Comparison of time domain maximum likelihood method and sample maximum likelihood method in errors-in-variables identification

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Abstract

The time domain maximum likelihood (TML) method and the Sample Maximum Likelihood (SML) method are two general approaches for identifying errors-in-variables models. In the TML method, an important assumption is that the noise-free input signal must be a stationary process with rational spectrum. For SML, the noise-free input needs to be periodic. In this report, numerical comparisons of these two methods are done under different situations. The results suggest that TML and SML have similar estimation accuracy at moderate or high signal-to-noise ratio (SNR).

1 Introduction

The dynamic errors-in-variables (EIV) identification problem has been a topic of active research for several decades. Till now, many solutions have been proposed with different approaches. For example, the Koopmans-Levin (KL) method [2], the Frisch scheme [1], the prediction error method [7], frequency domain methods [4], and methods based on higher order moments statistics [11], *etc*. See [10], [8] and references therein for comprehensive surveys in this respect. Among the available methods, the time domain maximum likelihood method (TML), also called the joint output approach, [7] and the sample maximum likelihood (SML) method [6] are two general approaches that can accommodate cases with rather arbitrarily correlated noises and provide good estimation accuracy.

In general, the TML method and the SML method work under different experimental situations. An essential assumption for the TML method is that the noise-free input signal is a stochastic stationary process with rational spectrum, so that it can be described as an ARMA process. Also, in the TML method, the input and output noises are usually described as ARMA processes. In contrast, the SML method works under arbitrary noise-free input signals and noise conditions, but with another necessary assumption: the noise-free signal is periodic. However, for a condition suitable for both approaches, which method gives the best estimates? Till now, the relation between the TML and SML methods is still an interesting but open problem.

In this report, we will numerically compare the two methods under different cases, such as, different order dynamic systems, distinct input signals, white or colored input and output noises, varied signal-to-noise ratios (SNR) etc. Based on these results, more complicated theoretical comparisons might be attempted in the future.

2 Problem statement

Let the noise-free input and output processes $u_0(t)$ and $y_0(t)$ be linked by a linear stable, discrete-time, dynamic system

$$A(q^{-1})\, y_o(t) = B(q^{-1})\, u_o(t), \tag{2.1}$$

where

$$\begin{aligned} A(q^{-1}) &= 1 + a_1\, q^{-1} + \ldots + a_{na}\, q^{-na} \\ B(q^{-1}) &= b_0 + b_1\, q^{-1} + \ldots + b_{nb}\, q^{-nb} \end{aligned} \tag{2.2}$$

are polynomials in the backward shift operator q^{-1}.

For errors-in-variables systems, the input and the output are measured with additive noises:

$$\begin{aligned} u(t) &= u_o(t) + \tilde{u}(t), \\ y(t) &= y_o(t) + \tilde{y}(t), \end{aligned} \tag{2.3}$$

where $\tilde{u}(t)$ and $\tilde{y}(t)$ are assumed to be mutually independent zero mean noise sequences both independent of $u_o(t)$ [1].

Problem: The task is to consistently estimate the system parameter vector

$$\theta = (a_1 \ldots a_{na} \quad b_0 \ldots b_{nb})^T \tag{2.4}$$

from the measured noisy data $\{u(t), y(t)\}_{t=1}^{N}$.

3 Review of the TML method

In the TML method, we consider the EIV system as a multivariable system with both $u(t)$ and $y(t)$ as outputs. An important assumption for this method is that the signal $u_o(t)$ is stationary with rational spectrum, so that $u_o(t)$ can be described as an ARMA process of the type

$$u_o(t) = \frac{C(q^{-1})}{D(q^{-1})}\; e(t)\;, \tag{3.1}$$

[1] For SML, $\tilde{u}$ and $\tilde{y}$ might be correlated. TML can also be extended to accommodate cases with rather arbitrarily correlated noises.

where $e(t)$ is a white noise with variance λ_e and the polynomials $C(q^{-1})$, $D(q^{-1})$ are relatively prime and asymptotically stable, with known degrees.

