PRECONDITIONERS BASED ON FUNDAMENTAL SOLUTIONS

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Abstract.

We consider a new preconditioning technique for the iterative solution of linear systems of equations that arise when discretizing partial differential equations. The method is applied to finite difference discretizations, but the ideas apply to other discretizations too.

If E is a fundamental solution of a differential operator P, we have $E * (Pu) = u$. Inspired by this, we choose the preconditioner to be a discretization of an approximative inverse K, given by

$$(Ku)(x) = \int_\Omega E(x-y)u(y)dy, \qquad x \in \Omega \subset \mathbb{R}^d,$$

where Ω is the domain of interest.

We present analysis showing that if P is a first order differential operator, KP is bounded, and numerical results show grid independent convergence for first order partial differential equations, using fixed point iterations.

For the second order convection-diffusion equation convergence is no longer grid independent when using fixed point iterations, a result that is consistent with our theory. However, if the grid is chosen to give a fixed number of grid points within boundary layers, the number of iterations is independent of the physical viscosity parameter. Also, if GMRES is used together with the proposed preconditioner, the numbers of iterations decrease as the grid is refined, also for fixed viscosity.

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1 Introduction

We are interested in the numerical solution of a partial differential equation (PDE)

$$Pu = f \quad \text{in } \Omega,$$

with some boundary conditions given on parts of the boundary $\partial\Omega$. Standard methods, such as finite differences, finite volumes, and finite elements, approximate the differential equation, including boundary conditions, by a large and typically sparse system of equations

$$Ax = b.$$

Memory requirement is often critical, and solving the system by employing an iterative method may be the only viable choice. To achieve fast convergence, both a well suited iterative method and a good preconditioner are normally required.

The convergence properties of the preconditioned problem

$$MAx = Mb,$$

may be very different from those of the original system. The preconditioner, here represented by the matrix M, should have the following properties:

1. Memory requirements for M is at most of the same order as for A.
2. The number of arithmetic operations required for constructing and applying M is comparable to that of A.
3. The convergence rate is significantly improved.

There are a number of preconditioning techniques available. Most of these techniques use approximate inverses as preconditioners. Some of them, such as ILU [10] [19], and SPAI [3] [9], approximate the inverse of the matrix. Preconditioners of this kind are independent of the problem in the sense that they do not use any information on the origin of the system of equations. Other techniques, such as the original multigrid method [11] and preconditioners based on fast transforms [12] [14] [15], are designed especially for systems of equations that arise when discretizing differential equations. Problem-specific preconditioners work only for a particular class of problems, but may on the other hand have an excellent performance for systems of equations within the class.

The preconditioning technique we introduce in this paper is problem-specific in the sense that it can only be applied when solving systems of equations arising from differential equations. In fact, it is our belief that using as much information as possible on the origin of the system of equations is the best way of achieving efficient preconditioners.

Preconditioning techniques are often hard to analyze. We believe that one of the main reasons for this is that discretized problems often are more complicated than the original ones. For example, a pth order ordinary differential equation has p linearly independent solutions, while a high order finite difference approximation allows for a number of extra, parasitic solutions. This is one of several phenomena that may complicate the picture. We therefore argue that one should divide the analysis into two parts, by first constructing a preconditioner for the differential equation, and then in a second step consider possible complications due to discrete phenomena. This is also the approach in this paper.

We construct a preconditioner for the differential equation by using well-known theory of partial differential equations, and in particular, of fundamental solutions. To the best of our knowledge, this has not been tried before. However, a similar train of thought can be found in [17] and [18], where knowledge about the Green's function is used for preconditioning the convection-diffusion equation and the Navier–Stokes equations. Here, information about the differential equation is used when constructing an approximate inverse of the matrix A, while we on the other hand approximate the inverse of the differential operator itself.

The boundary element method [5] is somewhat related to our approach. Here, one uses a fundamental solution to transform a differential equation into an integral equation that can be solved numerically. As we discuss later, one advantage of discretizing the preconditioner and the differential operator separately, is that it makes the technique applicable also to problems with varying coefficients.

Our approach is also related to preconditioners based on fast transforms, and in particular circulant preconditioners. When applied to a PDE problem, a circulant preconditioner is a convolution-operator with a specific discrete fundamental solution as kernel. These kinds of preconditioners are usually presented using matrices and other tools within the field of numerical linear algebra. This approach has proven itself useful both from an implementational and a theoretical point of view, but as an unfortunate consequence the relation to theory of partial differential equations is not widely understood.

The connection to preconditioners based on fast transforms, and the success of these preconditioners when applied to first order partial differential equations, suggest that fundamental solution based preconditioners should perform well on the same class of problems, which include a number of important applications, such as Maxwell's and Euler's equations. Actually, one of the main contributions of this paper is that we prove in some generality that a preconditioner based on a fundamental solution is a good approximation to the inverse of a first order operator. It is even possible to achieve grid independent convergence. Unfortunately, we also find that grid independent convergence cannot be achieved for higher order differential operators. This result is consistent with the findings of Chan and Chan [6], who reduced the condition number for an elliptic problem from $\mathcal{O}(h^{-2})$ to $\mathcal{O}(h^{-1})$ by using a circulant preconditioner.

In this initial study, we restrict our attention to linear differential operators with constant coefficients. However, the preconditioning technique described in this paper is in no way restricted to such problems. In a more general case, one could construct a constant coefficient approximation and use a fundamental solution of the resulting operator when constructing the preconditioner, a technique that is shown to work for circulant preconditioners in [13]. It is also possible to combine the averaging technique with a domain decomposition approach. Thus, the technique is applicable to problems for which a fundamental solution might not even exist.

We also restrict ourselves to scalar problems in this report. However, semicirculant preconditioners yield grid independent convergence also for first order hyperbolic systems [4], and we expect the same to be true for preconditioners based on fundamental solutions. This is a subject of current research.

The paper is organized as follows. In Section 2 we introduce the differential equation and the preconditioner K. In Section 3, preconditioned iterative methods are discussed, and in Section 4, convergence theory for preconditioned first order differential operators is developed. By a careful discretization, the theory also describes the behavior of the discretized problem. This is the topic of Section 5.

We illustrate our technique by several numerical experiments. Implementation aspects are discussed in Section 6, and in Section 7 preconditioning for the convection and the convection-diffusion equation is studied. Finally, we end the paper with a summary in Section 8.

2 Problem

We consider the scalar partial differential equation

$$Pu = f, \quad x \in \Omega, \tag{2.1}$$

where

$$P = \sum_{|\alpha| \le p} a_\alpha \partial^\alpha$$

is a pth order linear operator with real constant coefficients, $\alpha = (\alpha_1, \ldots, \alpha_d)$ is a multi-index and Ω is an open and bounded domain in $\mathbb{R}^d$. Also, q boundary conditions of the form

$$B_k u = g_k, \quad x \in \Gamma_k \subset \partial\Omega, \quad k = 1, \ldots, q, \tag{2.2}$$

are given, where

$$B_k = \sum_{|\alpha| \le p} b_{k,\alpha} \partial^\alpha, \quad k = 1, \ldots, q,$$

are linear partial differential operators. We assume (2.1) and (2.2) to have a unique solution in the classical sense, and we require $u \in C^p(\overline{\Omega})$, $f \in C(\overline{\Omega})$, and that the boundary $\partial\Omega$ is piecewise of class C^1.

