t \in \overline{U}_{2,2} \\ -\cos^{-1}(x_1) & t \in \overline{U}_{1,3} \\ -\sin^{-1}(x_2) & t \in \overline{U}_{2,4} \end{cases}. \tag{18}$$

Finally, $\cos\left(\cos^{-1}(x_1)\right) = x_1$, $t \in \overline{U}_{1,1}$, $\cos\left(\pi + \sin^{-1}(x_2)\right) = \cos\left(\mp \cos^{-1}(x_1)\right) = x_1$, $t \in \overline{U}_{2,2}$, $\cos\left(-\cos^{-1}(x_1)\right) = x_1$, $t \in \overline{U}_{1,3}$, $\cos\left(-\sin^{-1}(x_2)\right) = \cos\left(\pm \cos(x_1)\right) = x_1$. Note that when changing the dependence of *the argument* of the cosine function from x_2 to x_1, the fact that the coordinates determines a point on the unit circle is exploited. Further, note that the domains of definition of x_1 in (17) needs to be invoked to divide the intervals $\overline{U}_{2,2}$ and $\overline{U}_{2,4}$ in two cases. The midpoints of these two intervals follow trivially.

It now follows that the ODE of Theorem 1 becomes

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ \ddot{x}\left(h_{Period}(x_1, x_2)\right) \end{pmatrix} = \begin{pmatrix} x_2 \\ -\cos\left(h_{Period}(x_1, x_2)\right) \end{pmatrix} = \begin{pmatrix} x_2 \\ -x_1 \end{pmatrix}. \quad (19)$$

This is the linear ODE for the harmonic oscillator. Note that (19) has many solutions, a fact that is perfectly consistent with Theorem 1, since that theorem does not specify initial conditions. This is, however, done in Corollary 1 and Corollary 2.

B. When Order Two Is Not Enough

Example 1 of the last subsection indicates what the problem is when the order of the ODE used for modeling is not high enough. Referring to Example 1, order 4 is obviously enough for modeling of the signal in question. When trying to model the signal with a reduced order ODE, order two in this case, the signal is *projected* onto a subspace of lower dimension than the dimension of the signal. Then e.g. a spiraling movement along one particular dimension may be projected onto a lower dimensional hyper-plane in a way which can result in intersections in this lower dimensional space used for modeling.

A procedure can then be devised, when there are intersections in a specific low order space (like a 2-dimensional one). Simply increase the order of the ODE in steps of one by addition of another state variable. This state is selected as the next higher derivative of the signal. In case of extension from a second order ODE, the state variable $x_3 = \ddot{x}(t)$ is added. If there are still intersections in R^3, $x_4 = \dddot{x}(t)$ is added and so on.

The proof of Theorem 1 is given for the second order case. However, a closer look at the argumentation suggests that this does not have to be the case. Hence it could be conjectured that counterparts to Theorem 1 can be proved for higher order ODEs using the ideas of the present proof. As it turns out this is in fact the case.

In order to develop the counterpart to Theorem 1, the necessary modifications of the discussion leading to Theorem 1 are highlighted. In fact, the entire treatment of the general case parallels the development of the second order result of Theorem 1. However, the major modifications are discussed in order to enhance the clarity of the paper.

The starting point is the selection of the following $n+1$:th order tentative state vector

$$\begin{pmatrix} x_1 \\ \vdots \\ x_{n+1} \end{pmatrix} = \begin{pmatrix} x(t) \\ \vdots \\ x^{(n)}(t) \end{pmatrix}. \qquad (20)$$

The curve $\mathbf{x}(t)$ in the $n+1$dimensional space is now to be interpreted as $\mathbf{x}(t) = \begin{pmatrix} x(t) & \dots & x^{(n)}(t) \end{pmatrix}^T$.

As in the development of the second order result, the task is to prove that (20) represents a state. In order to do so, the conditions C1) and C2) remain unaffected. The condition C3) is replaced by

C3') The signal is $n+1$ times continuously differentiable.

The definition (2) of the set S is then changed to

$$S^{n+1} = \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_{n+1} \end{pmatrix} \in R^{n+1} \middle| \begin{pmatrix} x_1 \\ \vdots \\ x_{n+1} \end{pmatrix} = \begin{pmatrix} x(t) \\ \vdots \\ x^{(n)}(t) \end{pmatrix}, t \in R \right\} \qquad (21)$$

from which the following modification of C4) follows

C4') $\forall \begin{pmatrix} x \\ \vdots \\ x^{(n)} \end{pmatrix} \in S^{n+1} \subset R^{(n+1)}$:

1) $\begin{pmatrix} x(t_1) \\ \vdots \\ x^{(n)}(t_1) \end{pmatrix} = \begin{pmatrix} x(t_2) \\ \vdots \\ x^{(n)}(t_2) \end{pmatrix} \Rightarrow t_1 = t_2 + kT, k \in Z.$

2) $\exists \delta, L_1, L_2, \quad \delta > 0,\ 0 < L_1 \le L_2 < \infty \ \mid \ |t_1 - t_2| < \delta \ \Rightarrow \ L_1|t_1 - t_2| \le \left\| \begin{pmatrix} x(t_1) \\ \vdots \\ x^{(n)}(t_1) \end{pmatrix} - \begin{pmatrix} x(t_2) \\ \vdots \\ x^{(n)}(t_2) \end{pmatrix} \right\| \le L_2|t_1 - t_2|$.

