HIGH ORDER SUMMATION BY PARTS OPERATOR BASED ON A DRP SCHEME APPLIED TO 2D AEROACOUSTICS

Stefan Johansson

Division of Scientific Computing
Department of Information Technology
Uppsala University
P.O. Box 337
SE 75105 Uppsala
Sweden

Abstract

A strictly stable high order finite difference method based on Tam and Webb's dispersion relation preserving scheme in the interior has been verified for a 2D aeroacoustic problem. Results show that the method gives lower dispersion error than a similar method derived by Strand [11], which is based on standard sixth order difference approximation in the interior, when boundary effects are not important.

1 Introduction

Numerical simulation of wave propagation problems has been studied using computers for the last 50 years and is still actively studied. For the last 10 years computational aeroacoustics (CAA) has emerged as a subfield. In CAA sound generation and sound propagation are modeled by the Navier-Stokes or Euler equations. High accuracy is required due to long time integration, leading to the need of highly accurate numerical methods.

Tam and Webb introduced Dispersion Preserving Relation (DRP) schemes [13] that give a better approximation of the wave number than standard finite difference schemes, thus giving a smaller dispersion error.

One of the more difficult problems for finite difference approximation is how to apply numerical boundary conditions in a stable way. One approach proposed by Kreiss and Scherer called summation by parts (SBP) [7] is to devise methods that give the same energy estimate as for the continuous problem. Thus, the SBP methods are strictly stable by construction and one does not have to use the algebraically difficult GKS theory [5] to prove stability.

High accuracy and strict stability have been combined in [6], where theoretical results and experimental verification were presented for a model problem in 1D. The main contribution of the present paper is the application of the resulting method to aeroacoustic wave propagation in 2D

[12]. The propagation of acoustic, entropy and vorticity waves is computed using the linearized 2D Euler equations. In addition we discuss the application of numerical dissipation where the damping does not affect the better resolution of the DRP method compared to a standard finite difference method.

The outline of the paper is as follows. In Section 2, we introduce the 2D linearized Euler equations to model sound propagation. In Section 3, the theory for a scalar test case in 1D is recapitulated, using the energy method to give an energy estimate for the continuous problem and explaining summation by parts which gives the same estimate in a discrete norm, cf. [4]. A discussion on how to apply artificial dissipation to Dispersion Relation Preserving schemes is given as well. In Section 4 numerical results that verify the numerical method are presented for a problem in computational aeroacoustics.

2 2D Linearized Euler Equations

The linearized Euler equations in conservative form [1] are used as a model of sound propagation. Denoting density ρ, velocity in x-direction u, in y-direction v, specific total energy E, specific total entalphy H and pressure p. The equations are formulated in the variables ρ', $(\rho u)'$, $(\rho v)'$ and $(\rho E)'$ that are perturbations of a reference state ρ_0, $\rho_0 u_0$, $\rho_0 v_0$, $\rho_0 E_0$.

$$\frac{\partial}{\partial t}\begin{pmatrix} \rho' \\ (\rho u)' \\ (\rho v)' \\ (\rho E)' \end{pmatrix} + \frac{\partial}{\partial x}\begin{pmatrix} (\rho u)' \\ \rho_0 u_0 u' + (\rho u)' u_0 + p' \\ \rho_0 v_0 u' + (\rho v)' u_0 \\ \rho_0 H_0 u' + (\rho H)' u_0 \end{pmatrix} + \frac{\partial}{\partial y}\begin{pmatrix} (\rho v)' \\ \rho_0 u_0 v' + (\rho u)' v_0 \\ \rho_0 v_0 v' + (\rho v)' v_0 + p' \\ \rho_0 H_0 v' + (\rho H)' v_0 \end{pmatrix} = 0 \quad (1)$$

The acoustic quantities are defined by $\rho' = \rho - \rho_0$, $(\rho u)' = \rho u - \rho_0 u_0$, $(\rho v)' = \rho v - \rho_0 v_0$ and $(\rho E)' = \rho E - \rho_0 E_0$. With $(\rho H)' = (\rho E)' + p'$.

The variables u', v' and p' can be calculated using

$$\mathbf{u}' = \frac{(\rho \mathbf{u})' - \rho' \mathbf{u_0}}{\rho_0} \quad (2)$$

$$p' = (\gamma - 1)\left((\rho E)' - \frac{1}{2}\left(2\,(\rho \mathbf{u})' \cdot \mathbf{u}_0 - \rho' \mathbf{u}_0 \cdot \mathbf{u}_0\right)\right) \quad (3)$$

with $\mathbf{u} = (u, v)^T$

3 Theory

To make the paper self-contained the theory for a scalar test problem in 1D is recapitulated below.

3.1 Energy Method

A sufficient condition for well-posedness for an initial boundary value problem is to give a bound on the solution in terms of the initial value and boundary data, an energy estimate.