In this paper we consider the preconditioned problem

$$\begin{aligned} KPu &= Kf, \\ B_k u &= g_k, \quad k = 1, \ldots, q, \end{aligned} \tag{2.3}$$

where K is an integral operator given by

$$(Ku)(x) = \int_\Omega E(x-y) u(y)\, dy, \quad x \in \Omega, \tag{2.4}$$

and E is a fundamental solution of P. Since K is a convolution, the corresponding discrete operator can be applied using the fast Fourier transform, at least if the grid is structured. A comment on unstructured grids is found in Section 6, where implementation aspects are discussed.

A fundamental solution satisfies by definition the equation $PE = \delta$, where δ denotes the Dirac measure. Fundamental solutions play an important role in the theory of linear differential operators, and can for instance be used to study existence and regularity of solutions. We are interested in the relation

$$E * Pu = u. \tag{2.5}$$

This means that on $\mathbb{R}^d$, a convolution with a fundamental solution is a left inverse of a differential operator, provided that the convolution is well defined. The convolution is well defined e.g. if either E or Pu has compact support. The mild restriction $f \in \mathcal{E}$ implies $Pu \in \mathcal{E}$, and is hence sufficient for the convolution to be well defined. The space $\mathcal{E}$ consists of distributions defined on test-functions $\varphi \in C^\infty$.

By restricting the convolution to Ω, K is an approximate inverse, and we discuss this property further in Sections 3 and 4.

In this paper, we assume a fundamental solution to be known, something that is true for many important problems. This assumption simplifies the analysis but is not necessary for the implementation. For a constant coefficient problem, it is always easy to derive an expression for E in Fourier space, and an approximation of the inverse Fourier transform can be obtained numerically.

It is important to realize that in general, (2.4) must be interpreted in the sense of distributions. However, in many cases, the fundamental solution E is highly regular, and definition (2.4) can be used as it stands. For example, if P is an ordinary differential operator, E belongs to $C^\infty(\mathbb{R} \setminus 0)$. When P is elliptic, E is even analytic in $\mathbb{R}^d \setminus 0$, see e.g. [16]. Another interesting case, which we encounter in this paper, is when E is a positive distribution. Then, $E(x-y)$ is for every x a measure dE_x on some subset $\Omega_x \subset \Omega$, and we may replace (2.4) by

$$(Ku)(x) = \int_{\Omega_x} u \, dE_x.$$

3 Iterative methods and preconditioning

In this section we discuss the possibility of solving (2.1) and (2.2) with an iterative method. In practice, we do not solve the differential equation, but a discrete approximation of it. However, following the general approach of this paper, we begin by discussing iterative methods for the differential equation itself. In order to do this, we recast the differential equation into the form

$$u = F(u). \tag{3.1}$$

If F is a transformation of a space U into itself, a solution $u \in U$ to (3.1) is a fixed point of F. If U is a subspace of some Banach space B, and F is a contraction, that is, if there is a constant $C < 1$ such that

$$||F(u) - F(v)|| \leq C||u - v||,$$

for all $u, v \in U$, it follows from Banach's fixed point theorem that the fixed point iteration

$$u^{n+1} = F(u^n), \quad n = 0, 1, \ldots$$

converges for every initial guess $u^0 \in U$. We also know that the fixed point is unique. Hence, our mission is to choose suitable spaces B and U, and to recast (2.1) and (2.2) into the form (3.1), such that $F : U \to U$ is a contraction.

Since we are looking for classical solutions, it is natural to choose $C(\overline{\Omega})$ with norm

$$||u|| = \sup_{x \in \Omega} |u(x)|$$

as our Banach space B, and to let $U = C^p(\overline{\Omega})$.

A naive attempt to construct F would be to apply the forward Euler method directly to the unpreconditioned differential equation. In that case,

$$F(u) = u - \Delta t(Pu - f),$$

for $x \in \Omega$, and some $\Delta t > 0$. One also needs to incorporate the boundary conditions in some way. However, there are several problems with this choice of F already for inner points, and we begin by discussing these.

First of all, since F involves differentiation, $F(u)$ is in general less regular than u. Consequently, F does not map $C^p(\overline{\Omega})$ into itself. This problem is a technicality, and there are several ways of avoiding it. One could for instance make the space U smaller and consider infinitely smooth functions, that is, letting $U = C^\infty(\overline{\Omega})$. Besides, a finite difference or finite volume approximation P_h of P acts on functions defined on a discrete set of points, and in this case, differentiability is not an issue.

A more serious problem is that due to the nature of differentiation, F is not a contraction. Of course, every finite dimensional approximation P_h of P is bounded, and in the discretized case one can show that the forward Euler method yields a contraction if the matrix P_h is positive definite and Δt is small enough. The problem is that when the computational grid is refined, the norm of P_h increases and Δt has to be chosen even smaller. This usually means that the number of iterations required to achieve a certain accuracy increases, and this is the scenario we want to avoid.

In fact, any Runge-Kutta method has the same problem, and it is clear that these methods should be avoided, unless (2.1) and (2.2) first are transformed in some non-trivial way. This is why we introduce the preconditioner K.

To analyze the properties of KP, we use the Green formula

$$\int_\Omega (vPu - uP^*v)\,dx = \int_{\partial\Omega} J(u,v)\,ds, \tag{3.2}$$

where P^* is the formal adjoint of P,

$$P^* = \sum_{|\alpha| \le p} (-1)^{|\alpha|} a_\alpha \partial^\alpha,$$

and J is the bilinear form

$$J(u,v) = \sum_{0<|\alpha|\le p} a_\alpha \sum_{k=1}^{d} \sum_{l=1}^{\alpha_k} (-1)^{|\beta|} (\partial^\beta v) n_k (\partial^{\beta'} u).$$

where

$$\beta = (\alpha_1, \ldots, \alpha_{k-1}, l-1, 0, \ldots, 0) \text{ and}$$
$$\beta' = (0, \ldots, 0, \alpha_k - l, \alpha_{k+1}, \ldots, \alpha_d).$$

Here, the outward unit normal on $\partial\Omega$ is denoted by $n = (n_1, \ldots, n_d)$. We initially assume the fundamental solution E to be a function. From the Green formula (3.2) and the definition of E follows that

$$(KPu)(x) - u(x) = \sum_{0<|\alpha|\le p} a_\alpha \sum_{k=1}^{d} \sum_{l=1}^{\alpha_k} \int_{\partial\Omega} (\partial^\beta E)(x-y) n_k(y) (\partial^{\beta'} u)(y)\, ds(y). \tag{3.3}$$

Equation (3.3) shows precisely in what way KP fails to be the identity mapping. The important thing to note is that the right hand side of (3.3) involves differential operators acting on u of order up to $p-1$. Thus, for $p = 1$, KP is no longer a differential operator. According to the discussion above, this is a step in the right direction when aiming for grid independent convergence. Unfortunately, $p = 1$ is also a case where E is not a function, and it is not clear how to interpret equation (3.3). In the next section we examine this case more thoroughly.

If $p > 1$ the preconditioned operator KP is still a differential operator, although the order has been reduced to $p-1$ and only derivatives at the boundary of Ω influence the result. This indicates that preconditioners based on fundamental solutions improve the convergence rate, even though grid independent convergence cannot be achieved for differential equations of order $p > 1$. Once again we refer to the study by Chan and Chan [6] for comparison.