In this way the whole errors-in-variables model can be rewritten as a system with a two-dimensional output vector $z(t) = (u(t)\, y(t))^T$ and three mutually uncorrelated white noise sources $e(t)$, $\tilde{u}(t)$ and $\tilde{y}(t)$:

$$\begin{pmatrix} u(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} \frac{C(q^{-1})}{D(q^{-1})} & 1 & 0 \\ \frac{B(q^{-1})C(q^{-1})}{A(q^{-1})D(q^{-1})} & 0 & 1 \end{pmatrix} \begin{pmatrix} e(t) \\ \tilde{u}(t) \\ \tilde{y}(t) \end{pmatrix} . \tag{3.2}$$

By transforming the model to a general state space model and converting into the innovations form, we will get

$$z(t) = S(q^{-1}, \vartheta)\, \varepsilon(t, \vartheta) \;, \tag{3.3}$$

where $S(q^{-1})$ is a stable transfer function matrix which can be computed from the Riccati equation for Kalman filters, and $\varepsilon(t, \vartheta)$ is the prediction error $\varepsilon(t, \vartheta) = z(t) - \hat{z}(t|t-1; \vartheta)$, which depends on the data and the model matrices. Note that the parameter vector ϑ contains not only the system parameters θ but also the noise parameters and parameters of $u_o(t)$, *i.e.* the coefficients of the polynomials C and D.

The parameter vector ϑ is consistently estimated from a data sequence $z(t)_{t=1}^{N}$ by minimizing the loss function:

$$\hat{\vartheta}_N = \arg\min_{\vartheta} \frac{1}{N} \sum_{t=1}^{N} \ell(\varepsilon(t, \vartheta), \vartheta, t) \tag{3.4}$$

with

$$\ell(\varepsilon(t,\vartheta), \vartheta, t) = \frac{1}{2} \log\det Q(\vartheta) + \frac{1}{2} \varepsilon^T(t, \vartheta) Q^{-1}(\vartheta) \varepsilon(t,\vartheta) \tag{3.5}$$

where $Q(\vartheta)$ denotes the covariance matrix of the prediction errors. For Gaussian distributed data, the covariance matrix of the TML estimates parameters turns out to be asymptotically ($N \to \infty$) equal to the Cramér-Rao bound [9].

4 Review of the SML method

The ML estimate can also be computed in the frequency domain [5]. Let $U(w_k)$ and $Y(w_k)$, with $w_k = 2\pi k/N$, $k = 1, \ldots, N$, denote the discrete Fourier transforms of the input and output measurements, respectively. Write the transfer function as $G(e^{iw_k}) = B(e^{iw_k})/A(e^{iw_k})$ (there is no need to assume that A is stable as long as the system has stationary input and output signals, e.g. an unstable plant captured in a stabilizing feedback loop). The ML criterion in the frequency domain can be written as

$$\begin{aligned} V(\theta) \;\; = \;\; & \frac{1}{N} \sum_{k=1}^{N} |B(e^{iw_k}, \theta) U(w_k) - A(e^{iw_k}, \theta) Y(w_k)|^2 \\ & \times \{\sigma_U^2(w_k) |B(e^{iw_k}, \theta)|^2 + \sigma_Y^2(w_k) |A(e^{iw_k}, \theta)|^2 \\ & -2Re[\sigma_{YU}(w_k) |A(e^{iw_k}, \theta) B(e^{-iw_k}, \theta)]\}^{-1} \end{aligned} \tag{4.1}$$

where $\sigma_U^2(w_k)$, $\sigma_Y^2(w_k)$ and $\sigma_{YU}(w_k)$ are the variance or covariance of the input and output noise at frequency w_k, respectively. If these (co)variances of the noise are known a priori, it is easy to minimize the cost function (4.1) to get good estimates. However, knowing exactly the noise model is not realistic in many practical cases. Then we have to consider the (co)variances of the noises as additional parameters which should also be estimated from the data. In this case, a high dimensional nonlinear optimization problem should be solved, which leads to infeasible situations. Instead of doing so, another way is to replace the exact covariance matrices of the disturbances by sample estimates obtained from a small number (M) of repeated experiments. This is the fundamental idea of the sample ML method. An important assumption is utilized in SML, namely to have periodic excitation signals, where each period plays the role of an independent repeated experiment. The definitions for the sample (co)variances of $\hat{\sigma}_U^2(w_k), \hat{\sigma}_Y^2(w_k)$, and $\hat{\sigma}_{YU}(w_k)$ are:

$$
\begin{aligned}
\hat{\sigma}_U^2(w_k) &= \frac{1}{M-1}\sum_{l=1}^{M} |U_l(w_k) - \bar{U}(w_k)|^2, \\
\hat{\sigma}_Y^2(w_k) &= \frac{1}{M-1}\sum_{l=1}^{M} |Y_l(w_k) - \bar{Y}(w_k)|^2, \\
\hat{\sigma}_{YU}(w_k) &= \frac{1}{M-1}\sum_{l=1}^{M}\big(Y_l(w_k)-\bar{Y}(w_k)\big)\big(U_l(w_k)-\bar{U}(w_k)\big)^*;
\end{aligned}
$$

where $*$ indicates the complex conjugate and $\bar{U}(w_k)$ and $\bar{Y}(w_k)$ denote the sample mean of input and output which are similarly defined as:

$$
\bar{U}(w_k) = \frac{1}{M}\sum_{l=1}^{M} U_l(w_k), \quad \bar{Y}(w_k) = \frac{1}{M}\sum_{l=1}^{M} Y_l(w_k)\ .
$$

The parameter vector θ is estimated by minimizing,

$$
\begin{aligned}
\bar{V}(\theta) = \frac{1}{N}\sum_{k=1}^{N} & |B(e^{iw_k},\theta)\bar{U}(w_k) - A(e^{iw_k},\theta)\bar{Y}(w_k)|^2 \\
& \times\{\hat{\sigma}_{\bar{U}}^2(w_k)|B(e^{iw_k},\theta)|^2 + \hat{\sigma}_{\bar{Y}}^2(w_k)|A(e^{iw_k},\theta)|^2 \\
& -2Re[\hat{\sigma}_{\bar{Y}\bar{U}}(w_k)|A(e^{iw_k},\theta)B(e^{-iw_k},\theta)]\}^{-1}\ ,
\end{aligned} \tag{4.2}
$$

where $\hat{\sigma}_{\bar{U}}^2 = \hat{\sigma}_U^2/M$, $\hat{\sigma}_{\bar{Y}}^2 = \hat{\sigma}_Y^2/M$ and $\hat{\sigma}_{\bar{Y}\bar{U}} = \hat{\sigma}_{YU}/M$. The cost function (4.2) is an approximation of (4.1) by replacing the exact covariances of the noise by their sample estimates. The major advantage of this approach is that the plant parameters remain as the only unknowns to be estimated, which leads to a low dimension of the nonlinear optimization problem.

It is clear that this SML estimator is no longer an exact ML estimator. However, it was proved in [6] that the estimator is consistent if the number of experiments $M \geq 4$, and compared to the frequency domain maximum likelihood estimator assuming known noise variances, the loss in efficiency of SML is $(M-2)/(M-3)$ for $M \geq 6$, which is not a large factor even for small values of M.

5 Comparisons between TML and SML estimates

From the reviews above, it can be seen that the TML method and the SML method work under different assumptions. We assume here that NM periodic data are available, where M is the number of periods and N denotes the number of data points in each period. Also assume that in each period the noise-free input signal is one and the same realization of a stationary process. This experimental condition is suitable for both approaches.

The TML method uses all data points and the information that the input signal is an ARMA process, but does not exploit the periodicity of the data. However, the SML method uses the periodic information but disregards that the input signal is an ARMA process and does not use any parametric models of the noise terms. For comparison, we also give the asymptotic covariance matrix of the frequency domain maximum likelihood (FML) method calculated under the assumption of knowing the input-output noise variances and the period information, but using no assumptions on the input signals. See [6].