With this modification the following counterpart to Lemma 1 holds.

Lemma 1' : Let $x^{(n)} = m\left(x,\ldots,x^{(n-1)}\right)$ denote the multi-valued mapping that results when $x^{(n)}$ is considered as a function of $x,\ldots,x^{(n-1)}$ for $\left(x \quad \ldots \quad x^{(n)}\right)^T \in S^{n+1}$. If C3') and C4') holds, then $x^{(n)} = m\left(x,\ldots,x^{(n-1)}\right)$ is continuously differentiable in the interior of each separate single-valued branch. Further, the speed $\sqrt{\dot{\mathbf{x}}^T(t)\dot{\mathbf{x}}(t)}$ of the curve $\mathbf{x}(t)$ is strictly positive for all t.

The proof parallels that of Lemma 1.

Following section II.A, the tentative state vector of (20) is differentiated with respect to t to give

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \\ \dot{x}_{n+1} \end{pmatrix} = \begin{pmatrix} x_2 \\ x_3 \\ \vdots \\ x_n \\ x^{(n+1)}(t) \end{pmatrix}. \tag{22}$$

Again, the task is to express the last equation of (22) in terms of the tentative state variables. This is accomplished by solving each equation of (20) with respect to t using Lemma 2. This gives, using C3'), the *n+1 set of* relations (cf. (7) – (9))

$$t = h_{m,i}(x_m),\ \ t \in U_{m,i},\ \ x_m \in V_{m,i},\ \ i = 1,\ldots,I_m,\ \ m = 1,\ldots,n+1 \tag{23}$$

for all solution points (cf. the explanation of Fig. 2) $t_{m,i}$ and $x_{m,0}$ such that

$$\left.\frac{\partial\left(x_m - x^{(m-1)}(t)\right)}{\partial t}\right|_{x_{m,0},t_{m,i}} = x^{(m)}\left(t_{m,i}\right) \neq 0. \tag{24}$$

Extending the reasoning following (11) from 2 to *n+1* dimensional space, it can be concluded that there is again exactly one time t_0 corresponding to a given vector $\left(x_{1,0} \quad \ldots \quad x_{n+1,0}\right)^T$. Furthermore, by Lemma 1' at least one of the conditions of (24) are always fulfilled (since $\sqrt{\dot{\mathbf{x}}^T(t)\dot{\mathbf{x}}(t)}$ is strictly greater than zero). The corresponding relation of (23) is then used to solve for t in the relevant neighbourhoods of $\left(x_{1,0} \quad \ldots \quad x_{n+1,0}\right)^T$.

A set of points around the curve $\mathbf{x}(t)$ are then selected, so that the neighbourhoods in time covers one complete period of $\mathbf{x}(t)$ in the tentative state space. Denote the set of points by $\left\{\begin{pmatrix} x_{1,k} & \ldots & x_{n+1,k} \end{pmatrix}^T\right\}_{k=1}^{K}$ and define the generalized index $\delta(k)$ which takes values in the set $\{1,2,\ldots,n+1\}$, and where the value indicates which state equation that is selected to be solved for t for the point $\begin{pmatrix} x_{1,k} & \ldots & x_{n+1,k} \end{pmatrix}^T$. With this redefinition of $\delta(k)$, the conditions C5) and C6) do not need any modification from the treatment of the second order case. The treatment of the higher than second order case now be concluded as in the second order case. It follows from C5), C6), (23), (24) and Lemma 1' that there exist closed intervals $\overline{U}_{\delta(k),k} \subset U_{\delta(k),k}$ and corresponding *minimal* domains of definition $\overline{V}_{\delta(k),k} \subseteq V_{\delta(k),k}$ such that

$$\overline{U}_{\delta(k),k} \cap \overline{U}_{\delta(k+1),k+1} = \varnothing\,,\ \ k = 1,\ldots,K-1\,,\ \ \overline{U}_{\delta(K),K} \cap \overline{U}_{\delta(1),1} = \varnothing \tag{25}$$

$$\bigcup_{k=1}^{K} \overline{U}_{\delta(k)} = \left[\bar{t}, \bar{t}+T\right]$$

$$t \equiv h_{Period}\left(x_1,\ldots,x_{n+1}\right) = h^k_{m,i_{m1}(k)}\left(x_m\right),\quad t \in \overline{U}_{m,k}\,, x_m \in \overline{V}_{m,k}\,, \delta(k) = m\,,\ \ k = 1,\ldots,K\,.$$

Inserting the expression for t of (25) in (22) results in the following ODE of order *n+1*

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \\ \dot{x}_{n+1} \end{pmatrix} = \begin{pmatrix} x_2 \\ x_3 \\ \vdots \\ x_{n+1} \\ x^{(n)}\left(h_{Period}\left(x_1,\ldots,x_{n+1}\right)\right) \end{pmatrix}. \tag{26}$$

An insertion of the curve $\mathbf{x}(t)$ in (26) shows that $\mathbf{x}(t)$ indeed solves the ODE, a fact that proves the following generalization of Theorem 1.