A natural attempt to reduce the order even further is to restrict the space of functions to those satisfying homogeneous boundary conditions, and choosing a fundamental solution E that satisfies as many adjoint boundary conditions as possible. However, a closer investigation shows that E has to satisfy all the adjoint boundary conditions, and this can only be accomplished if $E(x-y)$ is replaced by a Green's function $G(x,y)$. If we do so, K will no longer be a convolution operator, and a fast application by the Fourier transform is no longer available.

We end this section by a short discussion about boundary conditions. There are at least two different ways of incorporating these extra conditions in the iteration. First, one may restrict the function space to a set where all elements satisfy the conditions. This is straight forward when a finite element method is used, but can be accomplished also for finite differences by eliminating boundaries. Another possibility is to define the fixed point iteration also for function values at the boundary, and to introduce the extra conditions in this way. In no case can grid independent convergence be achieved if the boundary conditions involve derivatives.

4 First order equations

In this section we consider the case where P is a first order partial differential operator, and the single boundary condition does not involve any derivatives. We derive a sharp convergence estimate of a preconditioned fixed point iteration. To do so, it is sufficient to examine a homogeneous problem

$$\begin{aligned} Pu &= 0, \quad x \in \Omega, \\ u &= 0, \quad x \in \Gamma. \end{aligned}$$

Since $p = 1$, we have $P = b \cdot \nabla + a$ for some $b \in \mathbb{R}^d$ and $a \in \mathbb{R}$. As before, the domain Ω is a bounded and open subset of $\mathbb{R}^d$ and we assume existence of a unique classical solution. This assumption poses additional requirements on Ω. Convexity is a sufficient requirement, and for simplicity we assume Ω to have also this property. Furthermore, Γ is the part of $\partial\Omega$ where $b \cdot n < 0$, n being the outward pointing normal at $x \in \partial\Omega$. This is the inflow boundary.

Without loss of generality, we choose the coordinate system such that the first basis vector $\hat{x}_1$ is aligned with the vector b. The origin is chosen so that all points in Ω have a positive first coordinate.

The operator P then takes the simple form

$$P = |b| \frac{\partial}{\partial x_1} + a, \tag{4.1}$$

where $|b|$ denotes the Euclidian length of b. This also means that if we pick an arbitrary $x \in \Gamma$, except possibly for corners, and increase the x_1-coordinate slightly, we end up inside Ω. This is only a way of simplifying notation. For the sake of implementation, any set of coordinates may be used.

A simple calculation shows that

$$E(x) = (c + H(x_1)/|b|)\, e^{-a x_1/|b|}\, \delta(x_2, \dots, x_d), \tag{4.2}$$

is a fundamental solution. Here, c is an arbitrary constant, and H is the Heaviside function. Both the support and the singular support of E coincides with the line $r(s) = s\,\hat{x}_1$, and E is clearly not a function. Still, we may formally define K according to equation (2.4). Alternatively, we can identify the expression $\delta(x_2, \dots, x_d)$ in (4.2) as a measure and define K to be

$$(Ku)(x) = \int_{I(x)} (c + H(x_1 - y_1)/|b|)\, e^{-a(x_1 - y_1)/|b|}\, u(y_1, x_2, \dots, x_d)\, dy_1, \tag{4.3}$$

where $I(x) = \{y_1 \,|\, (y_1, x_2, \dots, x_d) \in \Omega\}$. For later use, we denote the smallest and largest elements in $I(x)$ by $\underline{x}_1$ and $\overline{x}_1$, implying that $\underline{x}_1 \in \Gamma$ and $\overline{x}_1 \in \partial\Omega \setminus \Gamma$.

We use a fixed point iteration, with

$$F(u) = u - \varphi_\varepsilon K P u - (1 - \varphi_\varepsilon) u, \quad x \in \overline{\Omega}. \tag{4.4}$$

Strictly speaking, Pu can only be computed on the open set Ω. However, we are going to show that $F(u)(x)$ is continuous in x, and may therefore be extended to all of $\overline{\Omega}$.

In the definition of F, φ_ε denotes an infinitely smooth function that for $\varepsilon > 0$ satisfies

$$\varphi_\varepsilon(x) = \begin{cases} 1, & |x-y| > \varepsilon, \forall y \in \Gamma, \\ 0, & x \in \Gamma. \end{cases} \tag{4.5}$$

We also require $0 < \varphi_\varepsilon(x) \leq 1$ for all $x \in \Omega$. Functions of this kind are known as cut-off functions and are frequently used in the theory of distributions to construct smooth functions with compact support. Here, we use φ_ε to obtain sufficient smoothness of $F(u)$ while including the boundary condition in the iteration. This procedure is only a technicality, and the cut-off function will not be visible in the implementation and hence not affect the applicability or the performance of the method. By this construction,

$$F(u) = \varphi_\varepsilon(u - KPu),$$

and we see that $F(u)(x) = 0$ for $x \in \Gamma$.

The existence of a classical solution to the non-homogenous problem,

$$\begin{aligned} Pu &= f, \quad x \in \Omega, \\ u &= g, \quad x \in \Gamma, \end{aligned} \tag{4.6}$$

requires $f \in C(\Omega)$ and $g \in C^1(\Gamma)$. This case should be handled by modifying (4.4) according to

$$F(u) = u - \varphi_\varepsilon(KPu - Kf) - (1 - \varphi_\varepsilon)(u - \overline{g}), \quad x \in \overline{\Omega}, \tag{4.7}$$

where $\overline{g}$ is any smooth function defined on $\overline{\Omega}$, coinciding with g on Γ. The following theorem states that ε can be chosen so that the fixed point of F is arbitrarily close to the solution of (4.6).

THEOREM 4.1. *Assume F in (4.7) to be uniformly contractive in ε, i.e. there are constants $C < 1$ and $\varepsilon_0 > 0$ such that*

$$||F(u) - F(v)|| \leq C||u - v||, \quad \forall \varepsilon \in (0, \varepsilon_0).$$

Then the fixed point of F converges to the solution of (4.6) as $\varepsilon \to 0$.

PROOF. Let u^* solve (4.6) and let u_ε be the fixed point of F. Then

$$\begin{aligned} ||u_\varepsilon - u^*|| \leq& ||F(u_\varepsilon) - F(u^*)|| + ||F(u^*) - u^*|| \\ \leq& C||u_\varepsilon - u^*|| + ||F(u^*) - u^*||, \end{aligned}$$

implying

$$\begin{aligned} ||u_\varepsilon - u^*|| \leq& \frac{1}{1-C}||F(u^*) - u^*|| = \frac{1}{1-C}||(\varphi_\varepsilon - 1)(u^* - \overline{g})|| \\ \leq& \frac{1}{1-C} \sup_{x \in \operatorname{supp}(\varphi_\varepsilon - 1)} ||u^* - \overline{g}||. \end{aligned}$$

In the second last step we have used the definition of F and that u^* satisfies the PDE. The result now follows from uniform continuity of $u^* - \overline{g}$ and that $\overline{g} = u^*$ on Γ. ∎

We now turn back to the homogeneous case, and to derive a convergence estimate we need to compute the norm of F. In the following we assume that also the initial guess satisfies the boundary condition. This assumption simplifies the analysis, and as discussed above, such a guess can be constructed by performing one iteration. We begin our analysis by examining KP.