In the following subsections, we will show the numerical comparison results of the TML and SML methods for different circumstances, such as dynamic systems with different orders, different noise-free input signals, varied signal-to-noise ratios (SNR), and more general noise conditions. Although many systems with different orders and parameters have been tested, in order to save space, only a second order system and a sixth order system results are illustrated as representatives in this report. The polynomials of this second order system are

$$\begin{aligned} A(q^{-1}) &= 1 - 1.5\,q^{-1} + 0.7\,q^{-2} \\ B(q^{-1}) &= 2 + 1.0\,q^{-1} + 0.5\,q^{-2} \end{aligned} \tag{5.1}$$

and the sixth order system is

$$\begin{aligned} A(q^{-1}) &= 1 - 1.1\,q^{-1} + 0.5\,q^{-2} - 0.12\,q^{-3} \\ &\quad +0.23\,q^{-4} - 0.235\,q^{-5} + 0.175\,q^{-6}, \\ B(q^{-1}) &= 0.1 + 1.0\,q^{-1} + 0.85\,q^{-2} + 0.06\,q^{-3} \\ &\quad -0.534\,q^{-4} + 0.504\,q^{-5} + 0.324\,q^{-6}. \end{aligned} \tag{5.2}$$

The polynomials of the noise-free input signal model are

$$\begin{aligned} C(q^{-1}) &= 1 + 0.5\,q^{-1} \\ D(q^{-1}) &= 1 - 0.5\,q^{-1}. \end{aligned} \tag{5.3}$$

All the comparisons are based on the asymptotic case where the data number N is assumed to be large enough, and we assume $M = 6$ periods data are available. The standard deviations (std) are calculated from the theoretical covariance matrices of the estimation parameters, which have been proved to well meet their relevant Monte-Carlo simulations. Details on these formulas can be found in [9], [5], [6] and [3].

5.1 Comparison of dynamic systems with different order

Firstly, the estimation accuracy of TML and SML methods for systems with different system orders was analyzed. Comparison of the results show that for low order systems, the estimation accuracy of SML and TML method are quite similar. See Figure 1. For high order systems, in regions where the signal-to-noise ratio (SNR) is poor, the std of the TML method is larger than that of the SML method. See Figure 2. In Figure 3, new comparison results under the same condition as in Figure 2 are shown except adding the periodic information to TML by simple averaging of the data over the M periods. It can be seen that the difference of TML and SML estimation results in the low SNR area has disappeared. To show the comparisons more clearly the std of estimates by using different methods are plotted with linear scaling in Figure 4. According to these results, it seems that using the periodic information improve the TML estimates considerably for high order dynamic system, especially when the SNR is low.

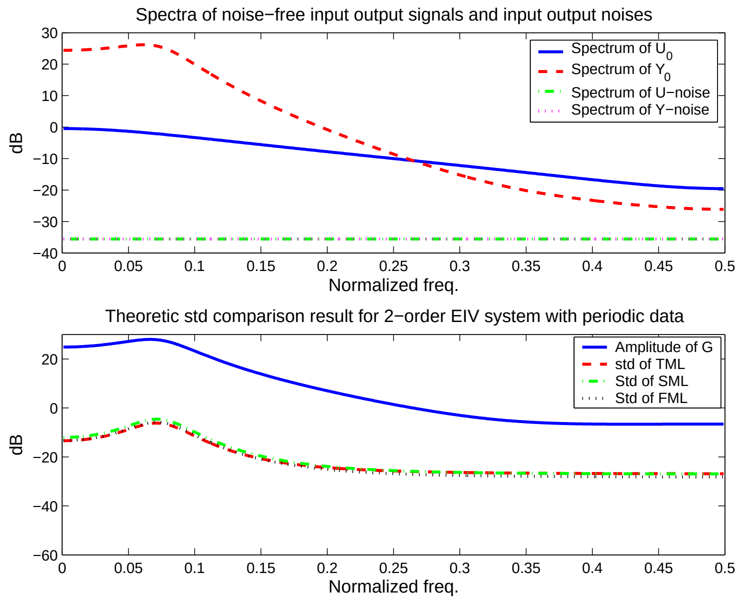

Figure 1: Spectra of noise free signals and noises (upper part), and comparison of standard deviation of the estimates of TML, SML and FML of a second order system with white measurement noise (lower part).

