HIGH ORDER FINITE DIFFERENCE OPERATORS WITH THE SUMMATION BY PARTS PROPERTY BASED ON DRP SCHEMES

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Abstract

Strictly stable high order finite difference methods based on Tam and Webb's dispersion relation preserving schemes have been constructed. The methods have been implemented for a 1D hyperbolic test problem, and the theoretical order of accuracy is observed.

1 Introduction

Computational aeroacoustics (CAA) has been given increased interest because of the need to better control noise levels from aircrafts, trains, cars, etc. due to increased transport and stricter regulations from authorities. Other applications range from simulating sound propagation in the atmosphere to improved design of musical instruments.

Much of the current effort in CAA involves the development of schemes for approximating derivatives in a way that better preserves the physics of wave propagation, a phenomenon of less significance in typical aerodynamic computations. An example of such a scheme is the Dispersion Preserving Relation (DRP) scheme proposed by Tam and Webb [5].

In scientific computing it is imperative that the numerical methods are stable. A good approximation of the wave propagation requires high order methods and for difference methods problems arise when applying boundary conditions. To accurately prescribe boundary conditions the simultaneous approximation term (SAT) method [1] can be used if the space derivatives are discretized by a summation by parts (SBP) operator proposed by Kreiss and Scherer [3].

In this paper SBP operators are derived for DRP type schemes, which together with the SAT method lead to strictly stable methods.

2 Theory

In recent years, central difference methods of the type proposed by Tam and Webb [5] called Dispersion Preserving Relation schemes (DRP) have attracted interest in aeroacoustics. The formal accuracy of the method is lowered to get a better approximation of the wave number.

Another development in the theory of finite difference methods is high order operators $Q = \frac{1}{h}H^{-1}B$ with the summation by parts property (SBP) developed by Strand [4], i.e. $B + B^T = diag(-1, 0, \ldots, 0, 1)$, with a discrete norm H and step size h. A consequence is that a stability proof done by the energy method for the continuous problem is valid for the semi-discrete problem if a SBP operators are used.

What we have done is to construct a SBP operator for difference schemes of DRP type. The motivation has been to combine the good wave resolution of DRP schemes with the good stability properties of SBP operators when using using SAT to prescribe boundary conditions in a strictly stable way, first shown by Carpenter et al. [1].

Consider the discrete function v approximating the continuous function u on $\{x_j\}_0^N$ where $x_j = hj$, $h = 1/N$ and let v_j denote $v(x_j)$. A classical central finite difference method approximating du/dx at x_m is

$$(Qv)_j = \frac{1}{h}\sum_{j=1}^{l} \alpha_j (v_{m+j} - v_{m-j}) \tag{1}$$

where $\alpha_k, k = 1 \ldots l$ are chosen such that the accuracy is of order $2l$.

Tam and Webb [5] proposed that the accuracy is lowered to $2(l-1)$ leaving a free parameter and then minimize the wave number error

$$\int_{-\pi/2}^{\pi/2} |hk - h\tilde{k}|^2 d(hk) \tag{2}$$

where k and $\tilde{k}$ are the exact and the approximate wave numbers, respectively.

A comparison between standard centered finite difference methods and DRP schemes is given in figure 1. Noticeable is that the fourth order DRP scheme derived from a sixth order standard scheme approximates the wave number about as well as the eighth order standard scheme. In figure 2 it can be seen that the DRP schemes in an interval near $kh = 1$ lies strictly above the line defining the exact group velocity. The approximation of the wave number and group velocity can be improved for $hk < 2$ by choosing a larger interval to optimize over in equation 2, at the expense of a poorer approximation for $kh > 2$.

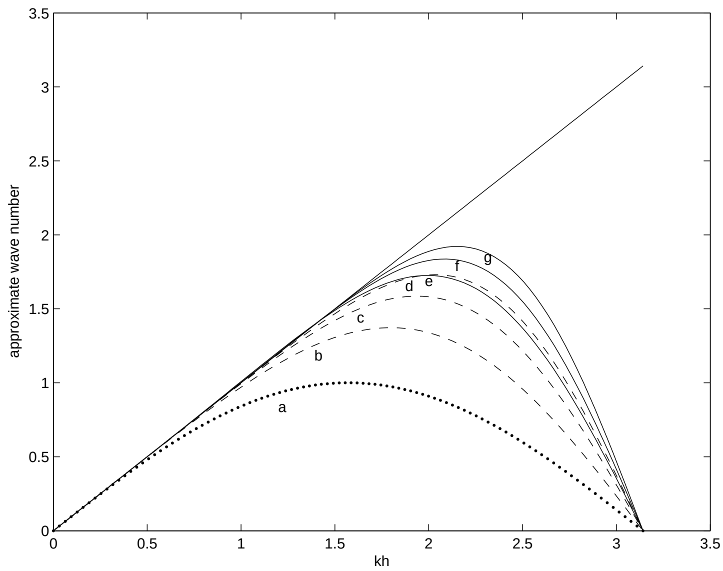

Figure 1: Approximate wave number vs exact for 2nd order standard centered difference method (SC2) (a) (dotted), SC4 (b), SC6 (c), SC8 (d) (dashed) and DRP schemes with order four (e) six (f) and eight (g) (solid).

