DIFFERENT METHODS THAT REDUCE COST IN MONOSTATIC RCS COMPUTATIONS FOR MOM ACCELERATED BY MLFMA

Martin Nilsson[(I)]

[(I)] Email: martin.nilsson@it.uu.se
Department of Information Technology, Scientific Computing, Uppsala University
SE-751 05 Uppsala
Sweden

The problem of computing the monostatic Radar Cross Section with the Method of Moments accelerated by the multilevel Fast Multipole algorithm is considered. For electrically large objects the problem becomes too expensive in terms of computational work. This paper suggests two methods that can reduce the work, the Minimal Residual Interpolation method and a new fast approximate method related to the Fast Physical Optics method. Numerical experiments show that the new method gives reasonable results, while reducing the work substantially for large objects.

1 Introduction

The Multilevel Fast Multipole Algorithm (MLFMA) [1] is an efficient method for computing the bistatic Radar Cross Section (BRCS) of an electrically large object. It reduces the computational work in the Method of Moments from $\mathcal{O}\left((\kappa D)^6\right)$, for a direct method, to $\mathcal{O}\left((\kappa D)^2 \log \kappa D\right)$. Here, κ is the wavenumber and D is the diameter of the smallest sphere enclosing the object. The memory requirement is reduced from $\mathcal{O}\left((\kappa D)^4\right)$ to $\mathcal{O}\left((\kappa D)^2\right)$. Unfortunately, in the worst case the problem of computing the monostatic RCS (MRCS) requires that $\mathcal{O}\left((\kappa D)^2\right)$ problems are solved in order to get enough resolution. This does not affect the estimates for the direct method but the estimate in MLFMA becomes $\mathcal{O}\left((\kappa D)^4 \log \kappa D\right)$. This implies that a direct method can be faster than MLFMA for quite large problems, because direct methods can be very efficiently implemented on computers and also have small constants in front of them. In this paper suggestions for reducing the cost when the MRCS is computed with MLFMA are given.

The rest of this paper is organized as follows. In section 2 the governing integral equations are given. The MLFMA is described in section 3. Section 4 describes a linear algebra method for interpolating between different right hand sides in order to compute the MRCS, called Minimal Residual Interpolation. Next, section 5 describes a fast asymptotic method that can obtain an approximate MRCS with computational work $\mathcal{O}\left(K\left(\kappa d_1\right)^2\left(\kappa D\right)^2 \log\left(\kappa D\right)\right) + \mathcal{O}\left(\left(\kappa D\right)^2 \log\left(\kappa D\right)\right)$, where $d_1 \leq D$. Section 6 contains numerical experiments that demonstrate the performance of the methods and conclusions are given in section 7.

2 The combined field integral equation

Time-harmonic electromagnetic scattering from a perfect electric conductor (PEC) is considered. The Electric Field Integral Equation (EFIE) and the Magnetic Field Integral Equation (MFIE) combined in a variational form yields the Combined Field Integral Equation (CFIE) [2]

$$\begin{aligned}
&\alpha \int_{\mathbf{x}\in\Gamma} \int_{\mathbf{y}\in\Gamma} G(\mathbf{x},\mathbf{y}) \left(\mathbf{J}\cdot\mathbf{J}' - \frac{1}{\kappa^2} \boldsymbol{\nabla}_\Gamma \cdot \mathbf{J} \boldsymbol{\nabla}_\Gamma \cdot \mathbf{J}' \right) d\Gamma(\mathbf{y})\, d\Gamma(\mathbf{x}) \\
&+ (1-\alpha)\frac{\imath}{\kappa} \int_{\mathbf{x}\in\Gamma} \mathbf{J}' \cdot \left(\frac{1}{2}\mathbf{J} + \hat{\mathbf{n}} \times \int_{\mathbf{y}\in\Gamma} \boldsymbol{\nabla}_{\mathbf{y}} G(\mathbf{x},\mathbf{y}) \times \mathbf{J} d\Gamma(\mathbf{y}) \right) d\Gamma(\mathbf{x}) \\
&= -\alpha \frac{1}{\imath\kappa Z} \int_{\mathbf{x}\in\Gamma} \mathbf{E}_a \cdot \mathbf{J}' d\Gamma(\mathbf{x}) + (1-\alpha)\frac{\imath}{\kappa} \int_{\mathbf{x}\in\Gamma} \hat{\mathbf{n}} \times \mathbf{H}_a \cdot \mathbf{J}' d\Gamma(\mathbf{x}),
\end{aligned} \tag{1}$$