5.2 Influence of different excitation signals

Secondly, different input signals were tried. The results show that for high order systems the input signal seems more influential than for low order systems. The difference between the estimation accuracy of the TML and SML methods can be observed more easily for high order dynamic systems. See Figure 5 and Figure 6. They show the comparison of TML and SML estimates with different input signals for a second and sixth order dynamic EIV system, respectively.

5.3 Influence of colored output noise

Several examples with colored output measurement noises were also studied. They give similar results as for the white noise cases, *i.e.*, if the periodic information has not been utilized, the std of the TML method is similar to that of the SML method except at the very low SNR regions, where the estimation uncertainty of TML is larger than that of SML. This phenomenon is more pronounced for high order dynamic systems. See

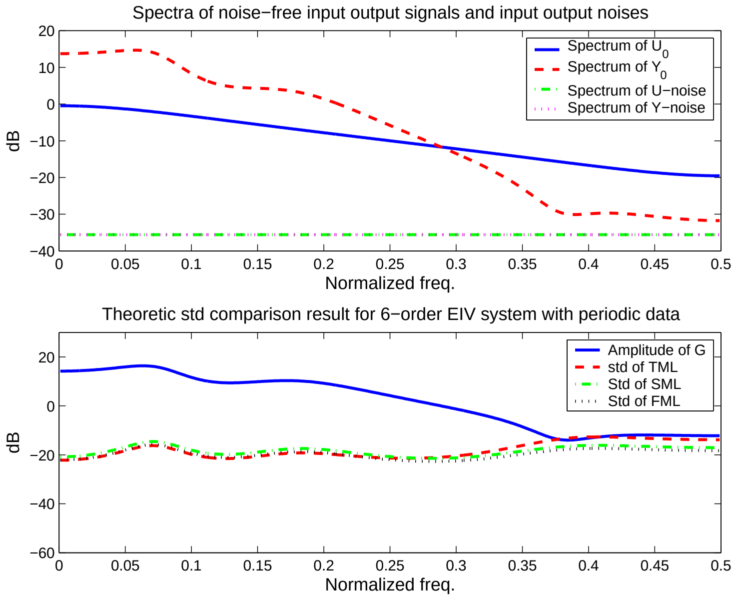

Figure 2: Spectra of noise-free signals and noises (upper part), and comparison of standard deviation of the estimates of TML, SML and FML of a sixth order system with white measurement noise (lower part).

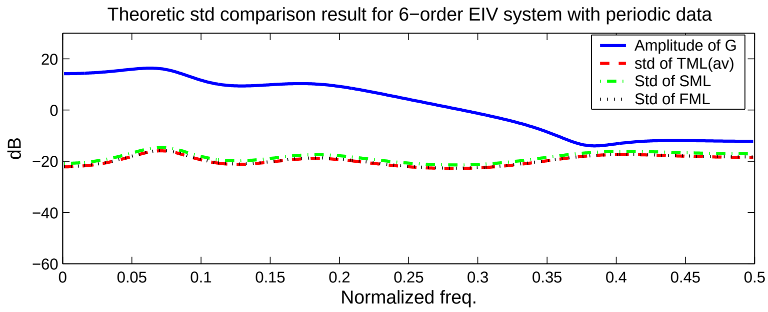

Figure 3: Comparison of the standard deviation of the estimates of TML (with averaging of the data), SML and FML under the same condition as in Figure 2.

Figures 7 and 8. Again, similar to the white noise case, the difference between TML and SML at low SNR will vanishes if the periodic information is utilized for TML. See Figures 9.

5.4 Influence of SNR

The above comparisons are made when the noise variance of $e(t)$ is $\lambda_e = 0.1$, and the noise variance at both input and output sides are equal to 0.1 (for white noises cases). In this subsection, we will further analyze the behavior of the TML method and the SML method for other SNR values.