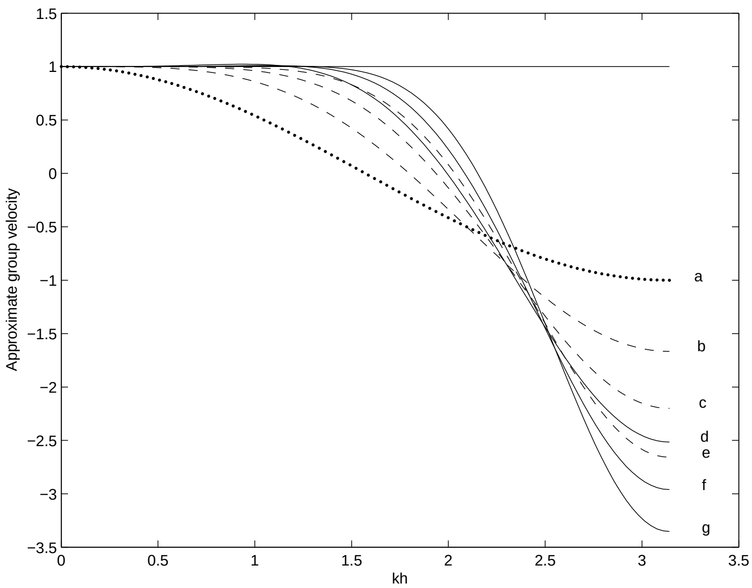

Figure 2: Approximate group velocity vs exact for 2nd order standard centered difference method (SC2) (a) (dotted), SC4 (b), SC6 (c), SC8 (d) (dashed) and DRP schemes with order four (e) six (f) and eight (g) (solid).

3 Summation by Parts Operators

When doing numerical calculations, we must establish an upper bound on the growth of the solution. In the continuous case the energy method is often used,

for example, with the usual L_2 scalar product and norm

$$(u,v)=\int_0^1 uv\,dx\,,\quad ||u||=(u,u)^{1/2}. \tag{3}$$

We have for the simplest hyperbolic equation, sometimes called the Kreiss equation,

$$\left\{\begin{array}{lll} u_t & =u_x & ,\, x\in[0,1],\, t\geq 0 \\ u(x,0) & =f(x) & ,\, x\in[0,1] \\ u(1,t) & =g(t) & t\geq 0 \end{array}\right. \tag{4}$$

the following energy growth

$$\frac{1}{2}\frac{d||u||^2}{dt}=(u,u_t)=(u,u_x)=\int_0^1 uu_x\,dx=\frac{1}{2}[u^2]_0^1=\frac{1}{2}(g(t)^2-u(0,t)^2) \tag{5}$$

If $g\equiv 0$ it follows that $\|u\|\leq\|f\|$ and the problem is well posed.

In the discrete case we want to find an operator Q that in a modified norm, determined by H, gives the same estimate as the continuous case. That is Q must satisfy

$$(u,Qv)_h=u_nv_n-u_0v_0-(Qu,v)_h \tag{6}$$

where

$$(u,v)_h=hu^THv. \tag{7}$$

The matrix Q is a banded matrix were row j approximate $\frac{du_j}{dx}$. Of course Q has to be modified at the l first and l last rows, where l is equal to or larger than the bandwidth of Q. The matrix H is a modified identity matrix, its l first elements are modified.

Since equation (6) is the discrete analogue of integration by parts, it is called the summation by parts property.

An example of such an operator Q and scalar product H is

$$Q=\frac{1}{h}\begin{pmatrix} -1 & 1 & 0 & \dots & 0 \\ -0.5 & 0 & 0.5 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & -0.5 & 0 & 0.5 \\ 0 & \dots & 0 & -1 & 1 \end{pmatrix},\ H=\begin{pmatrix} 0.5 & 0 & \dots & & 0 \\ 0 & 1 & 0 & & \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ & & 0 & 1 & 0 \\ 0 & & \dots & 0 & 0.5 \end{pmatrix}.$$

Operators if this kind were described by Kreiss and Scherer [3], and high order operators were later constructed by Strand [4]. High order operators Q have the following structure: in the interior a centered high order finite difference stencil is used and near the boundary where the stencil cannot be used, a small dense matrix is used instead.