THEOREM 4.2. *Let P and K be defined by (4.1) and (4.3), and assume $u(x)=0$ when $x \in \Gamma$. The preconditioned operator KP then satisfies*

$$(KPu)(x) = u(x) + c|b|e^{a(\overline{x}_1 - x_1)/|b|}\, u(\overline{x}_1, x_2, \ldots, x_d).$$

PROOF. From the definition of K and P,

$$\begin{aligned}
&(KPu)(x)\\
=&\int_{I(x)} (c + H(x_1 - y_1)/|b|)\, e^{-a(x_1-y_1)/|b|} \left(|b|\frac{\partial}{\partial y_1} + a\right) u(y_1, x_2, \ldots, x_d)\, dy_1\\
=&\int_{\underline{x}_1}^{x_1} (c|b| + 1)\frac{\partial}{\partial y_1}\left(e^{-a(x_1-y_1)/|b|}\, u(y_1, x_2, \ldots, x_d)\right) dy_1 +\\
&\int_{x_1}^{\overline{x}_1} c|b| \frac{\partial}{\partial y_1}\left(e^{-a(x_1-y_1)/|b|}\, u(y_1, x_2, \ldots, x_d)\right) dy_1\\
=&(c|b|+1)\left(u(x) - e^{-a(x_1-\underline{x}_1)/|b|}\, u(\underline{x}_1, x_2, \ldots, x_d)\right) +\\
&c|b|\left(e^{-a(x_1-\overline{x}_1)/|b|}\, u(\overline{x}_1, x_2, \ldots, x_d) - u(x)\right).
\end{aligned}$$

By assumption, $u(\underline{x}_1, x_2, \ldots, x_d) = 0$, and the result follows. □

According to Theorem 4.2,

$$F(u)(x) = -c|b|e^{a(\overline{x}_1 - x_1)/|b|}\, u(\overline{x}_1, x_2, \ldots, x_d)\, \varphi_\varepsilon(x), \quad x \in \Omega. \tag{4.8}$$

Relation (4.8) shows that $F : C^1(\Omega) \to C^1(\Omega)$. However, F can be extended in two different ways. First, $F : C^1(\overline{\Omega}) \to C^1(\overline{\Omega})$ since both u and $F(u)$ are continuous in x and therefore can be extended uniquely to all of $\overline{\Omega}$. Secondly, since C^1 is dense in C, there is a unique extension $F : C(\overline{\Omega}) \to C(\overline{\Omega})$. Note that formula (4.8) can be used to compute $F(u)$ also in this case.

We are now prepared to compute the norm of F, and from (4.8) we find

$$||F|| \equiv \sup_{||u||=1} ||F(u)|| = \begin{cases} |c||b|e^{a\eta/|b|}, & a \le 0,\\ |c||b|e^{a|\Omega|_1/|b|}, & a > 0, \end{cases}$$

for some $\eta \in (0, \varepsilon]$. Here,

$$|\Omega|_1 = \sup_{x\in\Omega} \overline{x}_1 - \underline{x}_1.$$

Thus, for $|c|$ small enough, $||F|| < 1$, and F is a contraction. Therefore, the fixed point iteration

$$u^{n+1} = F(u^n), \quad n = 0, 1, \ldots$$

converges for every $u^0 \in C(\overline{\Omega})$. We also know that the fixed point is unique, so our preconditioner does not introduce any new solutions.

In fact, we can do even better than this. The nth iterate satisfies

$$||u^n|| = ||F^n(u^0)|| \leq ||F^n||\,||u^0||,$$

where equality holds for some $u^0 \in C(\overline{\Omega})$. The power norm of F can easily be shown to satisfy

$$||F^n|| = \begin{cases} |c|^n|b|^n e^{a\eta/|b|}, & a \leq 0, \\ |c|^n|b|^n e^{a|\Omega|_1/|b|}, & a > 0, \end{cases}$$

for some $\eta \in (0, \varepsilon]$, yielding a sharp estimate of the rate of convergence.

Finally, we note that the best possible choice of c is $c = 0$, in which case K is an exact inverse. However, the iteration converges for a range of different values of c, implying a certain robustness of the method.

5 The discretized problem

We now consider a finite difference approximation of (2.1).

$$P_h u_h = f_h, \quad i \in \Omega_h. \tag{5.1}$$

Here, $h \in \mathbb{R}^d$ is the mesh parameter, and $\Omega_h = \{i \in \mathbb{Z}^d \,|\, (i_1h_1, \ldots, i_dh_d) \in \Omega\}$. Similarly, we approximate the boundary conditions (2.2) by

$$B_{k,h}u_h = g_{k,h}, \quad i \in \Gamma_{k,h}, \tag{5.2}$$

where $\Gamma_{k,h}$ is a discretization of Γ_k. We assume that equation (5.1) with boundary conditions (5.2) has a unique solution converging towards the solution of (2.1) and (2.2) when the grid is refined.

We construct a discretized version of the integral operator K by sampling the fundamental solution and approximating the integral in the simplest possible way,

$$(K_h u_h)_i = \sum_{j \in \Omega_h} E(ih - jh)(u_h)_j \, h_1 \cdot \ldots \cdot h_d, \quad i \in \Omega_h. \tag{5.3}$$

Here, $ih = (i_1h_1, \ldots, i_dh_d)$ and $jh = (j_1h_1, \ldots, j_dh_d)$.

One obvious problem with definition (5.3) is that E may not be well defined at certain points, and cannot be sampled. In practice, we therefore replace E in (5.3) by a function $\widetilde{E}_h$, which is the sampling of E everywhere, apart from close to these difficult points. A procedure for choosing the remaining values of $\widetilde{E}_h$ is given in the next section.

Using the definitions of the discretized operators and the mesh, we can now define the discretized iteration operator,

$$F_h u_h = \begin{cases} u_h - K_h P_h u_h + K_h f_h, & i \in \Omega_h, \\ u_h - B_{k,h}u_h + g_h, & i \in \Gamma_{k,h}. \end{cases} \tag{5.4}$$

Note that (5.4) is a consistent discretization of (4.7) if $\varepsilon \le \min_j h_j$.

In Sections 3 and 4 we discuss the properties of KP. We choose to do so since KP is easier to analyze than $K_h P_h$. However, if the two operators do not behave similarly, this is not a useful approach. In the remaining part of this section we discuss the criteria that need to be fulfilled to achieve good agreement.

The analysis in Section 3 is based on two important relations. The first is (2.5), stating that a convolution with a fundamental solution is a left inverse of a differential operator with constant coefficients. The corresponding discrete relation states that if E_h satisfies $P_h E_h = \delta_h$, then $E_h * P_h u_h = u_h$. Here,

$$(\delta_h)_i = \begin{cases} h_1^{-1} \cdot \ldots \cdot h_d^{-1}, & i = 0, \\ 0, & \text{otherwise}, \end{cases}$$

and

$$(u * v)_i = \sum_j u_{i-j} v_j \, h_1 \cdot \ldots \cdot h_d.$$

This relation has gained some interest. For instance, De Boor, Höllig and Riemenschneider proved existence of a bounded discrete fundamental solution in 1989 [7].