Theorem 2 [High order loop criterion]: Consider the periodic signal $x(t)$. Assume that C1), C2), C3'), C4'), C5) and C6) hold. Then there exists a function $h_{Period}\left(x_1,\ldots,x_{n+1}\right)$ and an *n+1:th order* ordinary differential equation

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \\ \dot{x}_{n+1} \end{pmatrix} = \begin{pmatrix} x_2 \\ x_3 \\ \vdots \\ x_{n+1} \\ x^{(n)}\left(h_{Period}\left(x_1,\ldots,x_{n+1}\right)\right) \end{pmatrix}$$

that has a solution given by $\begin{pmatrix} x_1 & \ldots & x_{n+1} \end{pmatrix}^T = \begin{pmatrix} x(t) & \ldots & x^{(n)}(t) \end{pmatrix}$.

Remark 5: The generalizations of Corollary 1 and Corollary 2 follow immediately. They are not reproduced here.

Remark 6: The possibility to use Theorem 2 in an order recursive mode to select the required order to model the periodic signal with an ODE is stressed.

III. A FREQUENCY DOMAIN CHECK FOR ORDER TWO SUFFICIENCY

How can a situation where C4) holds be characterized by simple means? One possibility, that is the basis for the development of this subsection, is to study the rotation (with time) of the point $\mathbf{x}(t)$. It is then clear that if the vector $\mathbf{x}(t)$ either rotates clockwise for all $t \in [\bar{t}, \bar{t}+T]$ or rotates counterclockwise for all $t \in [\bar{t}, \bar{t}+T]$, then loops cannot occur. This is further illustrated in Example 3.

Example 3: Phase plane plots were generated with a similar methodology as in Example 1. Here sums of solutions to $\ddot{x}+x=0$ and $\ddot{x}+4x=0$ were used. In Fig 3a the solution $\cos(t)+0.1\cos(2t)$ and (b) is shown, while Fig. 3b displays $\cos(t)+0.5\cos(2t)$. In Fig 3a, the overtone contents is small and it is obvious that the indicated vector, pointing to $\mathbf{x}(t) = \begin{pmatrix} x(t) & \dot{x}(t) \end{pmatrix}^T$, always rotates in the same direction when the time increases. In Fig 3b the overtone contents is higher and the indicated vector $\mathbf{x}(t) = \begin{pmatrix} x(t) & \dot{x}(t) \end{pmatrix}^T$ follows the additional loop of the curve. The vector does not always rotate in the same direction when time increases.

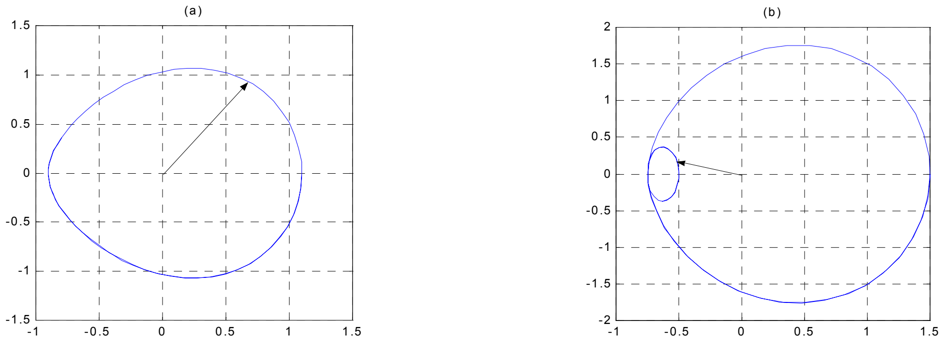

Figure 3: Phase plane plot generated as the sum of solutions to $\ddot{x}+x=0$ and $\ddot{x}+4x=0$. (a) corresponds to $\cos(t)+0.1\cos(2t)$ and (b) corresponds to $\cos(t)+0.5\cos(2t)$.

Example 3 indicates two things. First it is clear that a curve, where the vector pointing to $\mathbf{x}(t)=\begin{pmatrix} x(t) & \dot{x}(t) \end{pmatrix}^T$ rotates with an angular velocity with *constant sign*, do fulfill C4). Secondly, the overtone contents is of central importance. Both these observations are addressed by the following proposition.

Proposition 1: Assume that there is a point $\bar{\mathbf{x}}$ in the interior of the domain bounded by the closed curve $\mathbf{x}(t)=\begin{pmatrix} x(t) & \dot{x}(t) \end{pmatrix}^T$, such that the angular velocity $\dot{\theta}(t)$ of the vector $\mathbf{x}(t)-\bar{\mathbf{x}}$ is continuous and has a constant and strictly nonzero sign. Then, if C1), C2), C3), C5) and C6) hold, Theorem 1 holds.