Given a DRP scheme of order 2τ with coefficients α_j, $j=1\dots\tau+1$ a summation by parts operator Q which is accurate of order τ near the boundary and a diagonal norm matrix H with property that $h(HQ+(HQ)^T)=diag(-1,0,\dots,0,1)$ will be derived.

For simplicity we approximate $\frac{d}{dx}$ for $x=[0,\infty)$ and let $h=1$. The difference operator Q has to be modified for rows $1\dots2\tau$.

$$\begin{pmatrix} q_{0,0} & q_{0,1} & \dots & q_{0,2\tau-1} & & & & & \\ q_{1,0} & 0 & \ddots & \vdots & q_{\tau-1,2\tau} & 0 & \dots & & \\ \vdots & \ddots & \ddots & q_{2\tau-2,2\tau-1} & \vdots & \ddots & 0 & \dots & \\ q_{2\tau-1,0} & \dots & q_{2\tau-1,2\tau-2} & 0 & q_{2\tau-1,2\tau} & \dots & q_{2\tau-1,3\tau} & 0 & \dots \end{pmatrix}$$

In order for HQ to be nearly antisymmetric and have the SBP property the following relation must hold

$$\begin{array}{lllll} h_{0,0}q_{0,0} & = -1/2 & & \\ h_{i,i}q_{i,j} & = -h_{j,j}q_{j,i} & 0 \le i < 2\tau - 1 & i < j \le 2\tau - 1 \\ h_{i,i}q_{i,j} & = \alpha_{j-i} & \tau - 1 \le i < 2\tau & 2\tau \le j \le \tau + i + 1 \end{array}$$

and τth order accuracy near the boundary leads to the following system of equations, $j = 0, ..., \tau$. Note that $q_{i,i} = 0$ for $i \neq 0$

$$\begin{array}{ccccccccc} 0^j q_{0,0} & + & \dots & \alpha^j q_{0,\alpha} & & & & = & j * 0^{j-1} \\ \vdots & & \vdots & \vdots & & & & & \vdots \\ 0^j q_{\beta,0} & + & \dots & \alpha^j q_{\beta,\alpha} & & & & = & j * \beta^{j-1} \\ \vdots & & \vdots & \vdots & \ddots & & & & \vdots \\ 0^j q_{\gamma,0} & + & \dots & \gamma^j q_{\gamma,\gamma} & \dots & + & (3\tau)^j q_{\gamma,3\tau} & = & j * \gamma^{j-1} \end{array}$$

where $\alpha = max\{2, 2\tau - 1\}$, $\beta = max\{0, \tau - 2\}$, $\gamma = 2\tau - 1$ and $j = 0, ..., \tau$, note that $q_{i,i} = 0$ for $i \neq 0$. The accuracy conditions come from requiring that polynomials of degree τ and lower are exactly differentiated.

From these systems of equations, a system of equations with $h_{i,i}q_{i,j}$, $j > i$, $i = 0, \dots 2(\tau - 1)$ as unknowns can be derived. The solution is used to compute $h_{i,i}$, $i = 0, \dots, 2\tau - 1$, which is used to compute $q_{i,j}$, $i = 0, \dots 2\tau - 1$, $j = 0, \dots, 3\tau$.

To illustrate the derivation of H and Q we let $\tau = 1$. Then the following conditions yield the SBP property.

$$\begin{cases} h_{0,0}q_{0,0} & & = -\frac{1}{2} \\ h_{0,0}q_{0,1} & +h_{1,1}q_{1,0} & = 0 \\ h_{0,0}q_{0,2} & -\alpha_2 & = 0 \\ h_{1,1}q_{1,2} & -\alpha_1 & = 0 \\ h_{1,1}q_{1,3} & -\alpha_2 & = 0 \end{cases} \tag{8}$$

First order accuracy near the boundary leads to two equation systems

$$\begin{cases} q_{0,0} + q_{0,1} + q_{0,2} = 0 \\ q_{1,0} + q_{1,2} + q_{1,3} = 0 \end{cases} \tag{9}$$

$$\begin{cases} q_{0,1} + 2q_{0,2} = 1 \\ 2q_{1,2} + 3q_{1,3} = 1 \end{cases} \tag{10}$$