For example, introduce the L_2 scalar product and norm

$$(u,v) = \int_0^1 uv\,dx\,, \quad ||u||^2 = (u,u). \tag{4}$$

We have for the simplest hyperbolic equation, sometimes called the Kreiss equation

$$\left\{ \begin{array}{lll} u_t & = u_x & ,\, x \in [0,1],\, t \geq 0 \\ u(x,0) & = f(x) & ,\, x \in [0,1] \\ u(1,t) & = g(t) & t \geq 0 \end{array} \right. \tag{5}$$

the following energy estimate

$$\frac{d||u||^2}{dt} = (u,u_t)+(u_t,u) = (u_x,u)+(u,u_x) = [u^2]_0^1 = g^2(t)-u^2(0,t). \tag{6}$$

That leads to an estimate on the norm of u in terms of the initial and boundary data [4].

3.2 Summation by parts

In the semi-discrete case we want to find an operator Q approximating d/dx and a discrete norm, $\|u\|^2 = (u,u)_H = u^T H u$, such that the semi-discretization of (5) gives the same estimate as the continuous case, where u and v are grid functions on $[0,1]$ with step size h and H is a diagonal positive definite matrix. The energy method on $u_t = Qu$ gives

$$\frac{d||u||_H^2}{dt} = (u,u_t)_H+(u_t,u)_H = (u,Qu)_H+(Qu,u)_H = u^T(HQ+(HQ)^T)u \tag{7}$$

The same estimate as in the continuous is given if $(u,Qu)_H + (Qu,u)_H = [u^2]_0^1$, or equivalent $HQ + (HQ)^T = diag(-1,0,\ldots,0,1)$. This property is called summation by parts (SBP).

Operators with this property were proposed by Kreiss and Scherer [7] and extended to high order by Strand [11]. In [6] SBP operators for difference schemes of Dispersion Relation Preserving (DRP) schemes described in Section 3.3 type were developed.

3.3 Dispersion Relation Preserving Schemes

If the formal accuracy of a finite difference method is lowered by two orders as proposed by Tam and Webb [13] a free parameter is given that can be chosen such that the wave number is approximated better.

That free parameter ϕ is chosen such that error $E(\phi)$ in the approximation of the wave number is minimized

$$E(\phi) = \int_{-\pi/2}^{\pi/2} |hk - h\tilde{k}(\phi)|^2 d(hk) \tag{8}$$

where k and $\tilde{k}$ are the exact and the approximate wave numbers, respectively.

A comparison between standard centered finite difference methods and the DRP schemes is given in Figure (1). The straight line shows the exact non-dimensional wave number hk. The dotted, dotted-dashed and dashed lines show the approximate non-dimensional wave numbers $h\tilde{k}$ for different schemes. As seen the fourth order DRP scheme derived from the sixth order standard centered difference method gives a good approximation for the non-dimensional wave numbers up to about $\pi/2$ whereas the standard sixth order method gives a good approximation up to about 1.2. The details of the present fourth order DRP scheme and 6th, 8th order DRP schemes can be found in [6].

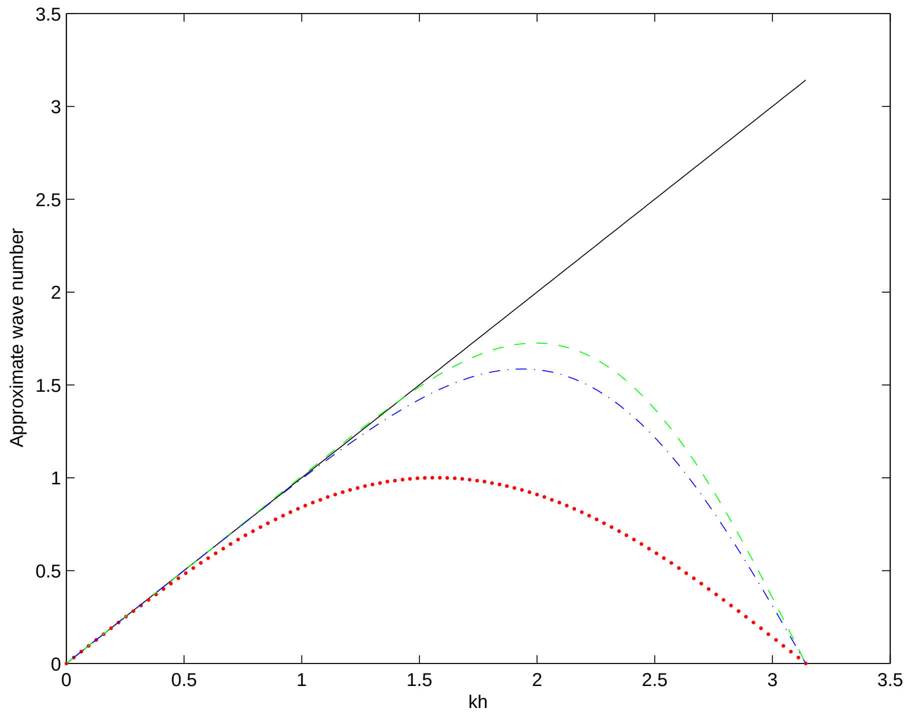


Figure 1: Approximate non-dimensional wave number vs exact for 2nd order standard centered difference method (SC2) (dotted), SC6 (dot-dashed), 4th order DRP scheme (dashed).