where, $\mathbf{J} = \mathbf{J}(\mathbf{y})$ is the unknown electric current on the surface Γ of the scatter, $\mathbf{J}' = \mathbf{J}'(\mathbf{x})$ is the test current, κ is the wavenumber, Z is the impedance in free space, $\hat{\mathbf{n}}$ is the unit normal pointing outward from Γ, and $\imath = \sqrt{-1}$. The function $G(\mathbf{x},\mathbf{y})$ is the free-space Green's function for Helmholtz' equation. The parameter α can vary between 0 (MFIE) and 1 (EFIE). The right hand side depends on the applied electric field $\mathbf{E}_a$ and the applied magnetic field $\mathbf{H}_a$.

RWG basis functions are used for discretization [3]. The discretization leads to a dense, complex system of equations of the form $\mathbf{A}\mathbf{x} = \mathbf{b}$. The unknowns in $\mathbf{x}$ are the coefficients for each basis function and the right hand side $\mathbf{b}$ depends on the applied fields $\mathbf{E}_a$ and $\mathbf{H}_a$. Only the right hand side is changed, if the wavenumber is constant and the applied fields are changed. In order to resolve the wavelength at least 10 basis functions per wavelength should be used. Thus, the total number of unknowns N depends on the frequency as $N = \mathcal{O}(f^2) = \mathcal{O}\left((\kappa D)^2\right)$, where D is the diameter of the smallest sphere enclosing the surface.

The applied electric and magnetic fields in (1) at $\mathbf{x}$ can be written

$$\mathbf{E}_a(\mathbf{x},\hat{\boldsymbol{\kappa}}_a) = \mathbf{E}_0 \exp(-\imath\kappa\hat{\boldsymbol{\kappa}}_a\cdot\mathbf{x}),\ \mathbf{H}_a(\mathbf{x},\hat{\boldsymbol{\kappa}}_a) = \mathbf{H}_0 \exp(-\imath\kappa\hat{\boldsymbol{\kappa}}_a\cdot\mathbf{x}), \tag{2}$$

for a plane wave traveling in the direction given by the unit vector $\hat{\boldsymbol{\kappa}}_a$. The electric and magnetic fields are coupled by $\mathbf{H}_0 = Z^{-1}\hat{\boldsymbol{\kappa}}_a \times \mathbf{E}_0$. Discretization with the test functions $\mathbf{j}_j$ and approximation of the integral with a quadrature rule with q positive weights w_{jk} gives the j:th component of the discretized right hand side

$$b_j(\hat{\boldsymbol{\kappa}}_a) = \sum_{k=1}^{q} w_{jk} \mathbf{K}(\hat{\boldsymbol{\kappa}}_a) \cdot \mathbf{j}_j(\mathbf{x}_{jk}) \exp(-\imath\kappa\hat{\boldsymbol{\kappa}}_a\cdot\mathbf{x}_{jk}) \tag{3}$$

where $\mathbf{K}(\hat{\boldsymbol{\kappa}}_a)$ depends on $\mathbf{E}_0$ and $\mathbf{H}_0$.

The Radar Cross Section (RCS) $\sigma(\hat{\boldsymbol{\kappa}},\hat{\boldsymbol{\kappa}}_a)$ is defined as

$$\sigma(\hat{\boldsymbol{\kappa}},\hat{\boldsymbol{\kappa}}_a) = \lim_{r\to\infty} 4\pi r^2 \frac{|\mathbf{E}_s(r\hat{\boldsymbol{\kappa}},\hat{\boldsymbol{\kappa}}_a)|^2}{|\mathbf{E}_a(r\hat{\boldsymbol{\kappa}},\hat{\boldsymbol{\kappa}}_a)|^2} \tag{4}$$