As shown in Table 1, we fix the variance of $e(t)$ to 0.1 and let the variance of the input noise, $\lambda_{\tilde{u}}^2$, vary from 0.01, 0.1 to 1 which corresponds to the SNR at the input side equal to 27.36dB, 7.36dB and -12.64dB, respectively. The variance of the output noise, $\lambda_{\tilde{y}}^2$, is chosen as 0.01, 0.1, 1, 10 and 100. This defines fifteen cases marked from I to XV. The estimation results of all these cases are shown in Figures 10-24 for the second

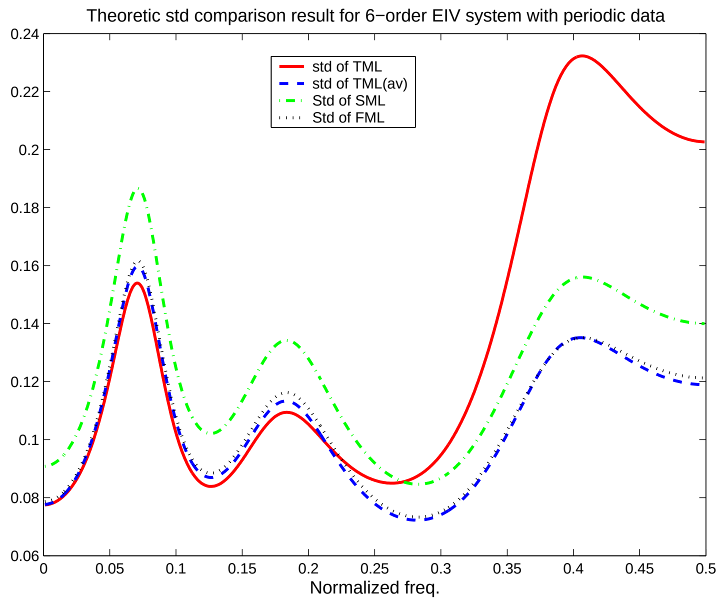

Figure 4: Comparison of the standard deviation of the estimates of TML, TML(with averaging of the data), SML and FML.

order system (5.1) and the sixth order system (5.2). The output SNR equals to 76.27dB, 56.27dB, 36.27dB, 16.27dB and -3.73dB for the second order system, and 54.43dB, 34.43dB, 14.43dB, -5.57dB and -25.57dB for the sixth order system, respectively.

Table 1: Fifteen cases with different variances of the input and output noise

	$\lambda_{\tilde{u}}^2 = 0.01$	$\lambda_{\tilde{u}}^2 = 0.1$	$\lambda_{\tilde{u}}^2 = 1$
$\lambda_{\tilde{y}}^2 = 0.01$	I	VI	XI
$\lambda_{\tilde{y}}^2 = 0.1$	II	VII	XII
$\lambda_{\tilde{y}}^2 = 1$	III	VIII	XIII
$\lambda_{\tilde{y}}^2 = 10$	IV	IX	XIV
$\lambda_{\tilde{y}}^2 = 100$	V	X	XV

Comparison results show that when the SNR at both the input and the output side are high, such as in cases I, II, III and IV, the two methods always give very similar performance both for low and high order systems. When the SNR becomes low, in cases V, IX, X, XIV and XV, the benefit of using periodic information in the SML method is more pronounced, which results in the SML having a lower covariance matrix than that of the TML. For moderate SNR, cases VI, VII and VIII, we will get similar results as in the subsection before, *i.e.*, the difference of the TML method and the SML method are observable only in the low SNR frequency regions especially for the high order dynamic systems. It is interesting to notice that when the input SNR is very low, as long as the output SNR is not too small (larger than 0 dB), the estimation accuracy by using TML is better than that of the SML method. See the middle parts of the Figures 20, 21 and 22 for cases XI, XII and XIII. The study of these low SNR cases suggests that neither of the two methods, TML or SML, is uniformly better than another when the noise is large. Besides, when TML uses the averaging data, the difference between the estimates accuracy of the two methods always disappears. See the lower parts of Figures 10-24.