Using the conditions for antisymmetry of HQ and multiplying the first equation in each system by $h_{0,0}$ and the second equation by $h_{1,1}$ yields,

$$\begin{cases} -\frac{1}{2} + h_{0,0}q_{0,1} + h_{0,0}q_{0,2} = 0 \\ -h_{0,0}q_{0,1} + \alpha_1 + \alpha_2 = 0 \end{cases} \tag{11}$$

$$\begin{cases} h_{0,0}q_{0,1} + 2h_{0,0}q_{0,2} = h_{0,0} \\ 2\alpha_1 + 3\alpha_2 = h_{1,1} \end{cases} \tag{12}$$

Using that $\alpha_1 = 1/2 - 2\alpha_2$, which is equivalent to requiring that the interior scheme is second order accurate, the solution can be computed.

$$h_{0,0} = \frac{1}{2} + \alpha_2,\; h_{1,1} = 1 - \alpha_2 \tag{13}$$

$$\begin{array}{llll} q_{0,0} = -\frac{1}{1+2\alpha_2} & q_{0,1} = \frac{1/2-\alpha_2}{h_{0,0}} & q_{0,2} = \frac{\alpha_2}{h_{0,0}} & \\ q_{1,0} = \frac{\alpha_2-1/2}{h_{1,1}} & q_{1,1} = 0 & q_{1,2} = \frac{\alpha_1}{h_{1,1}} & q_{1,3} = \frac{\alpha_2}{h_{1,1}} \end{array} \tag{14}$$

For $\tau > 2$ the system of equations for HQ is undetermined thus leading to free parameters, one for $\tau = 3$ and three for $\tau = 4$. The free parameters were initially chosen such that the bandwidth of Q was minimized, but as is seen in figure 3 the spectral radius for the fourth order method is very large and leads to a restrictive CFL condition. The spectral radius of Q can be made smaller if the three parameters are chosen as $[0.502, -0.1, 0.799]$, which are close to the values that minimize the bandwidth. These values were found by trail and error.

The eigenvalues in figure 3 were computed using MATLAB, and a computation in exact arithmetic using MAPLE showed that all eigenvalues are imaginary for SBP-2-4(6). Computations for the other SBP operators were unfortunately too time consuming to complete.

The original operators by Strand [4] are denoted by SBP-τ-2τ and the new ones SBP-τ-2τ-(σ). The number in parenthesis denote the order of the difference method modified for better wave approximation, in this article $\sigma = 2(\tau+1)$. The operators that have been derived are SBP-1-2(4), SBP-2-4(6) SBP-3-6(8) and SBP-4-8(10), where Tam and Webb's DRP scheme corresponds to SBP-2-4(6) in the interior. They can be found in the appendix.

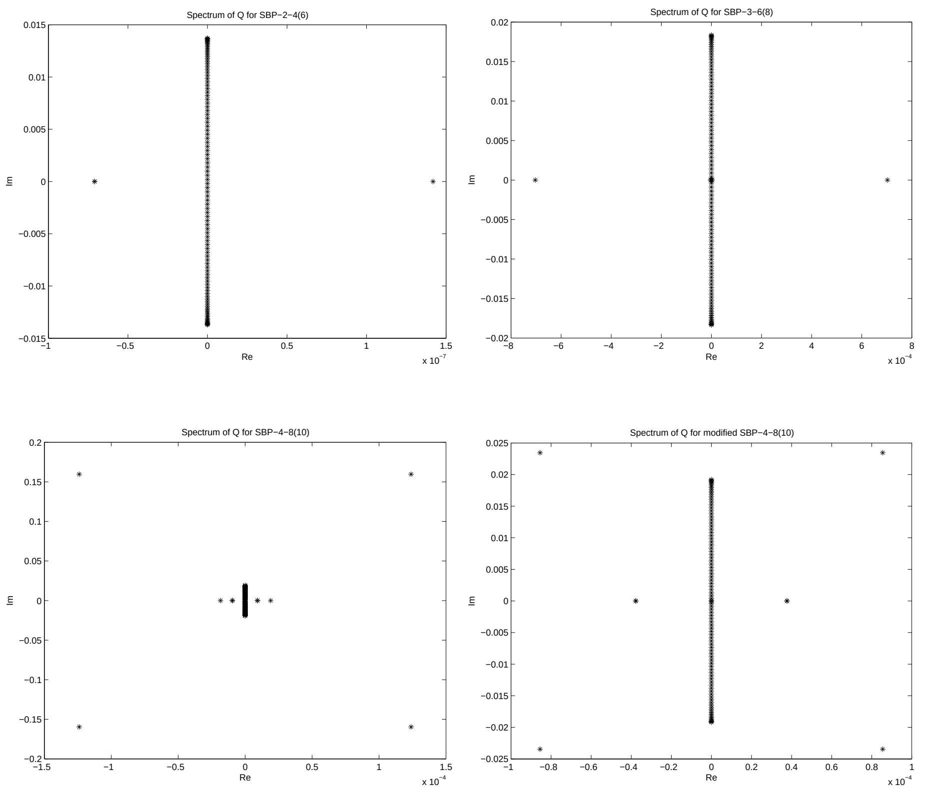

Figure 3: Spectrum for SBP-2-4(6), SBP-3-6(8), SBP-4-8(10) and a modified SBP-4-8(10), $h = 1/100$.