Here, $\mathbf{E}_s\left(r\hat{\boldsymbol{\kappa}}, \hat{\boldsymbol{\kappa}}_a\right)$ is the scattered electric field from the scatterer. It depends on the induced surface currents as

$$\mathbf{E}_s\left(r\hat{\boldsymbol{\kappa}}, \hat{\boldsymbol{\kappa}}_a\right) = \imath\kappa Z \frac{e^{\imath\kappa r}}{4\pi r} \hat{\boldsymbol{\kappa}} \times \left(\int_\Gamma e^{-\imath\boldsymbol{\kappa}\cdot\mathbf{x}} \mathbf{J}\left(\mathbf{x}, \hat{\boldsymbol{\kappa}}_a\right) d\Gamma\left(\mathbf{x}\right) \times \hat{\boldsymbol{\kappa}} \right) \tag{5}$$

where $\mathbf{J}\left(\mathbf{x}, \hat{\boldsymbol{\kappa}}_a\right)$ is the solution to (1) and (2). The case when $\hat{\boldsymbol{\kappa}} \neq \hat{\boldsymbol{\kappa}}_a$ is referred to as the bistatic RCS (BRCS). The special case $\sigma\left(\hat{\boldsymbol{\kappa}}_a, \hat{\boldsymbol{\kappa}}_a\right)$ is referred to as the monostatic RCS (MRCS). The RCS is often computed in decibels (dB) by the relation $\sigma_{\mathrm{dB}}\left(\hat{\boldsymbol{\kappa}}, \hat{\boldsymbol{\kappa}}_a\right) = 10\log_{10}\sigma\left(\hat{\boldsymbol{\kappa}}, \hat{\boldsymbol{\kappa}}_a\right)$.

In order to compute the MRCS for all possible $\hat{\boldsymbol{\kappa}}_a$ the number of samples M needed depend on the wavenumber as $M = \mathcal{O}\left(\left(\kappa D\right)^2\right)$. When one of the azimuthal or polar angle is fixed, the number of samples needed is $M = \mathcal{O}\left(\kappa D\right)$. Since (4) must be computed for each plane wave, each sample requires that a linear system of equations of the form $\mathbf{Ax} = \mathbf{b}$ is solved.

3 The multilevel Fast Multipole algorithm

A matrix element A_{kl} in the RWG discretized CFIE (1) can be approximated by the expression

$$A_{kl} \approx \sum_{j=1}^{K} \mathbf{R}_k\left(\hat{\boldsymbol{\kappa}}_j\right) \cdot \mathcal{T}_j^L\left(\kappa, \mathbf{X}_m - \mathbf{Y}_m\right) \mathbf{F}_l\left(\hat{\boldsymbol{\kappa}}_j\right) \tag{6}$$

where $\mathbf{F}_l\left(\hat{\boldsymbol{\kappa}}\right)$ is called the far field pattern and $\mathbf{R}_k\left(\hat{\boldsymbol{\kappa}}\right)$ the receiving pattern defined by

$$\begin{cases} \mathbf{F}_l\left(\hat{\boldsymbol{\kappa}}\right) = \hat{\boldsymbol{\kappa}} \times \int_\Gamma e^{\imath\boldsymbol{\kappa}\cdot\left(\mathbf{Y}_m - \mathbf{y}\right)} \mathbf{j}_l d\Gamma\left(\mathbf{y}\right) \times \hat{\boldsymbol{\kappa}} \\ \mathbf{R}_k\left(\hat{\boldsymbol{\kappa}}\right) = \alpha\hat{\boldsymbol{\kappa}} \times \displaystyle\int_\Gamma e^{\imath\boldsymbol{\kappa}\cdot\left(\mathbf{x} - \mathbf{X}_m\right)} \mathbf{j}_k' d\Gamma\left(\mathbf{x}\right) \times \hat{\boldsymbol{\kappa}} \\ \qquad + \left(1 - \alpha\right) \displaystyle\int_\Gamma e^{\imath\boldsymbol{\kappa}\cdot\left(\mathbf{x} - \mathbf{X}_m\right)} \mathbf{j}_k' \times \hat{\mathbf{n}}\left(\mathbf{x}\right) d\Gamma\left(\mathbf{x}\right) \times \hat{\boldsymbol{\kappa}} \end{cases} \tag{7}$$

and $\mathcal{T}_j^L\left(\kappa, \mathbf{X}_m - \mathbf{Y}_m\right)$ is the translation operator defined by

$$\mathcal{T}_k^L\left(\kappa, \mathbf{X}\right) = \frac{\imath\kappa}{16\pi^2} w_k \sum_{l=0}^{L} \imath^l \left(2l + 1\right) h_l^{(1)}\left(\kappa X\right) P_l\left(\hat{\boldsymbol{\kappa}}_k \cdot \hat{\mathbf{X}}\right) \tag{8}$$