3.4 Numerical Dissipation

To damp high frequency modes in the numerical solution that are underresolved artificial dissipation is often applied. The standard explicit type is $(-1)^{p+1}\beta(h^2D_+D_-)^p$, where $\beta > 0$ and $p \geq 1$.

But since DRP schemes are derived so that they give a good approximation for wave numbers less that $\pi/2$, very high order dissipation has to be used not to destroy that property. Figure 2 shows the Fourier transform of low pass filters for $p = 1 \dots 4$ which demonstrate that the damping is significant for wave numbers $< \pi/2$. Alternatives are to optimize the stencil in the artificial dissipation analogously to to use the idea proposed for filters by [2] or using an implicit artificial dissipation similar to the idea used for filters proposed by [14].

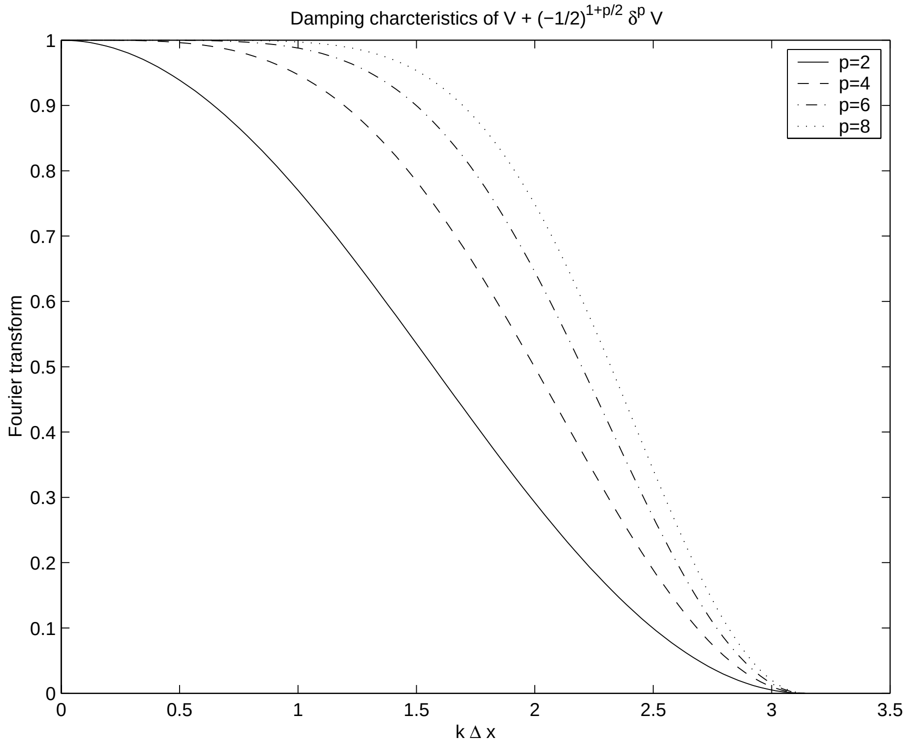


Figure 2: Fourier transform of low pass filters of orders 2, 4, 6, 8.

Assume that S is a dissipation operator, the energy method on the Kreiss equation in semi discrete form with dissipation added $u_t = Qu + H^{-1}Su$ gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_H = (u_t, u)_H + (u, u_t)_H = (Qu + Su, u)_H + (u, Qu + Su)_H = u^T(HQ + (HQ)^T + S + S^T)u \quad (9)$$

If the interior stencil of S is $(-1)^{p+1}\beta(h^2 D_+ D_-)^p$ a boundary closure that guarantees that $S + S^T leq 0$ can be found [3] [8].

Using an implicit dissipation $(-1)^{p+1}\beta P^{-1}(h^2 D_+ D_-)^p$, where P is a symmetric tridiagonal matrix with ones on the diagonal does not give an energy estimate with the standard boundary closure for $p = 1, 2, 3$, which was verified numerically. In general nothing can be said about the definiteness of $P^{-1}S$.

4 Application to a 2D aeroacoustic problem

The high order SBP operator based on the Tam and Webb DRP scheme was implemented in a FORTRAN program [10] and tested for an aeroacoustic test problem. The results were compared with those obtained with a SBP operator based on a standard central finite difference stencil.

The 2D linearized Euler equations described in Section 2 were solved using those SBP operators to discretize x and y derivatives and using the standard fourth order Runge-Kutta method to discretize the time derivative.

A benchmark problem was chosen from [12], choosing $u_0 = 0.5$, $v_0 = 0$ with the following initial condition

$$
\begin{aligned}
p'(x,y,0) &= e^{-log(2)\frac{x^2+y^2}{9}} && (10)\\
\rho'(x,y,0) &= e^{-log(2)\frac{x^2+y^2}{9}} + 0.1e^{-log(2)\frac{(x-67)^2+y^2}{25}} && (11)\\
u'(x,y,0) &= 0.04ye^{-log(2)\frac{(x-67)^2+y^2}{25}} && (12)\\
v'(x,y,0) &= -0.04(x-67)e^{-log(2)\frac{(x-67)^2+y^2}{25}} && (13)
\end{aligned}
$$

to the linearized Euler equations. There exists an analytical solution for verification of the numerical results [12].