The result is geometrically obvious and it is therefore not proved formally.

The next step is to proceed with the frequency domain analysis, using Proposition 1. To do so it is assumed that

C7) $\bar{\mathbf{x}}=\mathbf{0}$ is interior to the domain bounded by the closed curve $\mathbf{x}(t)=\begin{pmatrix} x(t) & \dot{x}(t) \end{pmatrix}^T$.

Proceeding from C7), the angle $\theta(t)$ of Proposition 1 can be written as

$$\theta(t)=\begin{cases} \tan^{-1}\left(\dot{x}(t)/x(t)\right), & x(t)>0 \\ \dfrac{\pi}{2}, & x(t)=0, \dot{x}(t)>0 \\ -\dfrac{\pi}{2}, & x(t)=0, \dot{x}(t)<0 \\ \pi+\tan^{-1}\left(\dot{x}(t)/x(t)\right), & x(t)<0 \end{cases} \quad (27)$$

Differentiation of the two right hand sides of (27) results in identical results. The angular velocity can be expressed compactly as

$$\dot{\theta}(t) = \frac{\ddot{x}(t)x(t) - \left(\dot{x}(t)\right)^2}{\left(\dot{x}(t)\right)^2 + \left(x(t)\right)^2} \tag{28}$$

The angular velocity of (28) has been defined as the value of the left and right hand limits when $x(t) = 0$. In order for Proposition 1 to hold, (28) must now have constant sign.

By C1), C2) and C3) it follows that $x(t)$ can be expanded in the following Fourier series

$$x(t) = x_F(t) + \varepsilon(t) \equiv A_0 + A_1\cos(\omega t + \phi_1) + \sum_{k=2}^{\infty} A_k \cos(k\omega t + \phi_k). \tag{29}$$

where the term including the fundamental frequency is $x_F(t) = A_1\cos(\omega t + \phi_1)$. Then note that in case $\varepsilon(t) = 0$, it follows easily that

$$\ddot{x}(t)x(t) - \left(\dot{x}(t)\right)^2 = \ddot{x}_F(t)x_F(t) - \left(\dot{x}_F(t)\right)^2 = -A_1^2\omega^2 \tag{30}$$

and hence $\dot{\theta}(t)$ has a constant sign in this case. The next step is to investigate how large $\varepsilon(t)$ can be allowed to be without causing $\dot{\theta}(t)$ to change sign. Towards that end, note that the denominator of (28) can be written as (the numerator is always greater than zero by C7))

$$\begin{aligned} \ddot{x}(t)x(t) - \left(\dot{x}(t)\right)^2 &= \ddot{x}_F(t)x_F(t) - \left(\dot{x}_F(t)\right)^2 + R\left(\varepsilon(t), x_F(t)\right) \\ R\left(\varepsilon(t), x_F(t)\right) &= \ddot{x}_F(t)\varepsilon(t) + x_F(t)\ddot{\varepsilon}(t) + \varepsilon(t)\ddot{e}(t) - 2\dot{x}_F(t)\dot{\varepsilon}(t) - \left(\dot{\varepsilon}(t)\right)^2. \end{aligned} \tag{31}$$

Straightforward calculations, using the definitions of (29) then give

$$\left|\ddot{x}_F(t)\varepsilon(t)+x_F(t)\ddot{\varepsilon}(t)\right| \le |A_0A_1|\omega^2+|A_1|\omega^2\sum_{k=2}^{\infty}|A_k|\left(1+k^2\right)$$

$$\left|\varepsilon(t)\ddot{\varepsilon}(t)\right| \le |A_0|\omega^2\sum_{k=2}^{\infty}k^2|A_k|+\omega^2\sum_{k_1=2}^{\infty}\sum_{k_2=2}^{\infty}|A_{k_1}||A_{k_2}|k_2^2 \qquad (32)$$

$$\left|2\dot{x}_F(t)\dot{\varepsilon}(t)\right| \le 2|A_1|\omega^2\sum_{k=2}^{\infty}|A_k|k$$

$$\left|\left(\dot{\varepsilon}(t)\right)^2\right| \le \omega^2\sum_{k_1=2}^{\infty}\sum_{k_2=2}^{\infty}|A_{k_1}||A_{k_2}|k_1k_2$$

Using (30) and (31), the denominator of (28) becomes

$$\ddot{x}(t)x(t)-\left(\dot{x}(t)\right)^2=\ddot{x}_F(t)x_F(t)-\left(\dot{x}_F(t)\right)^2+R\left(\varepsilon(t),x_F(t)\right)=-A_1^2\omega^2+R\left(\varepsilon(t),x_F(t)\right). \qquad (33)$$

Equation (33) shows that the denominator of (28) and hence $\dot{\theta}(t)$ of proposition 1, will have constant sign whenever $\left|R\left(\varepsilon(t),x_F(t)\right)\right| < A_1^2\omega^2$. Using the results of (32) and rearranging then gives

Proposition 2: Assume that C1), C2), C3), C5), C6) and C7) hold. Then, if the fundamental frequency signal $A_1\cos(\omega t+\phi_1)$, the bias A_0 and the overtone signal $\sum_{k=2}^{\infty}A_k\cos(k\omega t+\phi_k)$ fulfill

$$\frac{|A_0|}{|A_1|}\left(1+\sum_{k=2}^{\infty}\frac{k^2|A_k|}{|A_1|}\right)+\sum_{k=2}^{\infty}\frac{(1+k)^2|A_k|}{|A_1|}+\sum_{k_1=2}^{\infty}\sum_{k_2=2}^{\infty}\frac{k_2(k_1+k_2)|A_{k_1}||A_{k_2}|}{|A_1|^2}<1,$$

Theorem 1 holds.