In view of the discrete relation, we realize that a necessary requirement for a good agreement between K and K_h is that E is close to E_h at every point $i \in \Omega_h$. This may not be true for non-compact difference operators. In one of the examples in Section 7 we approximate a first order differential operator with a difference stencil that has three points in each spatial direction. Since two points are sufficient when approximating a first order derivative, such a difference stencil is non-compact. Our experiments verify that the performance of the preconditioner is degraded when the difference between E and E_h becomes large.

The second important relation is the integration by parts formula (3.3). Also this relation has a discrete counterpart, known as a summation by parts formula, which in general however, is not a direct analogue of the integration by parts formula.

The following theorem considers one possible discretization of

$$P = b_1 \frac{\partial}{\partial x_1} + b_2 \frac{\partial}{\partial x_2}, \quad x_1, x_2 \in (0,1),$$

and is meant to illustrate the two important relations.

THEOREM 5.1. *If*

$$P_h u_{i,j} = \begin{cases} \left(-\gamma h \left(D_+^{x_1} D_-^{x_1} + D_+^{x_2} D_-^{x_2}\right) + b_1 D_0^{x_1} + b_2 D_0^{x_2}\right) u_{i,j}, & i,j = 1 \ldots N-1, \\ \left(-\gamma h D_+^{x_2} D_-^{x_2} + b_1 D_-^{x_1} + b_2 D_0^{x_2}\right) u_{i,j}, & i = N, j = 1 \ldots N-1, \\ \left(-\gamma h D_+^{x_1} D_-^{x_1} + b_1 D_0^{x_1} + b_2 D_-^{x_2}\right) u_{i,j}, & i = 1 \ldots N-1, j = N, \\ \left(b_1 D_-^{x_1} + b_2 D_-^{x_2}\right) u_{i,j}, & i = j = N, \end{cases}$$

and

$$K_h u_{i,j} = \sum_{i'=1}^{N} \sum_{j'=1}^{N} \widetilde{E}_{i-i',j-j'}\, u_{i',j'}\, h^2,$$

then

$$\begin{aligned}
K_h P_h u_{i,j} &= \sum_{i'=1}^{N} \sum_{j'=1}^{N} u_{i',j'}\, P_h \widetilde{E}_{i-i',j-j'}\, h^2 \\
&\quad + \sum_{j'=1}^{N} \Bigg[\left(-\gamma - \frac{b_1}{2}\right) \widetilde{E}_{i-1,j-j'}\, u_{0,j'} \\
&\qquad + \left(\gamma - \frac{b_1}{2}\right) \widetilde{E}_{i,j-j'}\, u_{1,j'} \\
&\qquad + \left(\gamma - \frac{b_1}{2}\right) \widetilde{E}_{i-N,j-j'}\, u_{N-1,j'} \\
&\qquad + \left(\left(\gamma + \frac{b_1}{2}\right) \widetilde{E}_{i-N-1,j-j'} + 2\left(-\gamma + \frac{b_1}{2}\right) \widetilde{E}_{i-N,j-j'} \right) u_{N,j'} \Bigg] h \\
&\quad + \sum_{i'=1}^{N} \Bigg[\left(-\gamma - \frac{b_2}{2}\right) \widetilde{E}_{i-i',j-1}\, u_{i',0} \\
&\qquad + \left(\gamma - \frac{b_2}{2}\right) \widetilde{E}_{i-i',j}\, u_{i',1} \\
&\qquad + \left(\gamma - \frac{b_2}{2}\right) \widetilde{E}_{i-i',j-N}\, u_{i',N-1} \\
&\qquad + \left(\left(\gamma + \frac{b_2}{2}\right) \widetilde{E}_{i-i',j-N-1} + 2\left(-\gamma + \frac{b_2}{2}\right) \widetilde{E}_{i-i',j-N} \right) u_{i',N} \Bigg] h.
\end{aligned}$$

PROOF. The result follows from summation by parts. □

The following is trivially true.

COROLLARY 5.2. *If*

1. $P_h \widetilde{E}_h = \delta_h$,
2. $2\gamma = b_1 = b_2$,
3. $\widetilde{E}_{i,j} = 0$ *when* $i < 0$ *or* $j < 0$, *and*
4. $u_h = 0$ *when* $i = 0$ *or* $j = 0$,

then $K_h P_h u_h = u_h$.

It is easy to show that the third condition of the corollary can be satisfied if the second is. Note that Corollary 5.2 corresponds to Theorem 4.2 with $c = 0$. Thus, even though there are profound differences between the continuous and the discrete theory, the example above shows that there are cases with good agreement.

6 Implementation

As remarked in the previous section, the fundamental solution E may be undefined at certain points, such as the value of the Heaviside function at $x = 0$, or even singular on a set of measure zero. On the other hand, the discrete fundamental solution E_h is well defined and bounded at every grid point [7]. Numerical experiments and the discussion in Section 5 show that it is important to capture the singularity accurately. In the experiments in Section 7, we construct the preconditioner by sampling E away from the problematic points, and computing the remaining values by solving the equation $P_h \widetilde{E}_h = \delta_h$ in a small neighborhood of the singularity or discontinuity. It is important that the neighborhood is small compared to the computational domain, or the construction of the preconditioner will have a complexity of the same order as a direct solve of the original problem.

Waldén studied problems with singular source terms in [22] and found that it is possible to achieve convergence of expected order away from singularities. If fast convergence is achieved, our procedure will yield a preconditioner with a kernel that is close to the discrete fundamental solution E_h also nearby singularities.

We need an efficient way of applying K_h. From definition (5.3), it follows that E is needed at points in a domain larger than Ω. More precisely, we need to know $E(x - y)$ for all $x \in \Omega$ and $y \in \Omega$. This larger domain is denoted $\widetilde{\Omega}$.

One possible way of applying K_h is by use of the fast Fourier transform, FFT. We then choose a bounding box around $\widetilde{\Omega}$ and extend E with zeros inside the box. The target function u_h is also extended with zeros inside a box with the same size, see Figure 6.1. Both functions are now well defined and can be extended periodically to all of $\mathbb{R}^d$, and the convolution can be computed by the well known formula

$$\widehat{E * u} = \hat{E}\, \hat{u}. \tag{6.1}$$

There is a close relation to the method of multiplying a vector by a Toeplitz matrix in the fractional Fourier transform method, see e.g. [2].

The method described above is straight forward if the grid is uniform. However, non uniform grids can be handled using a fast Fourier transform for nonequispaced data, see [8]. Another possibility is to transform the grid into a uniform one and dealing with the differential operator as if it had variable coefficients, i.e. by averaging the coefficients as described in Section 1.

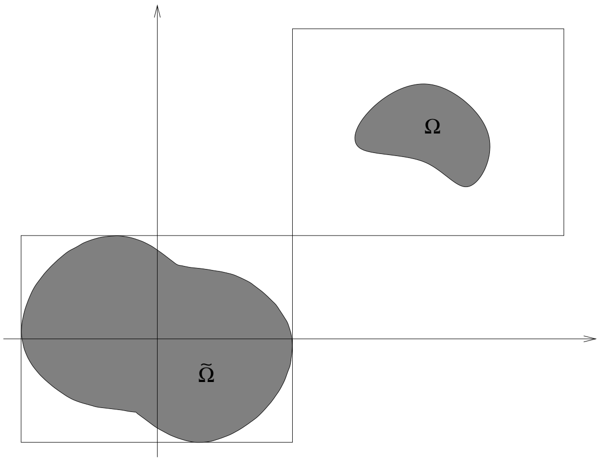

Figure 6.1: Values of E are needed in $\widetilde{\Omega}$. The rectangles are the bounding boxes on which E and u are extended with zeros in order to make the application using FFT possible.