The multilevel Fast Multipole algorithm (MLFMA) [1] uses (6) to compute a matrix-vector multiplication in $\mathcal{O}\left(N\log N\right)$ arithmetic operations. The key idea is the use of an interpolation step, called aggregation, a translation step and a transposed interpolation step, called disaggregation. In the aggregation step contributions from several far field patterns are added together, the translation step applies the translation operator to the aggregated far field patterns and the disaggregation step computes the contributions from the far field patterns to the receiving pattern. The reader is referred to [1, 4, 5] for more details. Note that the aggregation step is similar in structure to the method in section 5 and that the operations are performed in reversed order in the disaggregation step.

An iterative method like the GMRES method [6] with MLFMA for the matrix-vector multiplications is used to solve the system of equations. The major cost in the iterations is the multiplication of an arbitrary vector by the matrix. Thus, the total number of arithmetic operations is $\mathcal{O}\left(KN\log N\right)=\mathcal{O}\left(K\left(\kappa D\right)^{2}\log\left(\kappa D\right)\right)$, where K is the number of iterations in the iterative method. For M right hand sides the total number of arithmetic operations is $\mathcal{O}\left(KMN\log N\right)$, which implies that the problem of computing the MRCS requires $\mathcal{O}\left(K\left(\kappa D\right)^{4}\log\left(\kappa D\right)\right)$ arithmetic operations.

4 Minimal residual interpolation

In order to reduce the cost of computing the MRCS, a method for solving multiple right hand sides called Minimal Residual Interpolation (MRI) was developed in [7]. MRI requires that $M=\mathcal{O}\left(\left(\kappa D\right)^{2}\right)$ right hand sides are solved in order to compute the MRCS [8]. Thus, the complexity is not reduced in MRI, but the constant factor is usually reduced.

In MRI, the systems of linear equations that should be solved are

$$\mathbf{A}\mathbf{x}_{i}=\mathbf{b}_{i},\ i=1\ldots M,\ \mathbf{A}\in\mathbb{C}^{N\times N},\ \mathbf{x}_{i},\mathbf{b}_{i}\in\mathbb{C}^{N}. \tag{9}$$

The residual is defined as $\mathbf{r}_{i}=\mathbf{b}_{i}-\mathbf{A}\mathbf{x}_{i}$. An iterative method can solve the equations such that $\|\mathbf{r}_{i}\|\leq\varepsilon$ for some ε.

Assume that the solutions to $m<M$ right hand sides are known and are linearly independent. Let $\mathbf{s}_{i}$, $\mathbf{S}_{m}$, and $\mathbf{X}_{m}$ be defined by

$$\mathbf{A}\mathbf{x}_{i}=\mathbf{b}_{i}-\mathbf{r}_{i}\equiv\mathbf{s}_{i},\ i=1\ldots m,\ \mathbf{X}_{m}=\left[\mathbf{x}_{1}\,\mathbf{x}_{2}\,\ldots\,\mathbf{x}_{m}\right],\ \mathbf{S}_{m}=\left[\mathbf{s}_{1}\,\mathbf{s}_{2}\,\ldots\,\mathbf{s}_{m}\right]. \tag{10}$$

Let the **QR**-decomposition of $\mathbf{S}_{m}$ be given by $\mathbf{A}\mathbf{X}_{m}=\mathbf{S}_{m}=\mathbf{Q}_{\mathbf{S}_{m}}\mathbf{R}_{\mathbf{S}_{m}}$. Then, one can compute a minimal residual solution such that [7]

$$\mathbf{x}_{m+1}^{(0)}=\mathbf{X}_{m}\mathbf{R}_{\mathbf{S}_{m}}^{-1}\mathbf{Q}_{\mathbf{S}_{m}}^{H}\mathbf{b}_{m+1},\ \mathbf{r}_{m+1}^{(0)}=(\mathbf{I}-\mathbf{Q}_{\mathbf{S}_{m}}\mathbf{Q}_{\mathbf{S}_{m}}^{H})\mathbf{b}_{m+1}. \tag{11}$$

If $\|\mathbf{r}_{m+1}^{(0)}\|>\varepsilon$ the solution $\mathbf{x}_{m+1}^{(0)}$ can be used in an iterative method as initial guess.