Numerical solution obtained with the DRPSBP scheme for ρ', the acoustic density, with the two resolutions at time instant $T = 40$ can be found in Figure 3, near the center the acoustic pulse is seen, propagating with the speed of sound in all directions transported with the mean velocity. Near the right boundary the entropy and vorticity wave is seen propagating with the mean velocity.

The time instances were deliberately chosen such that the solution had not reached the boundary, since the non reflecting boundary condition implemented is not of high order and the purpose of this paper is to evaluate the new SBP method. Results for the first order Enquist-Majda boundary condition can be found in [10]

Figure 4 to Figure 7 show the numerical solution using SBP-3-6 (SBP operator based on the sixth order standard finite difference scheme with third order near the boundaries) compared to DRPSBP-2-4 (SBP operator based on a fourth order DRP scheme derived from a sixth order standard finite difference scheme with second order near the boundaries) of ρ' along the x-axis at time $T = 40$, compared to the exact solution for two different grid sizes, and with and without filtering. The filter is applied in each time step and of standard type $(-1)^{p+1}2^{-2p}(h^2D_+D_-)^p$, in this case $p = 3$ is chosen [9], with the boundary treatment according to [3].

Figure 4 shows that the method based on a DRP scheme has lower dispersion error and has more accurate peak values, and smaller amplitudes on the wiggles but more noticeable overshoots. The entropy and vorticity wave at $x = 87$ is better resolved than the acoustic waves at $x = -20$ and $x = 60$. Table 1 and Table 5 show that DRPSBP-2-4 has about a 60 % lower L_2-error along the x-axis for the lower resolution.

When refining the grid the differences between the methods are not seen in Figure 5. Table 2 and Table 6 show that the difference in L_2-error along the x-axis is about 10 %, but to advantage of the SBP method.

On the other hand when applying a standard type filter to damp the high frequency oscillations, there is less difference between the methods as shown in Figure 6 and Figure 7. Table 3, Table 4, Table 7 and Table 8 stating the L_2-error along the x-axis for the two grid sizes and the two time instances chosen, demonstrate that the L_2-error along the x-axis is larger than without filtering. The reason is that the filter imposes significant damping for wave numbers $hk < \pi/2$ cf. Figure 2, which affects the higher resolution of the DRP based SBP method.

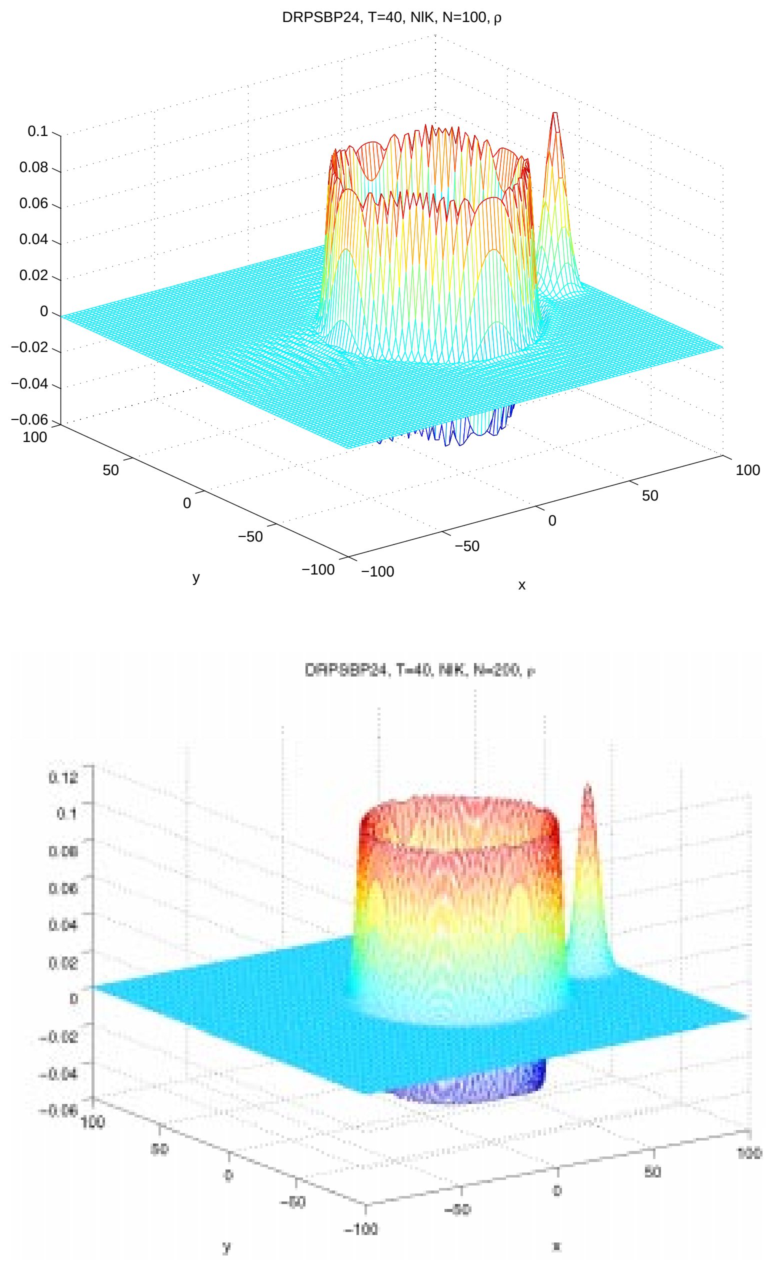
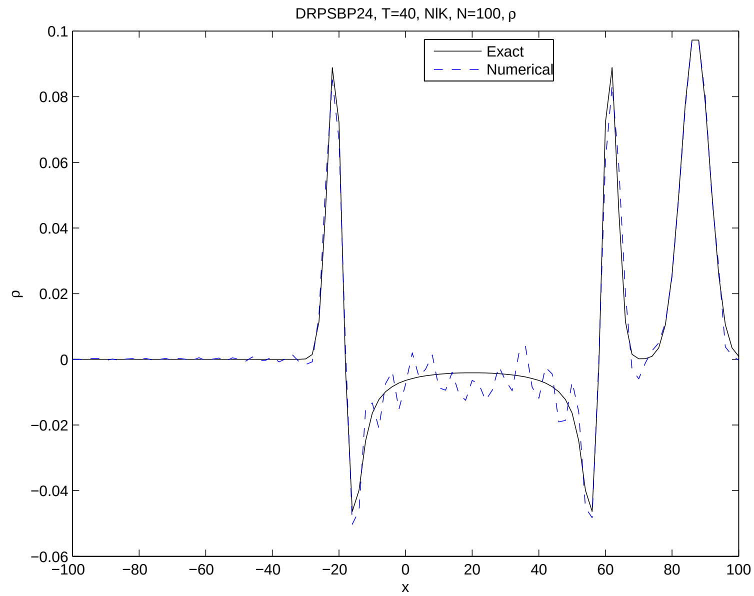