Remark 7: The inequality of Proposition 2 has interesting implications. First, $\sum k^2|A_k|$ needs to be convergent. This means that $|A_k|$ needs to decay at least as fast as $Ck^{-3-\delta}$, $\delta>0$ for large k. Secondly, $|A_1|$ needs to be sufficiently large as compared to the bias and overtone part of the signal. Put differently it may be stated that the high frequency roll-off of the spectrum needs to be larger than 60 dB / decade and that the overtone power needs to be sufficiently small as compared to the power of the fundamental frequency. It can also be noted that the result is independent of the frequency scale of the signal. Note also that Proposition 2 only gives sufficient conditions for Theorem 1 to hold. Finally, note that Theorem 1 does not necessarily imply the validity of Propositions 1 or 2.

Example 4: The fact that Proposition 2 gives sufficient conditions is illustrated in this example. The signal

$$x(t) = \cos(t) + A\cos(2t) \tag{34}$$

is studied and conditions on A that imply that

$$x(t)\ddot{x}(t) - \left(\dot{x}(t)\right)^2 < 0 \ , \ \ \forall t \tag{35}$$

are sought. Towards that end, straightforward calculations transform (35) to

$$-1 - 4A^2 + A\cos(t)\left(2\sin^2(t) - 5\right) < 0 \ , \ \forall t \ . \tag{36}$$

Denoting the time variable part of (36) by

$$f(t) = \cos(t)\left(2\sin^2(t) - 5\right) \tag{37}$$

and differentiating results in the following condition for a maximum/minimum of $f(t)$

$$\frac{df(t)}{dt} = \sin(t)\left(9 - 6\sin^2(t)\right) = 0 \ . \tag{38}$$

Hence it must hold that $\sin(t) = 0$, a fact that transforms (36) to

$$-1 - 4A^2 + 5|A| < 0 \ . \tag{39}$$

This leads to

$$\left(1 - |A|\right)\left(-1 + 4|A|\right) < 0 \tag{40}$$

and it can be concluded that

$$|A| < 0.25 \tag{41}$$

is required for Proposition 1 to hold. Note that the extreme value of $f(t)$ is attained so (41) is a necessary condition for (35) to hold.

This result can be compared to Fig. 3 and Proposition 2 (that is only sufficient). First, it follows immediately that (41) is consistent with Fig. 3 and Proposition 1. Secondly, an evaluation of the criterion of Proposition 2 results in the condition that

$$|A| < \frac{\sqrt{113}-9}{16} \approx 0.1019 \,. \tag{42}$$

Also (42) is consistent with Fig. 3 and hence Proposition 2 is effective with respect to the example of Fig. 3. However, (42) is more conservative than (41) since Proposition 2 only gives sufficient conditions for Theorem 1 to hold.

One remaining issue is the effect of the choice of coordinates of the state space, on the results of section II and III. Here, the effect of a linear transformation on Proposition 1 will be treated.

To discuss the effect of a linear transformation on the sign of the angular velocity, the new coordinates $\mathbf{z}(t) = \begin{pmatrix} z_1(t) & z_2(t) \end{pmatrix}^T$ are introduced. The linear transformation from $\mathbf{x}(t)$ to $\mathbf{z}(t)$ is assumed to be

$$\begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x(t) \\ \dot{x}(t) \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} . \tag{43}$$

The effect of $\begin{pmatrix} b_1 & b_2 \end{pmatrix}^T$ is to perform a pure translation, which moves the reference point $\bar{\mathbf{x}}$. This obviously constitutes a dramatic change of the properties of the sign of the angular velocity. At the same time it is easy to study the effect of a pure translation by graphical means, e.g. with the purpose of selection of a beneficial reference point $\bar{\mathbf{x}}$. The effect of the matrix multiplication is however not obvious. For these reasons the further study is constrained to the case where

C8) $\begin{pmatrix} b_1 & b_2 \end{pmatrix}^T = \begin{pmatrix} 0 & 0 \end{pmatrix}^T$.

With the assumption C8) it follows that (cf. (28))

$$\dot{\theta}_z(t) = \frac{z_1(t)\dot{z}_2(t) - \dot{z}_1(t)z_2(t)}{\left(z_1(t)\right)^2 + \left(z_2(t)\right)^2} \tag{44}$$

To proceed, assume that

C9) The transformation matrix $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ is nonsingular.