5 A fast asymptotic method

In [9] a fast method for computing the MRCS for the Physical Optics approximation (PO) is given. Instead of computing the scattered electric field for the entire scatterer, the scattered field is computed for subdomains of the scattering object. If PO is used the surface current depends on the illuminating plane wave as

$$\mathbf{J}_{\mathrm{PO}}\left(\mathbf{x},\hat{\boldsymbol{\kappa}}_{a}\right)=\tilde{\mathbf{J}}_{\mathrm{PO}}\left(\mathbf{x},\hat{\boldsymbol{\kappa}}_{a}\right)e^{-\imath\boldsymbol{\kappa}_{a}\cdot\mathbf{x}} \tag{12}$$

where $\tilde{\mathbf{J}}_{\mathrm{PO}}\left(\mathbf{x},\hat{\boldsymbol{\kappa}}_{a}\right)$ is a slowly varying function of $\hat{\boldsymbol{\kappa}}_{a}$. Thus, the scattered field in the monostatic direction is given by

$$\mathbf{E}_{s}\left(r\hat{\boldsymbol{\kappa}}_{a},\hat{\boldsymbol{\kappa}}_{a}\right)=\imath\kappa Z\frac{e^{\imath\kappa r}}{4\pi r}\hat{\boldsymbol{\kappa}}_{a}\times\left(\int_{\Gamma}e^{-2\imath\boldsymbol{\kappa}_{a}\cdot\mathbf{x}}\tilde{\mathbf{J}}_{\mathrm{PO}}\left(\mathbf{x},\hat{\boldsymbol{\kappa}}_{a}\right)d\Gamma\left(\mathbf{x}\right)\times\hat{\boldsymbol{\kappa}}_{a}\right) \tag{13}$$

The computed field of each subdomain can be smoothed by multiplying with the factor $e^{2\imath\boldsymbol{\kappa}_{a}\cdot\mathbf{X}_{m}}$, which approximately removes the phase dependence. Here, $\mathbf{X}_{m}$ is the midpoint of the subdomain. This makes it possible to compute the fields approximately by interpolation. The number of samples needed in each subdomain is now $\mathcal{O}\left(\left(\kappa d_{1}\right)^{2}\right)$ where d_{1}

is the diameter of the smallest sphere enclosing the subdomain. The total field is computed in a way that is similar to the interpolation step in MLFMA. First the samples are interpolated onto a denser set of samples. Then the phase factor is restored and the fields of adjacent subdomains are aggregated. The number of samples required after the new phase factor is removed is $\mathcal{O}\left((\kappa d_2)^2\right)$ where d_2 is the diameter of the smallest sphere enclosing the aggregated subdomains. This process is continued until the field of the entire scatterer is computed. The number of arithmetic operations required in the aggregation step is $\mathcal{O}\left((\kappa D)^2 \log(\kappa D)\right)$.

In this paper the scheme is applied to MoM instead. This is motivated by the fact that the inverse of the impedance matrix usually have large elements in positions corresponding to basis functions that are geometrically close. Since the surface current of one basis function is $x_j = \sum_{i=1}^{N} A_{ji}^{-1} b_i$ it can be expected that the phase variation is dominated by the self interaction

$$A_{jj}^{-1} b_j = A_{jj}^{-1} \sum_{k=1}^{q} w_{jk} \mathbf{K}(\hat{\boldsymbol{\kappa}}_a) \cdot \mathbf{j}_j(\mathbf{x}_{jk}) \exp(-\imath\kappa\hat{\boldsymbol{\kappa}}_a \cdot \mathbf{x}_{jk}) \tag{14}$$

Thus, when the scattered electric field is multiplied by the phase factor $e^{2\imath\boldsymbol{\kappa}_a \cdot \mathbf{X}_m}$ it is expected to become smoothly varying as in the case of PO. This allows for interpolation to be used and gives an approximate method for computing the MRCS. The approximation comes from the fact that the field does not need to be smooth, i. e. there could be reflections from basis functions far away in some directions that lead to large elements in the impedance matrix. A more detailed analysis will be given in a forthcoming paper. Since only $\mathcal{O}\left((\kappa d_1)^2\right)$ equations have to be solved the total number of arithmetic operations of the method is $\mathcal{O}\left(K(\kappa d_1)^2 (\kappa D)^2 \log(\kappa D)\right) + \mathcal{O}\left((\kappa D)^2 \log(\kappa D)\right)$, which gives a significant reduction depending on how small d_1 is. Note that d_1 will depend on the accuracy required.