7 Numerical experiments

In this section, three model problems arising from the convection and the convection-diffusion equation in two dimensions are studied. First, the convection equation is solved using an upwind scheme, and then we add second order artificial viscosity. Finally, we also treat physical viscosity, i.e. the standard convection-diffusion equation.

All of the model problems are solved on $\Omega = \{(x_1, x_2) \mid x_1, x_2 \in (0,1)\}$, using fixed point iteration, where F_h is given by (5.4). We iterate until the 2-norm of the residual is reduced by a factor of 10^{-7}. The grids are uniform with N grid points in each dimension, and we apply K_h using FFT according to (6.1).

In all examples, the forcing functions for the PDEs are identically zero. The iterations are started with a random initial guess. Each problem is solved for a number of initial guesses and the iteration counts presented in the tables represent maximum values. The difference in the numbers of iterations for different initial guesses does not exceed a few percent.

7.1 Convection equation

The analysis in Section 4 suggests that for first order differential operators, our method should give favorable convergence properties. We study a pure convection problem,

$$\begin{aligned} Pu &\equiv b \cdot \nabla u = 0, && x \in \Omega, \\ Bu &\equiv u = \begin{cases} 0, & x_1 = 0, \\ \sin^2(\pi x_1), & x_2 = 0, \end{cases} \end{aligned}$$

where b is set to $(1,1)^T$.

A fundamental solution for this problem is given by (4.2). Here, we have $a = 0$ and $b = (1,1)^T$, and we use

$$E(x_1, x_2) = \left(c + \frac{1}{\sqrt{2}} H(\frac{x_1 + x_2}{\sqrt{2}})\right) \delta(\frac{x_1 - x_2}{\sqrt{2}}),$$

where H is the Heaviside step function and c is an arbitrary constant.

The operator that determines the behavior of the error under the fixed point iteration is given by (4.8). Using the analysis from Section 4, we find that

$$F(u)(x_1, x_2) = \begin{cases} -c\sqrt{2}u(1, x_2 - x_1 + 1)\varphi_\varepsilon(x_1, x_2), & x_1 > x_2, \\ -c\sqrt{2}u(x_1 - x_2 + 1, 1)\varphi_\varepsilon(x_1, x_2), & x_1 < x_2. \end{cases}$$

Using $c = 0$, we see that K is an exact inverse to the differential operator *with* its boundary conditions, and we expect convergence in one iteration.

In this section, P is discretized using one sided difference approximations,

$$P_h u_{i,j} = (D_-^{x_1} + D_-^{x_2})u_{i,j}, \quad (i,j) \in \Omega_h.$$

Note that this problem can be solved by stepping through the domain, starting from the bottom left corner. Hence, the solution can be computed in $\mathcal{O}(N^2)$ arithmetic operations. Recall that N is the number of grid points in each dimension. Thus, there is really no need for an iterative solver. However, the iterative scheme serves as a relevant model for more complicated problems, and is therefore worth studying in the context of our preconditioning technique.

Since E is singular along the line $x_1 = x_2$ and zero elsewhere, we use the difference operator P_h to compute the values of $\widetilde{E}_h$ in a neighborhood, consisting of r grid points on each side of that line, see Figure 7.1.

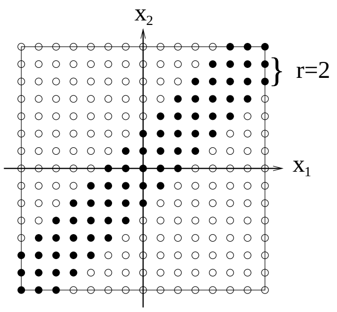

Figure 7.1: Values of $\widetilde{E}_h$ in shaded grid points are computed using the computational stencil.

In practice, we do not want r to grow as the grid is refined, or the construction of the preconditioner will become to costly. However, we perform experiments both for constant and growing r to obtain a better understanding of the behavior of the method, and to make the connection to the analysis more visible.

Table 7.1: Numbers of iterations for the convection problem.

h^{-1}	$r = N$	$r = N/8$	$r = 4$
16	1	10	6
32	1	9	9
64	1	7	14
128	1	5	22
256	1	4	36
512	1	3	62

Table 7.1 shows the numbers of iterations for a wide range of grid sizes. Even though only the first three conditions of Corollary 5.2 are satisfied, we obtain a preconditioner that is an exact inverse in the case $r = N$, i.e. if $\widetilde{E}_h = E_h$.

Using $r = N/8$, the iteration numbers are decreasing, and for $r = 4$, the iteration count increases slowly. These phenomena are explained using Figure 7.2, which shows plots of normalized discrete fundamental solutions E_h for three different h. The dotted lines represent borders of an area with fixed physical size, i.e. the region in which non-zero values of $\widetilde{E}_h$ are computed when $r = N/8$. The solid lines represent borders of an area with a constant number of grid lines, i.e. the region in which non-zero values of $\widetilde{E}_h$ are computed in the case $r = 4$.

We see in Figure 7.2 that more of the non-zero elements of E_h are captured in the physically constant area as the grid is refined. Even though the width of the area of E_h with values significantly larger than zero decrease, there is an increasing number of non-zero values that are not captured in the case $r = 4$. The conclusion is that $\widetilde{E}_h$ approximate E_h better as the grid is refined when $r = N/8$ and worse when $r = 4$.

It is also possible to include a zero order term in the differential operator, i.e. using $Pu = b\cdot\nabla u + au$. The fundamental solution is given in (4.2). Numerical experiments show that for $-4 \leq a \leq 1$, the convergence properties are almost unaffected, while $a > 1$ yields improved convergence. For $a \ll 0$, the PDE-problem is very ill conditioned, and the convergence rate is decreased.

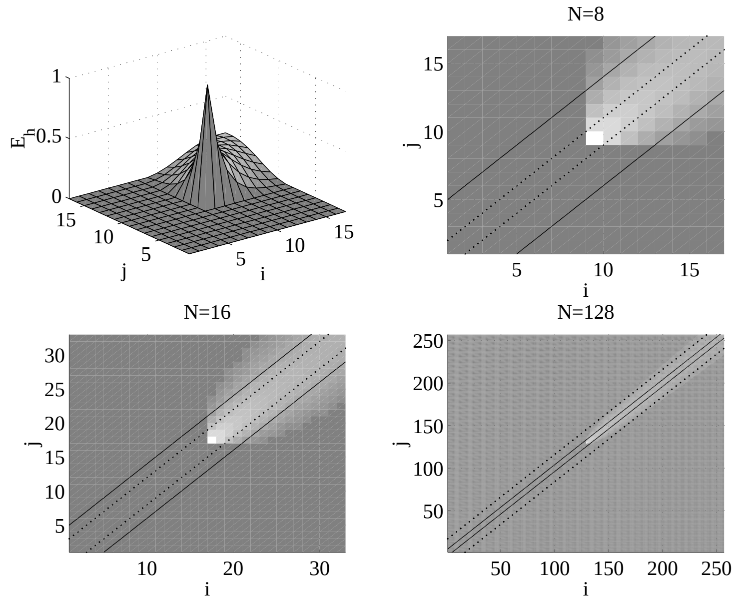

Figure 7.2: Discrete fundamental solutions, E_h, of P_h. Solid lines are boundaries for the computation of $\widetilde{E}_h$ when $r = 4$ and dotted lines are boundaries for $r = N/8$.