The assumptions C8), C9) and (43) implies that $\begin{pmatrix} z_1 & z_2 \end{pmatrix}^T = 0 \Leftrightarrow \begin{pmatrix} x & \dot{x} \end{pmatrix}^T = 0$. Hence, exactly as in the analysis following Proposition 1 it is sufficient to study the sign of the numerator. Using (43), (44), C8), C9), and the definition

$$sign(\theta) = \begin{cases} -1, & \theta < 0 \\ 0 & \theta = 0, \\ 1 & \theta > 0 \end{cases} \tag{45}$$

it follows that

$$sign\left(\dot{\theta}_z(t)\right) = sign\left(z_1(t)\dot{z}_2(t) - z_2(t)\dot{z}_1(t)\right)$$

$$= sign\left(\left(a_{11}x(t) + a_{12}\dot{x}(t)\right)\left(a_{21}\dot{x}(t) + a_{22}\ddot{x}(t)\right) - \left(a_{21}x(t) + a_{22}\dot{x}(t)\right)\left(a_{11}\dot{x}(t) + a_{12}\ddot{x}(t)\right)\right)$$

$$sign\left(\det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\left(x(t)\ddot{x}(t) - \left(\dot{x}(t)\right)^2\right)\right) = sign\left(\det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\right) sign\left(x(t)\ddot{x}(t) - \left(\dot{x}(t)\right)^2\right) \tag{46}$$

$$= \pm\, sign\left(\dot{\theta}_x(t)\right).$$

This proves

Proposition 3: Assume that C1), C2), C3), C5), C6), C7), C8) and C9) hold. Then the sign of the angular velocity of the solution in the phase plane plot remains constant after the linear state transformation

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} x \\ \dot{x} \end{pmatrix}$$

provided that the sign of the angular velocity was constant before the state transformation.

The conclusion is hence that the results of Propositions 1 and 2 are invariant with respect to linear state transformations without a translation term.

IV. NUMERICAL EXAMPLE

The purpose with the present section is to illustrate the previous results numerically. A particular intention is to illustrate the possibility to approximate a higher than second order linear ODE by a nonlinear ODE of second order. This observation opens up e.g. for model reduction applications, by pinpointing the possibility to trade order of ODEs on one hand against degrees of the nonlinearity on the other hand. Which of these two choices that is most efficient is obviously dependent on the specific signal and application, see [6] and [7] for some further examples.

Example 5: In this example modeling of the signal $\cos(t)+(1/10)\cos(3t)$ is studied. Following Example 1, it is clear that the signal can be described by a fourth order linear ODE, which is a minimal (linear) realization of the signal. The phase plane plot of the signal appears in Fig. 5a, and it can be seen that there are no intersections. Hence Theorem 1 holds and it can be concluded that there exists a second order ODE that has a solution equal to the signal. Note also that in this specific example the RHS of the inequality of Proposition 2 equals 1.78>1 and so the frequency domain result of section III does not apply in this case. Note, however, that Proposition 2 does only give sufficient conditions for the validity of Theorem 1.

In order to model the signal the extended Kalman filter (EKF) algorithm of [6] was applied. To describe that method briefly, it is first noted that it is based on the following second order continuous time ODE model

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ \boldsymbol{\varphi}^{\mathbf{T}}(x_1, x_2)\boldsymbol{\theta} \end{pmatrix}$$
$$x = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \qquad (47)$$

where

$$\boldsymbol{\varphi}(x_1, x_2) = \begin{pmatrix} 1 & \dots & x_2^M & x_1 & \dots & x_1 x_2^M & \dots & x_1^N & \dots & x_1^N x_2^M \end{pmatrix}^T$$
$$\boldsymbol{\theta} = \begin{pmatrix} \theta_{00} & \dots & \theta_{0M} & \theta_{10} & \dots & \theta_{1M} & \dots & \theta_{N0} & \dots & \theta_{NM} \end{pmatrix}^T. \qquad (48)$$

Here $x(t)$ is the measured signal and $\boldsymbol{\theta}$ denotes the unknown parameter vector. The model is hence a polynomial one. The EKF algorithm is derived as follows. The model (47), (48) is first discretized by an Euler numerical integration scheme. Discrete time Gaussian measurement and system noise are then modeled by the conventional covariance matrices, allowing e.g. for tuning of the estimator bandwidth. The result is a discrete time nonlinear state space model that immediately lends itself to application of the extended Kalman filter, see e.g. [9] and [10] for details on the use of the EKF to solve recursive identification problems.

In the numerical experiment 10000 samples of the noise-free signal were generated, using a sampling period of 0.025 s. The initial value of the state covariance matrix was selected as $\mathbf{P}(0) = 10\mathbf{I}$. The bandwidth of the EKF was tuned by selection of the noise covariance matrices according to $\mathbf{R}_1 = 10^{-5}\mathbf{I}$ and $R_2 = 1$. The initial state vector was selected as $\mathbf{x}(0) = (-0.5 \quad 0.5)^T$ and the initial parameter vector was selected as $\boldsymbol{\theta}(0) = \mathbf{0}$.

Two runs were then performed, one with $M = N = 2$ and one with $M = N = 3$. Results appear in Fig. 4, Fig. 5 and in Table 1.