6 Numerical experiments

In this section some numerical results are presented in order to verify the methods in section 4 and section 5.

In the first experiment a generic aircraft model called RUND depicted in figure 1 is used. The model is approximately $0.8 \times 0.8 \times 0.25$ m. It is discretized with 16218 edges at the frequency 1.5 GHz. The MRCS is computed in the plane spanned by the wings for all angles with MRI and compared to the fast approximate method using 4, 8, 16, 32 and 64 samples respectively. Figure 2 shows the result for the different number of samples. In figure 3 the relative error of the computed MRCS defined as

$$\text{Error} = \frac{\sqrt{\sum_{\tilde{m}=1}^{\tilde{M}} \left|\sigma_{dB}^{\text{MRI}} - \sigma_{dB}^{s}\right|^2}}{\sqrt{\sum_{\tilde{m}=1}^{\tilde{M}} \left|\sigma_{dB}^{\text{MRI}}\right|^2}} \tag{15}$$

is plotted. Here, σ_{dB}^{MRI} is the MRCS obtained in MRI in dB, σ_{dB}^{s} is the MRCS obtained with the fast method and s samples in dB and $\tilde{M}$ is the number of of total values computed. The speedup compared to solving all $\tilde{M} \approx 225$ right hand sides is also plotted in figure 3. In this case MRI needed 55 solutions before the complete MRCS could be computed. The speedup for all cases is only modest, approximately 5–15 times, since a large part of the solution time comes from the setup of the problem.

The second experiment also uses the model aircraft RUND. This time it is modelled with 64959 edges. The MRCS at 6 GHz is shown in figure 4. Again the solution obtained by MRI is compared to the solution obtained by the fast approximate method using 4, 8, 16, 32, 64 and 128 samples respectively. In this case measurements from FOI, the Swedish Defence Research Agency, are plotted for comparison. A comparison between different solvers was done in [7], where it was found that the results obtained by MRI are satisfactory. In figure 5 the relative error defined in (15) is plotted. The speedup compared to solving all $\tilde{M} \approx 900$ right hand sides is also plotted in figure 5. In this case MRI needed 139 solutions before the complete MRCS could be computed. The solution time is dominated by the iterative solver so the speedup of the fast method with four samples is an order of magnitude greater than in the previous case.