7.2 Convection with artificial viscosity

Note that the one sided difference approximation used in the previous problem can be seen as a centered approximation with a specific amount of second order artificial viscosity added. The centered difference approximation of a first derivative allows for rapidly oscillating solutions, why one must include some viscosity.

We now study the convection problem once more. The operators P and B are the same as in Section 7.1, but in the discretization we use centered differences with second order artificial viscosity of arbitrary strength. The fast solution method mentioned in the previous section is no longer available since we impose boundary conditions on all boundaries. The problem is therefore more realistic in the sense that the iterative method is essential.

Using ϵ_h to specify the amount of artificial viscosity, we get the discretization

$$P_h u_{i,j} = \begin{cases} \left(-\epsilon_h \left(D_+^{x_1} D_-^{x_1} + D_+^{x_2} D_-^{x_2}\right) + b_1 D_0^{x_1} + b_2 D_0^{x_2}\right) u_{i,j}, & i,j = 1 \ldots N-1, \\ \left(-\epsilon_h D_+^{x_2} D_-^{x_2} + b_1 D_-^{x_1} + b_2 D_0^{x_2}\right) u_{i,j}, & i = N, j = 1 \ldots N-1, \\ \left(-\epsilon_h D_+^{x_1} D_-^{x_1} + b_1 D_0^{x_1} + b_2 D_-^{x_2}\right) u_{i,j}, & i = 1 \ldots N-1, j = N, \\ \left(b_1 D_-^{x_1} + b_2 D_-^{x_2}\right) u_{i,j}, & i = j = N. \end{cases}$$

When constructing the preconditioner we have two obvious choices for the fundamental solution. The first is the same as in the previous problem, since we are dealing with the same differential operator. The only difference is that we use another stencil for the computation of $\widetilde{E}_h$ in the neighborhood of the singularity.

The second choice is the fundamental solution of the convection-diffusion operator [20], given by

$$E(x) = \frac{1}{2\pi\epsilon} e^{b\cdot x/2\epsilon} K_0 \left(\frac{|b||x|}{2\epsilon}\right), \tag{7.1}$$

letting $\epsilon = \epsilon_h$. Here, K_0 is the modified Bessel function of the second kind. This function is singular at the origin. Therefore, $\widetilde{E}_h$ is a sampling of E at all grid points, except for in a small neighborhood of the singularity. To determine these values, the equation $P_h \widetilde{E}_h = \delta_h$ is solved in a neighborhood, where the number of grid points on each side of the singularity is denoted by s. The boundary values for this small region are determined by (7.1). Figure 7.3 shows an example where $s = 2$.

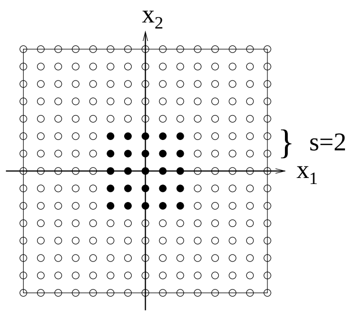

Figure 7.3: Values of $\widetilde{E}_h$ in shaded grid points are computed using the computational stencil when the fundamental solution is based on the convection-diffusion operator.

For highly resolved problems, ϵ_h is very small, and the numerical evaluation of E difficult. This is overcome by the use of an asymptotic formula for K_0 [1].

One major difference between the fundamental solutions described above is that the preconditioner based on the pure convection operator involves a solve of a linear system in approximately $(2r + 1)(2N + 1)$ grid points, while the one based on the convection-diffusion operator uses a fixed number, $(2s+1)^2$, of grid points. In the latter case, the work required to construct the preconditioner is proportional to the number of unknowns.

Results from using the two fundamental solutions for preconditioning are displayed in Tables 7.2 and 7.3. It is clear that the preconditioner based on the fundamental solution of the convection-diffusion operator gives more favorable results, probably since the viscous behavior is taken into account in all of Ω. For this $\widetilde{E}_h$, we note that a linear relation between ϵ_h and h gives constant iteration counts.

Table 7.2: Numbers of iterations for $\widetilde{E}_h$ based on the convection operator, $b = (1,1)^T$, $r = 4$.

h^{-1}	$\epsilon_h = h$	$\epsilon_h = h/2$	$\epsilon_h = h/3$	$\epsilon_h = h/4$
16	18	5	13	26
32	20	9	13	25
64	26	14	14	25
128	39	22	18	23
256	65	36	27	29
512	113	61	44	39

Table 7.3: Numbers of iterations for $\widetilde{E}_h$ based on the convection-diffusion operator, $b = (1,1)^T$, $s = 4$.

h^{-1}	$\epsilon_h = h$	$\epsilon_h = h/2$	$\epsilon_h = h/3$	$\epsilon_h = h/4$
16	17	5	13	27
32	17	5	13	25
64	19	5	12	24
128	20	5	11	23
256	19	5	11	22
512	19	5	10	20

Note that when $\epsilon_h = h/2$, the centered difference approximation turns into the one-sided approximation used in Section 7.1. For this parameter value we see, in Table 7.2, that the numbers of iterations are essentially the same as for the case $r = 4$ in Table 7.1, since both the coefficient matrices and the preconditioners actually are the same. In Table 7.3, we find that 5 iterations is enough, independent of the grid size. This is an improvement compared to the first model problem, since a fundamental solution computed on a constant number of grid lines gives increasing iteration counts.

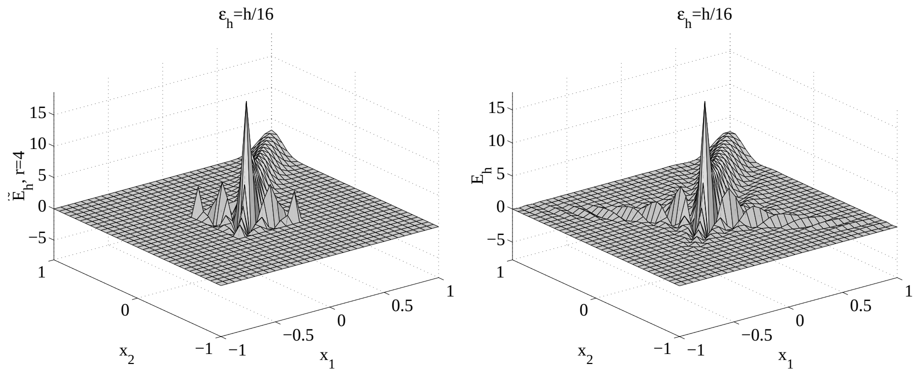

Figure 7.4: Plots of $\widetilde{E}_h$ and E_h. Note the difference along $x_1 = -x_2$.