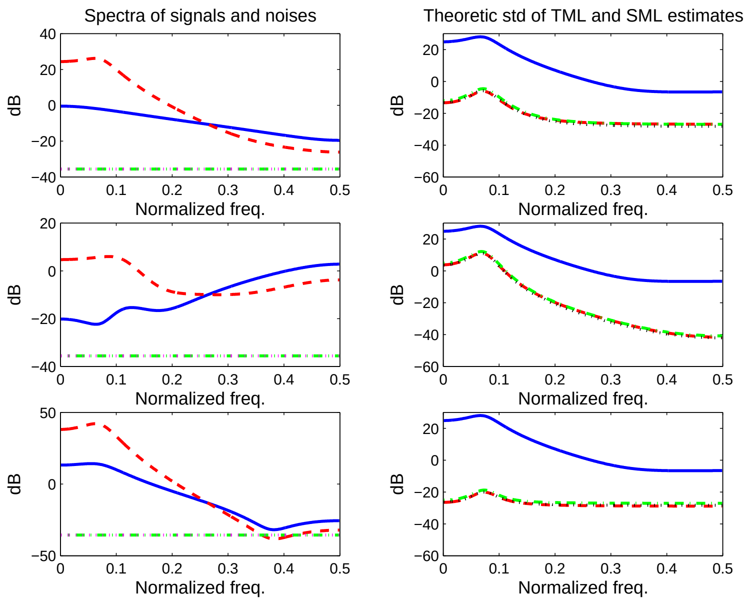

Figure 5: Comparison of TML and SML estimates for different input signals applied to a second order dynamic EIV system. The left three plots are the spectra of the noise free input signals (solid line), the noise free output signals (dashed line) and the noises (dotted line), where both input and output noise are white Gaussian noises with the same variance. The right three plots are the amplitude of the system G (solid line), the standard deviation of the estimates of TML (dashed line), SML (dash-dotted line) and FML (dotted line).

6 Concluding Remarks

In this paper, the asymptotic covariance matrices of the TML method and the SML method for estimating the EIV systems have been numerically analyzed and compared. Based on a large number of comparisons for different cases, it was shown that the two methods have similar estimation accuracy as long as the SNR at both input and output sides are not very low. The notable accuracy difference can be observed at low SNR regions especially for high order dynamic systems. The smaller the SNR, the more distinct this phenomenon becomes. When the SNR is very low (less than 0 dB), it seems that the benefit of using the periodic information is more important than knowing that both the signals and noises have rational spectra. From the efficiency point of view, SML and TML with the data averaged over the periods have similar estimation accuracy (TML is slightly better).

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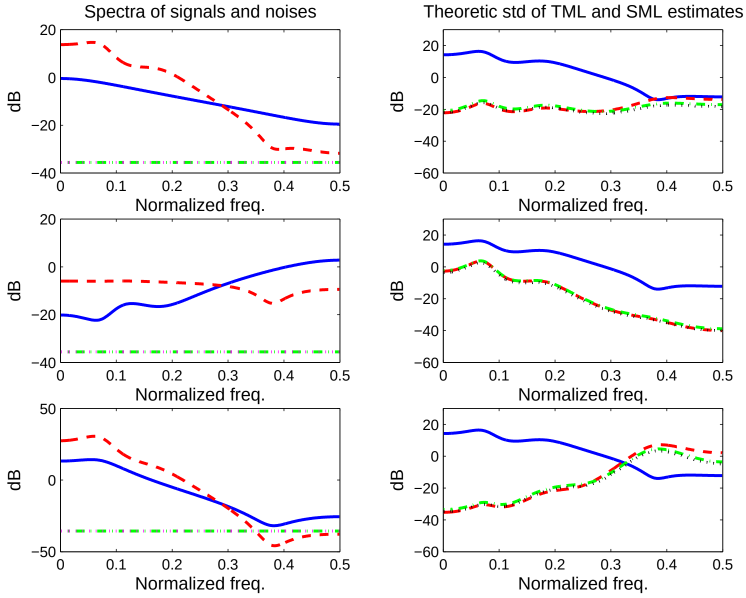

Figure 6: Comparison of TML and SML estimates for different input signals applied to a sixth order dynamic EIV system. See caption of Figure 5 for the detail descriptions.