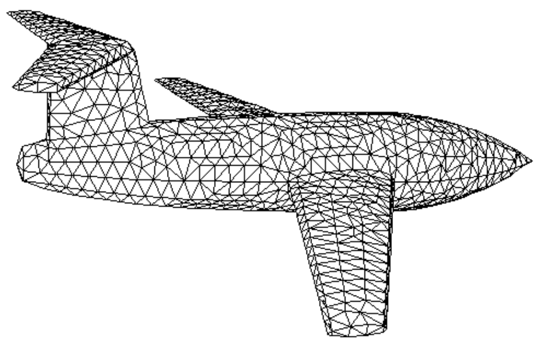

Figure 1: *A triangulation of a generic aircraft model called RUND.*

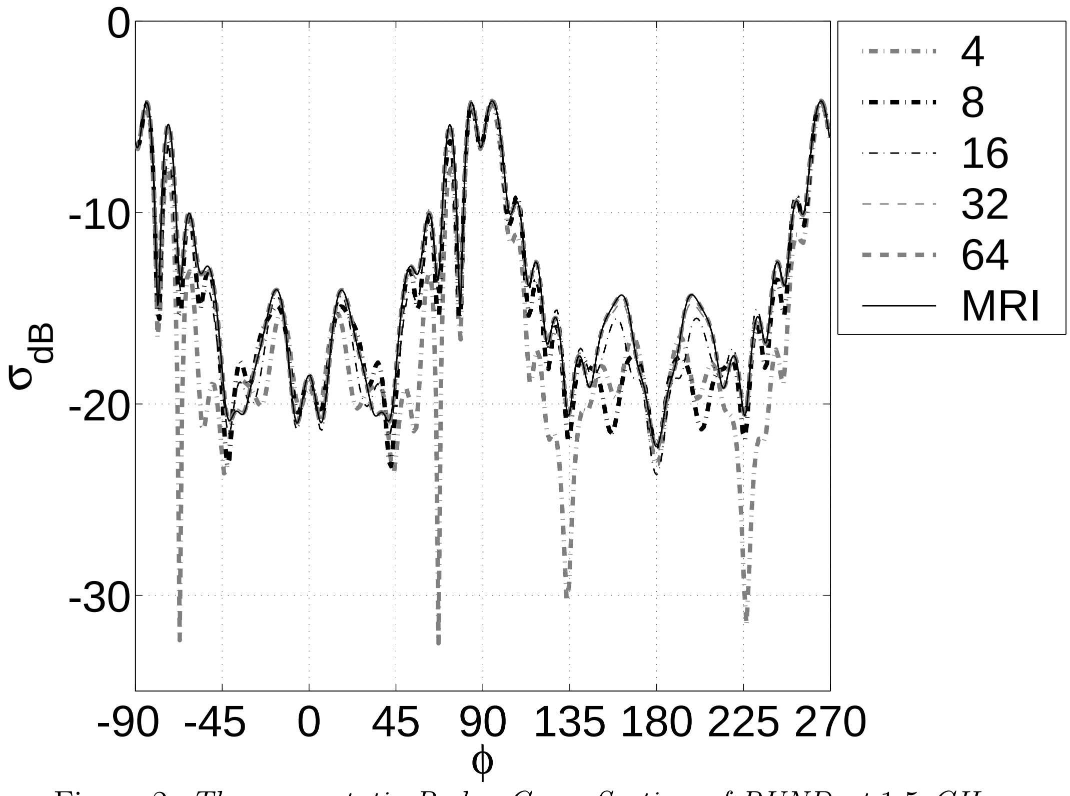

Figure 2: *The monostatic Radar Cross Section of RUND at* 1.5 *GHz.*

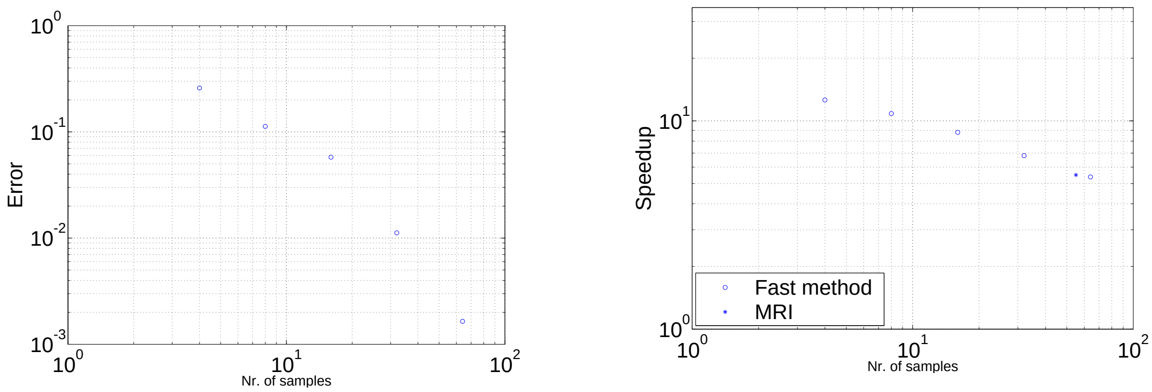

Figure 3: The relative error (left) and the speedup compared to the standard method (right) at 1.5 GHz.