Figure 3: Surface plot of acoustic density at time 40, without filtering, top: N=101, bottom: N=201.



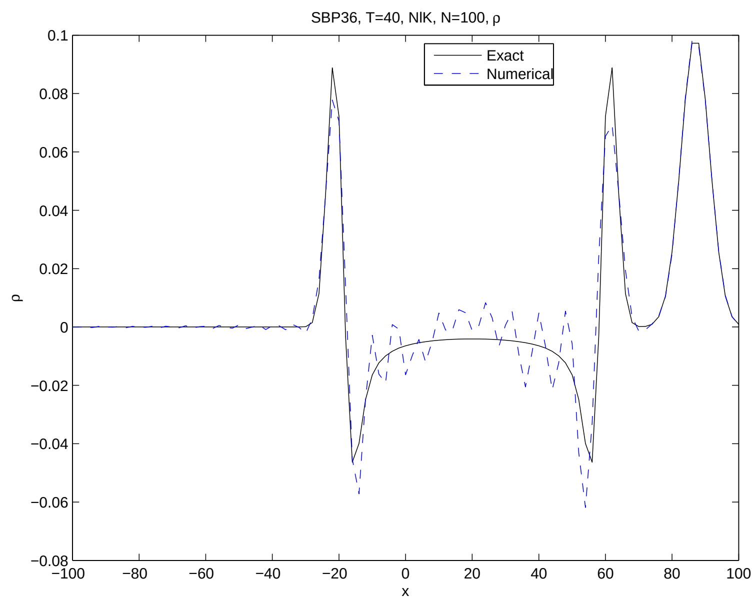


Figure 4: Numerical solution of ρ' at time 40, $N = 101$, without filtering, top: DRPSBP-2-4, bottom: SBP-3-6.