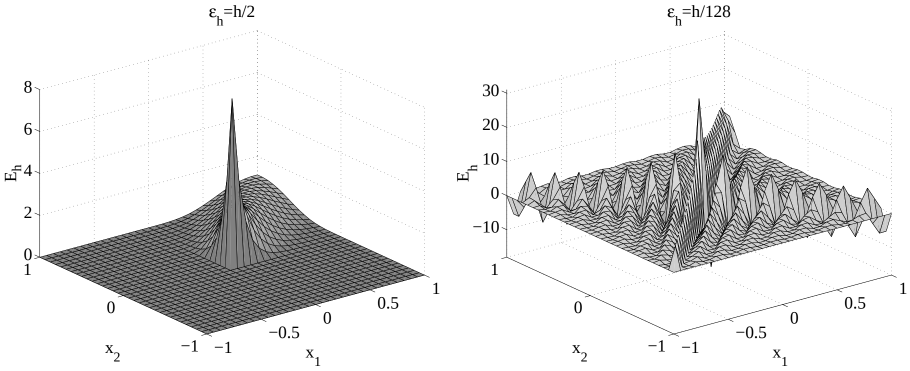

Figure 7.5: Two plots of E_h. In the left, where $\epsilon_h = h/2$, E_h is well approximated by $\widetilde{E}_h$. The right, where $\epsilon_h = h/128$, illustrates a behavior that can not be captured by $\widetilde{E}_h$.

For $\epsilon_h \ll h$, the simple iterative method used here yields very slow convergence. In this case, $\widetilde{E}_h$ is no longer a good approximation of the discrete fundamental solution E_h, as shown in Figure 7.4, where $\epsilon_h = h/16$. For even smaller ϵ_h, the oscillating parts becomes more pronounced, see Figure 7.5. Consequently, K_h fails to approximate the inverse of P_h. Also, K_h and K will no longer have the same behavior and the theory in Section 4 does not apply to the discretized problem.

7.3 Convection-diffusion equation

We now turn to the standard convection-diffusion equation. The definition of the differential equation is

$$\begin{aligned} Pu \equiv& -\epsilon\Delta u + b\cdot\nabla u = 0, \quad x \in \Omega, \\ Bu \equiv& u = \begin{cases} 0, & x_1 = 0, \\ 0, & x_1 = 1, \\ \sin^2(\pi x_1), & x_2 = 0, \\ 0, & x_2 = 1. \end{cases} \end{aligned}$$

The fundamental solution is given in equation (7.1). Using centered difference approximations, we get the discretization

$$P_h u_{i,j} = \left(-\epsilon\left(D_+^{x_1}D_-^{x_1} + D_+^{x_2}D_-^{x_2}\right) + b_1 D_0^{x_1} + b_2 D_0^{x_2}\right) u_{i,j}, \quad (i,j) \in \Omega_h.$$

The preconditioner is constructed in the same way as in the previous problem, i.e. by sampling E, except for in a neighborhood of the origin, consisting of $(2s+1)^2$ grid points. We first examine the influence of s on the number of iterations. Table 7.4 shows that there is no gain in using a large s.

Table 7.4: Numbers of iterations, $b = (1,1)^T$.

	$\epsilon = 1/200$	$\epsilon = 1/200$
h^{-1}	$s = 4$	$s = N$
16	19	20
32	16	16
64	9	10
128	9	8
256	66	65

Guided by the results in Table 7.4, we employ $s = 4$ and show the results for different ϵ. The iteration counts are presented in Table 7.5.

Table 7.5: Numbers of iterations, $s = 4$, $b = (1,1)^T$.

h^{-1}	$\epsilon = 1/100$	$\epsilon = 1/200$	$\epsilon = 1/400$	$\epsilon = 1/800$
16	15	19	>200	>200
32	10	16	19	>200
64	9	9	15	20
128	60	9	9	15
256	div	66	8	9
512	div	div	67	8

One interesting property seen in Table 7.5 is that the iteration count along the diagonals varies very little. Guided by that observation, we also solved the problem for $\epsilon = 1/1600$ on the finest mesh. As expected, the solution converged in 9 iterations. This is a satisfying result, since a specific ϵ requires a mesh parameter that yields sufficient resolution of the boundary layers that would occur if e.g. a non-zero forcing function were employed. The thickness of exponential boundary layers in the solution of the convection-diffusion equation is of order ϵ [21]. Hence, a resolution criterion in a practical computation is that ϵ/h is constant. The results in Table 7.5 show that the number of iterations is independent of the viscosity coefficient ϵ, if such a resolution criterion is employed.

For a fixed ϵ, the increase in the number of iterations seen in Table 7.5 is in correspondence with the analysis in Section 4. The iteration operator is unbounded, since it involves derivatives.

Remember that we have used a very simple iterative method so far. Table 7.6 shows iteration counts for preconditioned GMRES, for a wider range of ϵ. The preconditioner is the same as in Table 7.5. We observe that the numbers of iterations decrease as the grid is refined and that GMRES, together with the fundamental solution preconditioner, is much less sensitive to the parameter ϵ. The nice convergence properties come at the price of memory, since we use the unrestarted version of GMRES.

Table 7.6: Numbers of iterations, GMRES, $s = 4$, $b = (1,1)^T$.

h^{-1}	$\epsilon = 5$	$\epsilon = 1/2$	$\epsilon = 1/20$	$\epsilon = 1/200$	$\epsilon = 1/2000$
16	24	38	25	41	>200
32	24	31	29	30	>200
64	17	26	30	26	166
128	15	18	23	20	82
256	13	16	15	13	30
512	10	17	13	7	12

8 Summary and conclusions

We introduce a new preconditioning technique for the iterative solution of linear systems of equations

$$Ax = b, \tag{8.1}$$

arising when discretizing partial differential equations

$$Pu = f, \quad x \in \Omega.$$

In this paper, we consider the finite difference method, but any method may by used.

The preconditioner is constructed in two steps. First, we approximate the inverse of the differential operator P by K,

$$Ku(x) = \int_{x\in\Omega} E(x-y)u(y)\,dy, \quad x \in \Omega,$$

where E is a fundamental solution of P. We prove KP to be bounded for first order differential equations, which is a favorable when striving for grid independent convergence for Runge–Kutta methods. We also give a sharp estimate of the rate of convergence for such equations.

In the second step, K is discretized and used as a left preconditioner for (8.1). We find that the discretized operator K_hP_h shows the same behavior as the original operator KP, if E is close to the discrete fundamental solution E_h. This is not true at points where E is singular, and we discuss how to overcome this problem. We also show how to apply K_h in an efficient way by the use of FFT.

Numerical experiments for the convection equation with artificial viscosity shows that the preconditioner should be based on the fundamental solution of the convection-diffusion operator. With this setting, our technique gives convergence in a fixed number of iterations, independent of the number of grid points.

We also consider a convection-diffusion equation. For this problem, the number of iterations is independent of the physical viscosity, if the grid has a fixed number of grid points within boundary layers. However, for a fixed viscosity, the number of iterations is no longer independent of the grid, a result that is consistent with the theory.

The above discussion concerns experiments with fixed point iterations. If preconditioned GMRES is used instead, remarkably beneficial convergence results are obtained, both in terms of robustness with respect to parameters and to grid size. These results are not explained by the analysis.

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