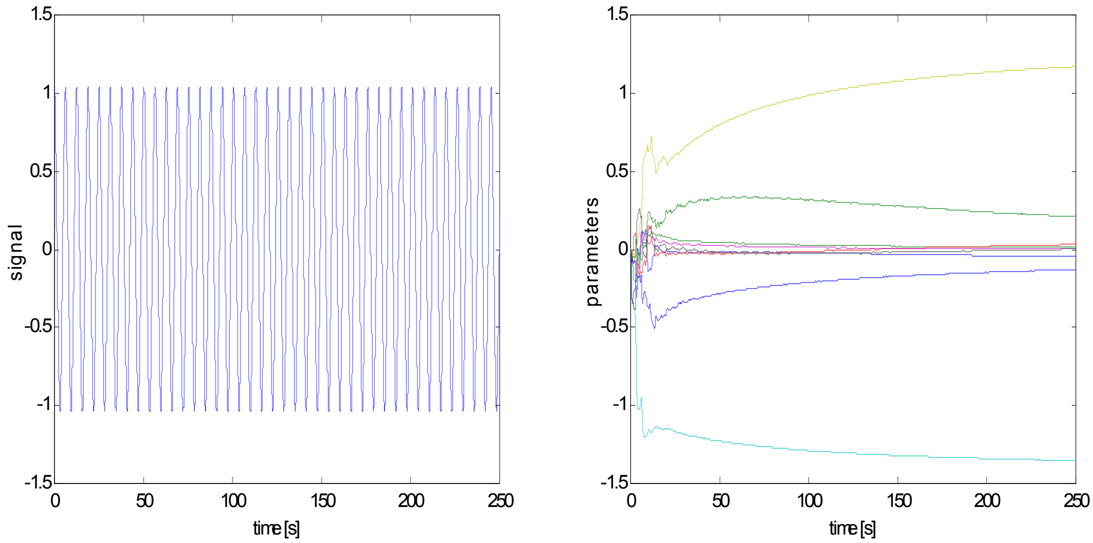

Figure 4: The signal and the parameter estimates obtained by the EKF using a degree of 2 in x_1 and a degree of 2 in x_2.

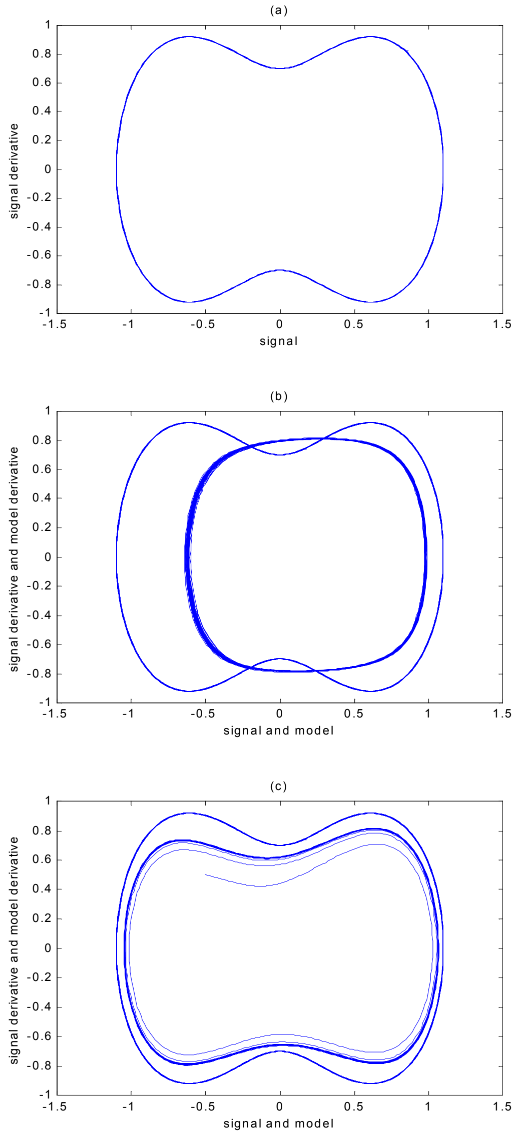

Figure 5: Phase plane plots of the signal (a), of the model obtained with the EKF using a degree of 2 in x_1 and a degree of 2 in x_2 (b), and of the model obtained with the EKF using a degree of 3 in x_1 and a degree of 3 in x_2 (c).

Parameter	Degree 2,2	Degree 3,3
θ_{00}	0.6361	0.0814
θ_{01}	0.0491	0.1844
θ_{02}	-0.9143	-0.1219
θ_{03}		-0.2579
θ_{10}	-2.1289	0.7197
θ_{11}	0.0324	0.1947
θ_{12}	2.9355	0.5784
θ_{13}		-0.1038
θ_{20}	-0.5068	-0.0622
θ_{21}	-0.3473	-0.2511
θ_{22}	0.6407	0.0729
θ_{23}		0.1463
θ_{30}		-2.0826
θ_{31}		-0.0532
θ_{32}		-0.8220
θ_{33}		-0.2946

Table 1: Parameter estimates at the end of the runs.