4 Numerical experiments

The summation by parts operators for DRP schemes were tested for the hyperbolic system

$$u_t + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} u_x = 0,\ u = \begin{pmatrix} u^I \\ u^{II} \end{pmatrix},$$
$$0 \le x \le 1,\ t \ge 0,$$

With the initial data

$$u^I(x,0) = \sin 2\pi x,\ u^{II}(x,0) = -\sin 2\pi x,\ 0 \le x \le 1,$$

and boundary conditions

$$u^I(0,t) = u^{II}(0,t),\ u^{II}(1,t) = u^I(1,t),\ t \ge 0,$$

the exact solution is

$$u^I = \sin 2\pi(x-t),\ u^{II} = -\sin 2\pi(x+t)\,.$$

The boundary conditions were imposed using the simultaneous approximation term (SAT) method [1].

In the SAT method one does not impose the exact boundary conditions (b.c.) which might destroy the SBP property and therefore strict stability. Instead the boundary conditions are imposed as a penalty term at the same accuracy as the discretization.

In our case, the SAT formulation reads

$$v_t + Q \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} v = -\frac{1}{h} H^{-1} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} P \tag{15}$$

where $v = (v_0^I, v_0^{II}, \dots, v_N^I, v_N^{II})^T$ denotes the numerical approximation of the exact solution u, and the matrix P which is imposing the boundary conditions and reads

$$P = \begin{pmatrix} 1 & -1 & 0 & \dots & 0 & 0 \\ 0 & 0 & & \dots & 0 & 0 \\ \vdots & \vdots & & & \vdots & \vdots \\ 0 & 0 & & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & -1 & 1 \end{pmatrix} \tag{16}$$

Table 1: Order of accuracy for u^I using SBP 2-4 and SBP 2-4(6) for the 1D test case at $t = 1.5$.

# of grid points /	SBP 2-4	SBP 2-4(6)
101		
202	3.0134	3.0137
401	3.0091	3.0106
801	3.0068	3.0083

Table 2: Order of accuracy for u^I using SBP 3-6(8) for the 1D test case at $t = 1.5$.

# of grid points /	SBP 3-6(8)
101	
202	3.9141
401	4.1361
801	4.3108

Table 3: Order of accuracy for u^I using SBP 4-8(10) for the 1D test case at $t = 1.5$.

# of grid points /	SBP 4-8(10)
101	
202	4.6758
401	4.6000
801	4.5678

The orders of accuracy for SBP-2-4 (SBP operator, 2nd order near boundaries, 4th order standard scheme in interior) and SBP-2-4(6) (SBP operator, 2nd order near boundaries, 4th order DRP scheme derived from 6th order standard scheme in interior) agree with theory, saying that the global order of accuracy for a finite difference scheme is at most one order higher than the accuracy at the boundaries [2].

5 Conclusions

New strictly stable high order finite difference methods have been developed. They are based on dispersion relation preserving type schemes proposed Tam and Webb [5] and strictl stability is guaranteed by requiring that the methods have the summation by parts property introduced by Kreiss and Scherer [3].

The methods have been developed using MAPLE and implemented in MATLAB. Numerical experiments show that the order of accuracy corresponds to theory.

6 Appendix

6.1 SBP operators with diagonal norm and minimal bandwidth

6.1.1 Second order at the boundary, SBP-2-4(6)

A SBP operator with a fourth order DRP finite difference operator in the interior and second order near the boundary (SBP 2-4(6)) has the following centered finite difference operator in the interior of hQ

$$hQ(j,j-3) = -\alpha_3, \quad hQ(j,j-2) = -\alpha_2, \quad hQ(j,j-1) = -\alpha_1,$$
$$hQ(j,j) = 0,$$
$$hQ(j,j{+}1) = \alpha_1, \quad hQ(j,j{+}2) = \alpha_2, \quad hQ(j,j{+}3) = \alpha_3 \quad \text{for } j = 7, \ldots, N{-}6$$

where α_j has the following values

$\alpha_1 = 0.79926642697415587 \quad \alpha_2 = -0.18941314157932453 \quad \alpha_3 = 0.026519952061497799$

H is a diagonal matrix with ones in the diagonal except for the first and last four elements of the diagonal. In this case the first four elements read

$$h_{0,0} = 0.34532668264616756, \quad h_{1,1} = 1.2556866187281647,$$
$$h_{2,2} = 0.86931338127183581, \quad h_{3,3} = 1.0296733173538330$$

and in reverse order for $h_{n-3,n-3}, \ldots, h_{n,n}$.