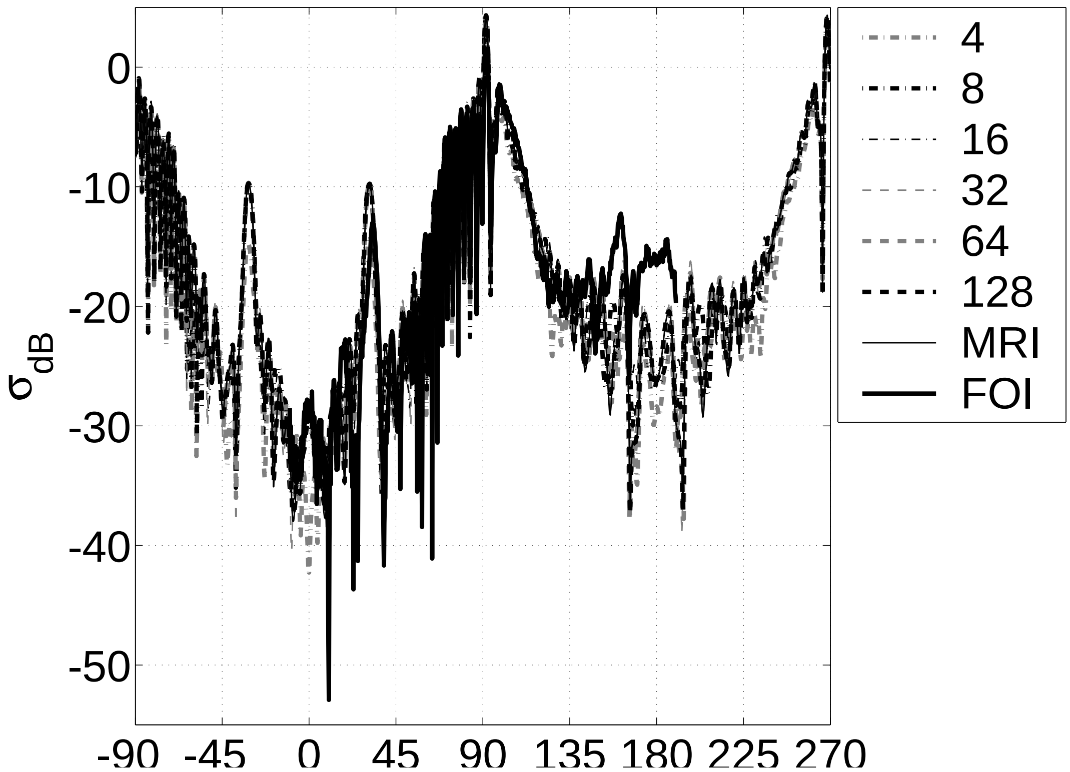

Figure 4: *The monostatic Radar Cross Section of RUND at* 6.0 *GHz.*

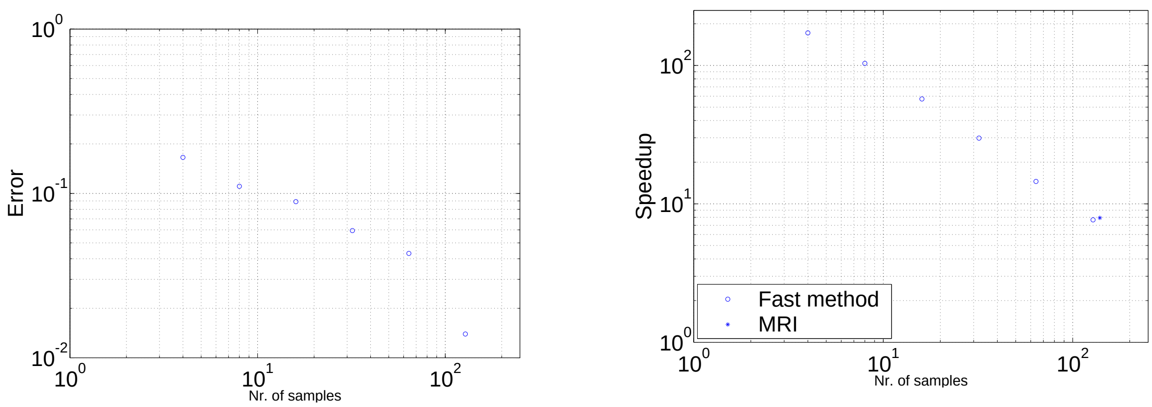

Figure 5: The relative error (left) and the speedup compared to the standard method (right) at 6.0 GHz.

7 Conclusions

For electrically large objects the standard methods of computing the MRCS become too expensive. In this case the new fast approximate method shows great promise. As the numerical experiments demonstrate it only requires a few samples of the monostatic field in order to compute a fairly accurate MRCS. This could be especially important when one wants to minimize the MRCS with an optimizing algorithm as in [10].

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