It can be seen in Fig. 5 b and 5 c that when the degree of the polynomial expansion increases, then the second order nonlinear ODE is quite successful in modeling the signal. The fit is not perfect, a fact that is natural considering the fact that a series expansion is used for modeling, and hence modeling errors affect the estimator. The phase plane plots of Fig. 5 have been obtained by off-line simulation of the estimated ODE, using the parameters obtained at the end of the runs. The conclusion of this experiment is that a second order nonlinear ODE has been estimated and that this ODE has a stable orbit that closely approximates the estimated signal (the signal needs a fourth order linear ODE for its realization). This supports the theoretical results presented previously in this paper.

V. CONCLUSIONS

An analysis of the required order of an ODE, used for periodic signal modeling, has been presented. The analysis relates to nonlinear differential equations, identifiability analysis and the realization of signals from nonlinear

ODEs. Since the results above indicate when a specific order is sufficient, they may also be used for indication of model reduction possibilities.

It was shown that a second order ODE is sufficient for modeling a certain class of periodic signals. These signals are characterized by phase plane plots that lack intersections. Building on the geometric approach of the paper, a method for selection of the order of an ODE used for periodic signal analysis can be formulated: Simply differentiate the signal repeatedly and add dimensions to the state space until the corresponding state space trajectory of the signal does not intersect itself at any point. Using this observation the theoretical results were extended from order two to an arbitrary order of the ODE. A criterion for assessment of the possibility to use a second order ODE for periodic signal analysis was defined. The criterion relates to properties of the overtone contents of the signal. It was shown that the criterion is insensitive to linear state space transformations. The fact that second order ODEs are sufficient in many cases is practically important. The reason is that the algorithms of [6] and [7], that are sometimes used for initialization of more advanced schemes, require the $n-1$:*th* derivative of the signal in case an n:*th* order ODE model is used. Such numerical differentiation requirements would have limited applicability for $n > 2$. In order to illustrate the results, the harmonic oscillator was constructed analytically from the signal $\cos(t)$. The model order reduction property was also illustrated numerically by approximation of a two-frequency signal requiring a fourth order linear ODE for its description. This signal was successfully approximated by a second order nonlinear ODE by application of a previously published EKF algorithm.

There are several topics for further research that could be attempted. The results obtained for arbitrary order of the ODE could be further developed into methods that allow a systematic choice of the order of the ODE that is used for periodic signal modeling. Estimation algorithms handling such cases could also be developed and analyzed. Tools that can be used to determine the stability of orbits estimated with such algorithms are of special interest. The reason is the experimental observation that simulated orbits are often only "weakly" stable. In such situations it may not be clear if the stability or instability is caused by the discretization of the numerical integration scheme, i.e. it may be difficult to reach firm conclusions based on simulation only. Related to this is the remaining problem of construction of an ODE that has an asymptotically stable orbit that equals a given signal. Further work is also needed on the effects of the choice of coordinate system. Such work, which was only pursued for linear transformations in the present paper, needs to take at least two directions. First, the result of Theorem 1 needs to be analyzed. Secondly, transformations more general than a linear one should be analyzed. One interesting special case that should be studied is the class of conformal mappings, see e.g. [8] .

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REFERENCES

[1] K. J. Åström and B. Wittenmark, *Adaptive Control*. Reading, MA: Addison-Wesley, 1989.

[2] H. K. Khalil, *Nonlinear Systems - second edition*. Upper Saddle River, NJ: Prentice Hall, 1996.

[3] P. Stoica and R. Moses, *Introduction to Spectral Analysis*. Upper Saddle River, NJ: Prentice Hall, 1997.

[4] R. Kumaresan and D. W. Tufts, "Estimating the parameters of exponentially damped sinusoids and pole-zero modeling in noise", *IEEE Trans. Acoust. Speech, Signal Processing,* vol. ASSP-30, pp. 8333-840, 1976.

[5] A. Nehorai and B. Porat, "Adaptive comb filtering for harmonic signal enhancement", *IEEE Trans. Acoust. Speech, Signal Processing,* vol. ASSP-34, pp. 1124-1138, 1986.

[6] T. Wigren, E. Abd-Elrady and T. Söderström, "Harmonic signal analysis with Kalman filters using periodic orbits of ODEs", *Proc. ICAASP 2003*, Hongkong, China, 2003.

[7] T. Wigren, E. Abd-Elrady and T. Söderström, "Least squares harmonic signal analysis using periodic orbits of ODEs", *to appear at SYSID 2003,* Rotterdam, the Netherlands, August 27-29, 2003.

[8] R. V. Churchill, J. W. Brown and R. F. Verhey, *Complex Variables and Applications - third edition.* Auckland, New Zealand: Mc Graw Hill International, 1974.

[9] A. H. Jazwinski, *Stochastic Processes and Filtering Theory.* New York, NY: Academic Press, 1970.

[10] L. Ljung and T. Söderström, *Theory and Practice of Recursive Identification.* Cambridge, MA: MIT Press, 1983.