Below is the boundary operator for hQ for the SBP 2-4(6) operator. The same operator is located in the bottom right half corner with opposite sign and the elements are in a different order as this MATLAB code indicates.

```
Q(N-3:N,N-6:N) = -Q(3:-1:0,6:-1:0);
```

q00 = -1.4479043326991200
q01 = 1.8437129980973616
q02 = -0.34371299809736176
q03 = -0.052095667300879370
q04 = 0

q10 = -0.50703996036005625
q11 = 0
q12 = 0.54223976216033715
q13 = -0.056319682880449493
q14 = 0.021119881080168560

q20 = 0.13653680246090086
q21 = -0.78324253158386005
q22 = 0
q23 = 0.83408716489415180
q24 = -0.21788821575736760
q25 = 0.030506779986175047
q26 = 0

q30 = 0.017471584109301608
q31 = 0.068681853722050198
q32 = -0.70418755285702870
q33 = 0
q34 = 0.77623301828214610
q35 = -0.18395459840223802
q36 = 0.025755695145768824

6.1.2 Third order at the boundary, SBP-3-6(8)

A SBP operator with a sixth order DRP finite difference operator in the interior and third order near the boundary (SBP 3-6(8)) has the following centered finite difference operator in the interior of hQ

$$hQ(j,j{-}4) = -\alpha_4, \quad hQ(j,j{-}3) = -\alpha_3, \quad hQ(j,j{-}2) = -\alpha_2, \quad hQ(j,j{-}1) = -\alpha_1,$$
$$hQ(j,j) = 0,$$
$$hQ(j,j{+}1) = \alpha_1, \quad hQ(j,j{+}2) = \alpha_2, \quad hQ(j,j{+}3) = \alpha_3 \quad hQ(j,j{+}4) = \alpha_4 \quad \text{for } j = 9, \ldots, N{-}8$$

where α_j has the following values

$$\alpha_1 = 0.8331572598964345 \quad \alpha_2 = -0.2331572598964345$$
$$\alpha_3 = 0.05230549233656718 \quad \alpha_4 = -0.005939804278316752$$

H is a diagonal matrix with ones in the diagonal except for the first and last six elements of the diagonal. In this case the first six elements read

$$h_{0,0} = 0.3153550936462424, \quad h_{1,1} = 1.393363420657677,$$
$$h_{2,2} = 0.6216064920179795, \quad h_{3,3} = 1.246449063537576,$$
$$h_{4,4} = 0.9087199126756564, \quad h_{5,5} = 1.014506017464869$$

and in reverse order for $h_{n-5,n-5}, \ldots, h_{n,n}$.

Below is the boundary operator for hQ for the SBP 3-6(8) operator. The same operator is located in the bottom right half corner with opposite sign and the elements are in a different order as this MATLAB code indicates.

```
Q(N-5:N,N-9:N) = -Q(5:-1:0,9:-1:0);
```

q00 = -1.585514266533103	q01 = 2.008723732799078
q10 = -0.4546274514421262	q11 = 0
q20 = 0.006638621722402879	q21 = -0.4773187801117445
q30 = 0.1664613995014180	q31 = -0.4554757153982624
q40 = -0.08600120227144731	q14 = 0.2769005367876363
q50 = 0	q15 = -0.02035922973188162

q02 = -0.01308559919861748348336738140004	q03 = -0.657942933867588344344421746018
q12 = 0.2129411810879291959290559542 61	q13 = 0.407450971157474941475221424938
q22 = 0	q23 = 0.307082033336054068764476763891
q32 = -0.153142387513228968467498042145	q33 = 0
q42 = -0.172627548513855444351101578662	q43 = -0.660390739163292203649761743354
q52 = 0.0485326149573270703688045031784	q53 = 0.0939867209644063450256722219284
q04 = 0.247819066800230419419438769975	q05 = 0
q14 = -0.180588228368106206106416068749	q15 = 0.0148235275648283216283776183329
q24 = 0.252362375273239592208687307356	q25 = -0.0792086803303361725384538561848
q34 = 0.481455867214575438317175291842	q35 = -0.0764973850672778007026589857013
q44 = 0	q45 = 0.847673568210825495387209342755
q54 = -0.759283668722722050186941866401	q55 = 0
q06 = 0	q07 = 0
q16 = 0	q17 = 0
q26 = -0.00955556988961586894406770497530	q27 = 0
q36 = 0.0419636019366228644454343113704	q37 = -0.00476538067384707719478469156032
q46 = -0.256577694231350970143666364671	q47 = 0.0575595313880133315974641085544
q56 = 0.821244276084627463843621091279	q57 = -0.229823437103968213426853599491
q08 = 0	q09 = 0
q18 = 0	q19 = 0
q28 = 0	q29 = 0
q38 = 0	q39 = 0
q48 = -0.00653645220651922535583776530795	q49 = 0
q58 = 0.0515575969349816747689080909099	q59 = -0.00585487338277167148733298046745