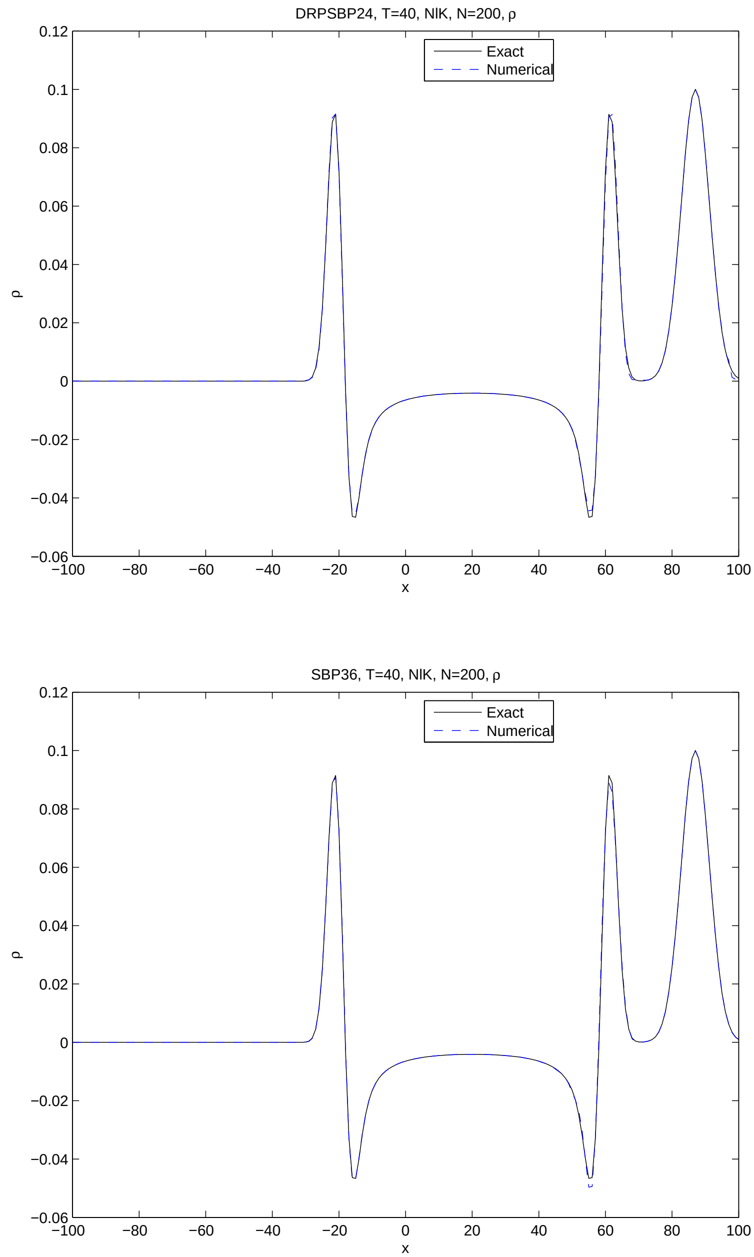
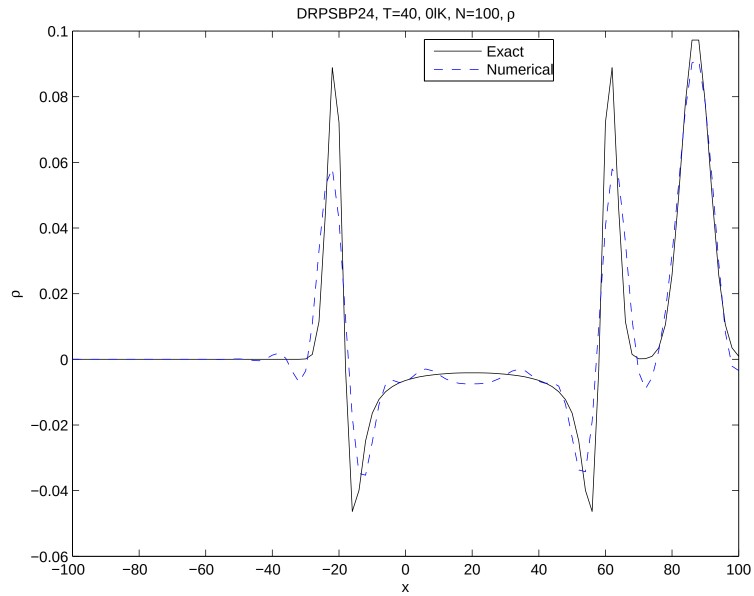


Figure 5: Numerical solution of ρ' at time 40, $N = 201$, without filtering, top: DRPSBP-2-4, bottom: SBP-3-6.