6.1.3 Fourth order at the boundary, SBP-4-8(10)

A SBP operator with a eight order DRP finite difference operator in the interior and second order near the boundary (SBP 4-8(10)) has the following centered finite difference operator in the interior of hQ

$$\begin{gathered} hQ(j,j-5) = -\alpha_5, \quad hQ(j,j-4) = -\alpha_4, \quad hQ(j,j-3) = -\alpha_3, \\ hQ(j,j-2) = -\alpha_2, \quad hQ(j,j-1) = -\alpha_1, \quad hQ(j,j) = 0, \\ hQ(j,j+1) = \alpha_1, \quad hQ(j,j+2) = \alpha_2, \quad hQ(j,j+3) = \alpha_3 \\ hQ(j,j+4) = \alpha_4, \quad hQ(j,j+5) = \alpha_5 \quad \text{for } j = 9, \dots, N-8 \end{gathered}$$

where α_j has the following values

$$\begin{gathered} \alpha_1 = 0.85710439841851208608 \quad \alpha_2 = -0.26526216962115666981 \\ \alpha_3 = 0.07480520850713872 2005 \quad \alpha_4 = -0.01444845684162134 9730 \\ \alpha_5 = 0.00135962853377409728 77 \end{gathered}$$

H is a diagonal matrix with ones in the diagonal except for the first and last eight elements of the diagonal. In this case the first eight elements read

$$\begin{gathered} h_{0,0} = 0.294851829648342276, \quad h_{1,1} = 1.52599254960446488, \\ h_{2,2} = 0.25663709986386517, \quad h_{3,3} = 1.79947333003289182, \\ h_{4,4} = 0.411348429226366286, \quad h_{5,5} = 1.2793004001361369051, \\ h_{6,6} = 0.9230236540992415309, \quad h_{7,7} = 1.009372707388694828 \end{gathered}$$

and in reverse order for $h_{n-7,n-7}, \dots, h_{n,n}$.

Below is the boundary operator for hQ for the SBP 4-8(10) operator. The same operator is located in the bottom right half corner with opposite sign and the elements are in a different order as this MATLAB code indicates.

```
Q(N-7:N,N-12:N) = -Q(7:-1:0,12:-1:0);
```

q00 = -1.6957669911573197987	q01 = 2.0621682891199350282
q10 = -0.39845154764838537791	q11 = 0
q20 = -1.0060547060384519478	q21 = 2.6156550656068261280
q30 = 0.41657226346831719539	q31 = -1.6072314212261918344
q40 = -1.2098265916012883296	q41 = 5.8230938981601276548
q50 = 0.089325888656450236921	q51 = -0.72026816640720284437
q60 = 0	q61 = 0.15057109202244840663
q70 = 0	q71 = 0
q02 = 0.87566342176012784417	q03 = -2.5423300884267936392
q12 = -0.43989345194087425217	q13 = 1.8952714273978855804
q22 = 0	q23 = -10.439577939423169752
q32 = 1.4888706386814724978	q33 = 0
q42 = -10.496233449667786180	q43 = 7.2472876183287816283
q52 = 2.0946589947956824426	q53 = -2.7302904871947353579
q62 = -0.70025901086696570588	q63 = 1.2248064281916752937
q72 = 0.017998153763907768034	q73 = -0.080090074114033601879

q04 = 1.6878317108800632472	q05 = - 0.38756634217601260433
q14 = - 1.5696803558226220357	q15 = 0.60382952310539918663
q24 = 16.823791823568735151	q25 = -10.441584991462022668
q34 = - 1.6566849467542517715	q35 = 1.9410466687451647647
q44 = 0	q45 = - 2.9362325326725905861
q54 = 0.94412120877136880657	q55 = 0
q64 = - 0.83167376571312553306	q65 = - 0.48635033688049237125
q74 = 0.10696008003992586465	q75 = 0.097204485324945230782
q06 = 0	q07 = 0
q16 = -0.091075595091402694223	q17 = -0.070788070790662511317
q26 = 2.5185588185387488660	q27 = 0.044924664119343000785
q36 = -0.62825343729493118302	q37 = -0.26246018679458559520
q46 = 1.8661905666948906603	q47 = -0.076694695407211351188
q56 = 0.35090496733375371850	q57 = 0.86342625760635763892
q66 = 0	q67 = 0
q76 = -0.78956252086787838706	q77 = 0
q08 = 0	q09 = 0
q18 = 0	q19 = 0
q28 = 0	q29 = 0
q38 = 0.00075557026107702591326	q39 = 0
q48 = 0.0033052965252138828252	q49 = -0.035124618972764619908
q58 = 0.0010627906734254224064	q59 = -0.011294029799477757896
q68 = 0.0014730159164782497904	q69 = -0.015653398238988014629
q78 = 0.0013470034644502469271	q79 = -0.014314293160353361934
q010 = 0	q011 = 0
q110 = 0	q111 = 0
q210 = 0	q211 = 0
q310 = 0	q311 = 0
q410 = 0	q411 = 0
q510 = 0.058473528577946443849	q511 = 0
q610 = 0.081043652754640918266	q611 = -0.28738393479202890305
q710 = 0.074110591617504789877	q711 = -0.26279903119968949743
q012 = 0	
q112 = 0	
q212 = 0	
q312 = 0	
q412 = 0	
q512 = 0	
q612 = 0	
q712 = 0.84914560513122095062	

6.2 Modified SBP-4-8(10) operator with diagonal norm and smaller spectral radius

q00 = -1.695766991157320	q01 = 2.291032605468909
q10 = -0.4426726432456110	q11 = 0
q20 = 0.5330534182148309	q21 = -2.407040818496168
q30 = -0.1184427737693269	q31 = 0.6618843657314521
q40 = 1.830612773966382	q41 = -8.288032061663463
q50 = -0.6245601110692797	q51 = 2.709330688538044
q60 = 0.3850559271659521	q61 = -1.714360699440487
q70 = -0.05705219972778544	q71 = 0.2742239481189889

q02 = -0.4639662012147939	q03 = 0.7228532812132124
q12 = 0.4048092994116597	q13 = - 0.7805039834619138
q22 = 0	q23 = 11.03207597325678
q32 = -1.573371461527950	q33 = 0
q42 = 14.02365705116605	q43 = -10.10065328335495
q52 = -4.343572255913729	q53 = 3.002892932574174
q62 = 2.860630607831792	q63 = - 2.082622244166414
q72 = -0.4973385909142549	q73 = 0.3800614179644372
q04 = - 2.553891864910215	q05 = 2.70983565186940
q14 = 2.234132119994110	q15 = - 2.27133994516961
q24 = -22.47769049395616	q25 = 21.6521061372589
q34 = 2.308946619504642	q35 = - 2.13484805031142
q44 = 0	q45 = 3.77017058952430
q54 = - 1.212267071714534	q55 = 0
q64 = 0.6108231151820283	q65 = - 0.690362730945967
q74 = - 0.06793041339961766	q75 = 0.0990714324530388
q06 = - 1.20540452249925	q07 = 0.1953080412300543
q16 = 1.03696146986553	q17 = -0.1813863173941616
q26 = -10.2885737022032	q27 = 1.956069485925045
q36 = 1.06826234200632	q37 = -0.2131866118947915
q46 = - 1.37062437516525	q47 = 0.1666886279744916
q56 = 0.498101251671513	q57 = -0.07816772353808257
q66 = 0	q67 = 0.8513566887329923
q76 = - 0.778525470347940	q77 = 0
q08 = 0	q09 = 0
q18 = 0	q19 = 0
q28 = 0	q29 = 0
q38 = 0.0007555702610770257	q39 = 0
q48 = -0.03512461897276456	q49 = 0.003305296525213877
q58 = 0.05847352857794642	q59 = -0.01129402979947775
q68 = -0.2873839347920289	q69 = 0.08104365275464091
q78 = 0.8491456051312209	q79 = -0.2627990311996895
q010 = 0	q011 = 0
q110 = 0	q111 = 0
q210 = 0	q211 = 0
q310 = 0	q311 = 0
q410 = 0	q411 = 0
q510 = 0.001062790673425422	q511 = 0
q610 = -0.01565339823898801	q611 = 0.00147301591647824960
q710 = 0.0741105916175047861	q711 = -0.0143142931603533612
q012 = 0	
q112 = 0	
q212 = 0	
q312 = 0	
q412 = 0	
q512 = 0	
q612 = 0	
q712 = 0.001347003464450247	

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