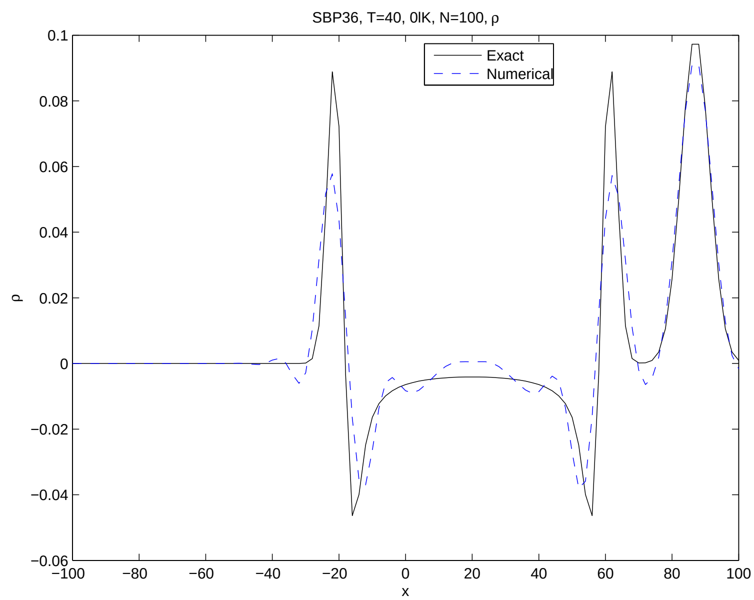
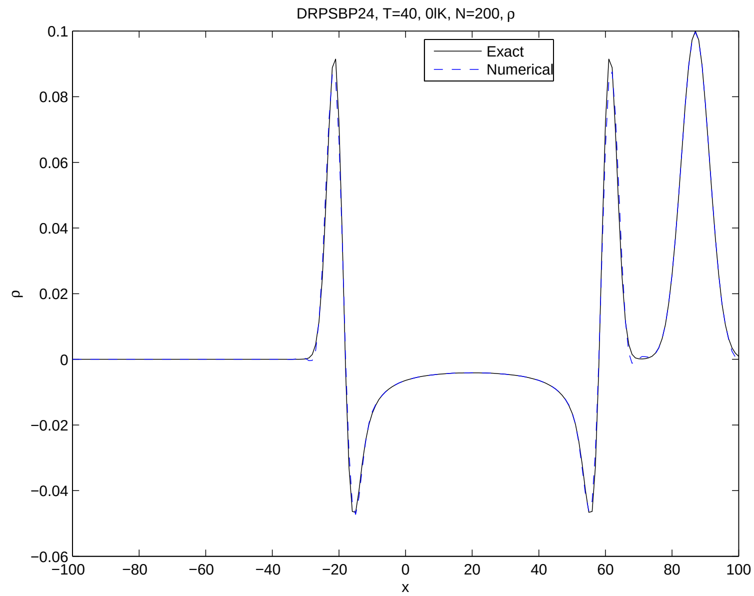


Figure 6: Numerical solution of ρ' at time 40, $N = 101$, sixth order filter, top: DRPSBP-2-4, bottom: SBP-3-6.



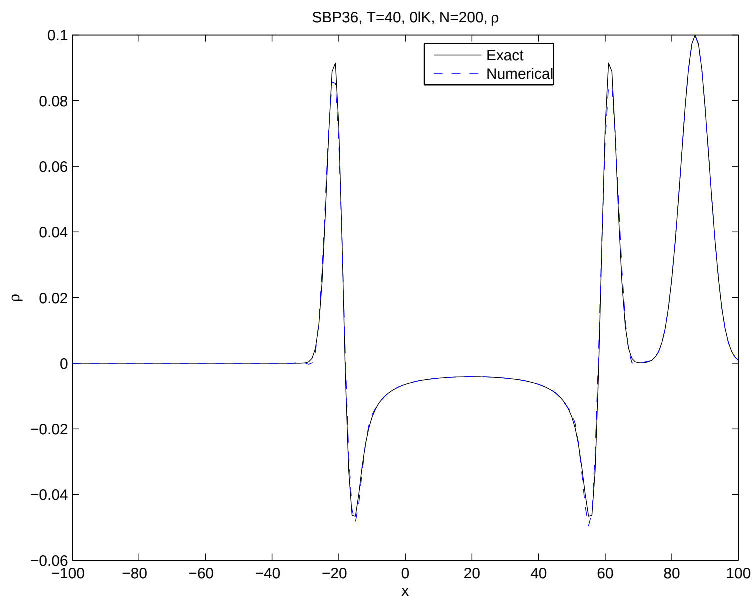


Figure 7: Numerical solution of ρ' at time 40, $N = 201$, sixth order filter, top: DRPSBP-2-4, bottom: SBP-3-6.

	DRPSBP-2-4	SBP-3-6
L2-error along the x-axis in ρ	0.00431254	0.00788948
L2-error along the x-axis in p	0.00431147	0.00788902
L2-error along the x-axis in u	0.00428112	0.00649726

Table 1: u_0=0.5, T=20, N = 101, no filter

	DRPSBP-2-4	SBP-3-6
L2-error along the x-axis in ρ	0.00049776	0.00047037
L2-error along the x-axis in p	0.00049762	0.00047037
L2-error along the x-axis in u	0.00049358	0.00046720

Table 2: u_0=0.5, T=20, N = 201, no filter

	DRPSBP-2-4	SBP-3-6
L2-error along the x-axis in ρ	0.01022241	0.01020396
L2-error along the x-axis in p	0.01018905	0.01017082
L2-error along the x-axis in u	0.00991698	0.00986553

Table 3: u_0=0.5, T=20, N = 101, sixth order filter

	DRPSBP-2-4	SBP-3-6
L2-error along the x-axis in ρ	0.00118599	0.00114419
L2-error along the x-axis in p	0.00118562	0.00114392
L2-error along the x-axis in u	0.00117612	0.00113471

Table 4: u_0=0.5, T=20, N = 201, sixth order filter

	DRPSBP-2-4	SBP-3-6
L2-error along the x-axis in ρ	0.00444214	0.00723430
L2-error along the x-axis in p	0.00436382	0.00723236
L2-error along the x-axis in u	0.00372037	0.00676451

Table 5: M=0.5, T=40, N = 101, no filter

	DRPSBP-2-4	SBP-3-6
L2-error along the x-axis in ρ	0.00074894	0.00063326
L2-error along the x-axis in p	0.00071616	0.00063313
L2-error along the x-axis in u	0.00071466	0.00063204

Table 6: u_0=0.5, T=40, N = 201, no filter

	DRPSBP-2-4	SBP-3-6
L2-error along the x-axis in ρ	0.00903172	0.00910053
L2-error along the x-axis in p	0.00885448	0.00895898
L2-error along the x-axis in u	0.00879172	0.00881355

Table 7: u_0=0.5, T=40, N = 101, sixth order filter

	DRPSBP-2-4	SBP-3-6
L2-error along the x-axis in ρ	0.00148481	0.00138574
L2-error along the x-axis in p	0.00147850	0.00138488
L2-error along the x-axis in u	0.00147532	0.00138194

Table 8: u_0=0.5, T=40, N = 201, sixth order filter

The numerical results for the 2D linearized Euler equations show that the better approximation of the wave number for the DRP based SBP operator of fourth order (second order near the boundary) results in a smaller L_2-error than for a SBP operator based on standard sixth order (third order near the boundary) stencil. The results are valid when the problem is not fully resolved, which usually is the case for large scale applications.

Application of a sixth order filter affects the good wave approximation of the DRPSPB operator. That problem can be solved by using a filter or artificial dissipation that apply less dissipation for wave numbers $hk < \pi/2$ than the present filter.

5 Conclusions and Future Work

A strictly stable finite difference method with the summation by parts property (SBP) based on Tam and Webb's dispersion relation preserving (DRP) scheme in the interior developed in [6] has been tested on a 2D aeroacoustic problem. The results show that the SBP method based on the DRP scheme yields smaller errors for low resolutions without using a filter. But the advantage disappears with high resolution and/or using a sixth order explicit filter. However, using other types of artificial dissipation or filters, that are tailored to match the better wave number resolution of the DRP scheme, should lead to better results. Also in realistic applications it is often not possible to use the resolution needed, which shows the advantage of the DRPSBP method. The development and implementation of accurate non reflecting boundary conditions is future work. We also plan to develop high order numerical dissipation that will replace the filter used in this paper.

References

[1] M. Billson, L.-E. Eriksson, and L. Davidson. Acoustic Source Terms for the Linear Euler Equations on Conservative Form. In *8th AIAA/CAES Aeroacoustics Conference*, Breckenridge, Colorado, (2002). AIAA Paper 2002-2582.

[2] C. Bogey and C. Bailly. A family of low dispersive and low dissipative explicit schemes for flow and noise computations. *J. Comp. Phys.*, 194(1):194–214, (2004).

[3] L.-E. Eriksson. Boundary conditions for artificial dissipation operators. Technical Report FFA TN 1984-53, The Aeronautical Reaserch Institute of Sweden, (1984).

[4] B. Gustafsson, H.-O. Kreiss, and J. Oliger. *Time Dependent Problems and Difference Methods*. John Wiley & Sons, New York, (1995).

[5] B. Gustafsson, H.-O. Kreiss, and A. Sundström. Stability Theory of Difference Approximations for Mixed Initial Boundary Value Problems. II. *Math. Comp.*, 26(119):649–686, (1972).

[6] S. Johansson. High Order Finite Difference Operators with the Summation by Parts Property Based on DRP Schemes. Technical Report 2004-035, Uppsala University, Department of Scientific Computing, 2004.

[7] H.-O. Kreiss and G. Scherer. Finite Element and Finite Difference Methods for Hyperbolic Partial Differential Equations. In C. De Boor, editor, *Mathematical Aspects of Finite Elements in Partial Differential Equations.*, pages 195–211. Academic Press, Inc., (1974).

[8] K. Mattsson, M. Svärd, and J. Nordström. Stable and Accurate Artificial Dissipation. *J. Sci. Comp.*, 21(1):57–79, (2004).

[9] B. Müller and S. Johansson. Strictly Stable High Order Difference Approximations for Low Mach Number Aeroacoustics. In P. Neittaanmäki, T. Rossi, K. Majava, and O. Pironneau, editors, *ECCOMAS 2004*, 2004.

[10] C. L. Müller. *High order accurate numerical solution of the linearized Euler equations for sound propagation in the atmosphere.* Master's thesis, Uppsala University, Department of Scientific Computing, (2004).

[11] B. Strand. Summation by Parts for Finite Difference Approximations for d/dx. *J. Comp. Phys.*, 110:47–67, (1994).

[12] C.K.W. Tam. Benchmark Problems and Solutions. In J.C. Hardin, J.R. Ristorcelli, and C.K.W. Tam, editors, *ICASE/LaRCWorkshop on Benchmark Problems in Computational Aeroacoustics (CAA)*, (1995). NASA CP 3300.

[13] C.K.W Tam and J. C. Webb. Dispersion-Relation-Preserving Finite Difference Schemes for Computational Acoustics. *J. Comp. Phys.*, 107:262–281, (1993).

[14] M. R. Visbal and D. V. Gaitonde. High-Order-Accurate Methods for Complex Unsteady Subsonic Flows. *AIAA J.*, 37(10):1231–1239, (1999).