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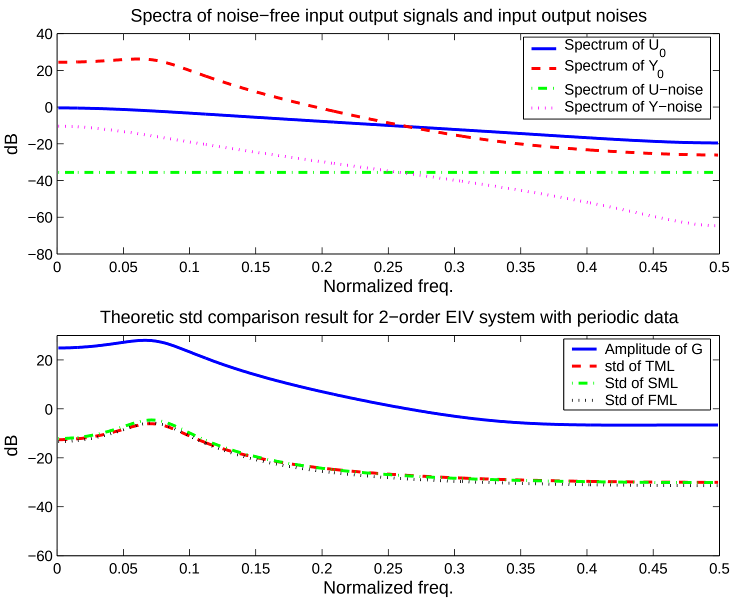

Figure 7: Spectra of the noise-free signals and noises (upper part), and comparison of standard deviation of the estimates of TML, SML and FML of a second order system with the white input noise and the colored output noise (lower part).

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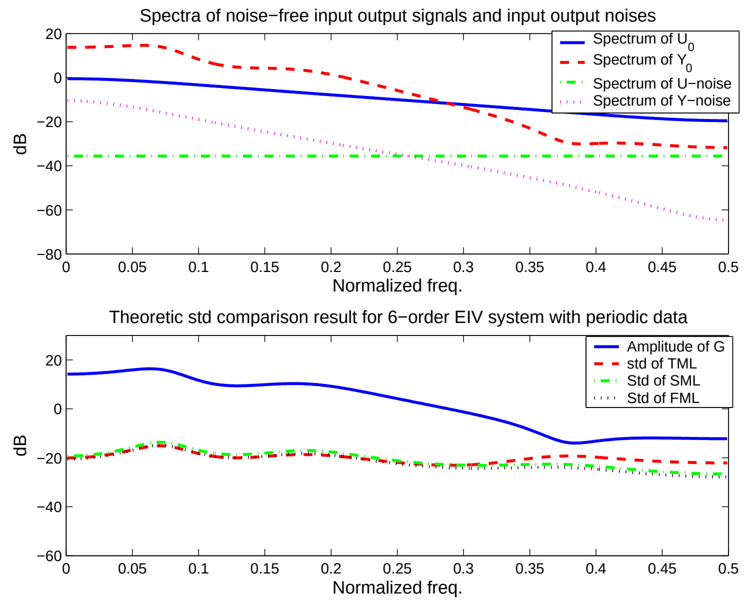

Figure 8: Spectra of the noise-free signals and noises (upper part), and comparison of standard deviation of the estimates of TML, SML and FML of a sixth order system with the white input noise and the colored output noise (lower part).

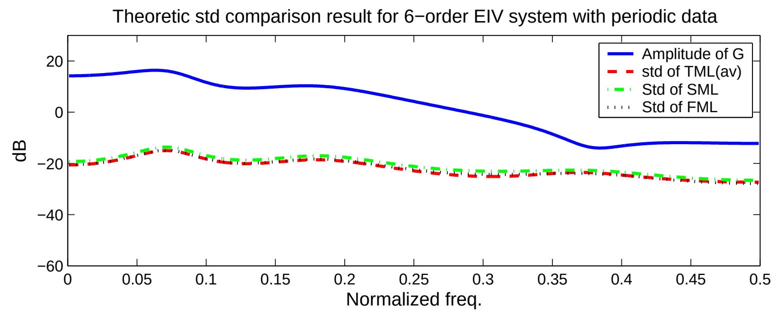

Figure 9: Comparison of the standard deviation of the estimates of TML (with averaging of the data), SML and FML under the same condition as in Figure 8.

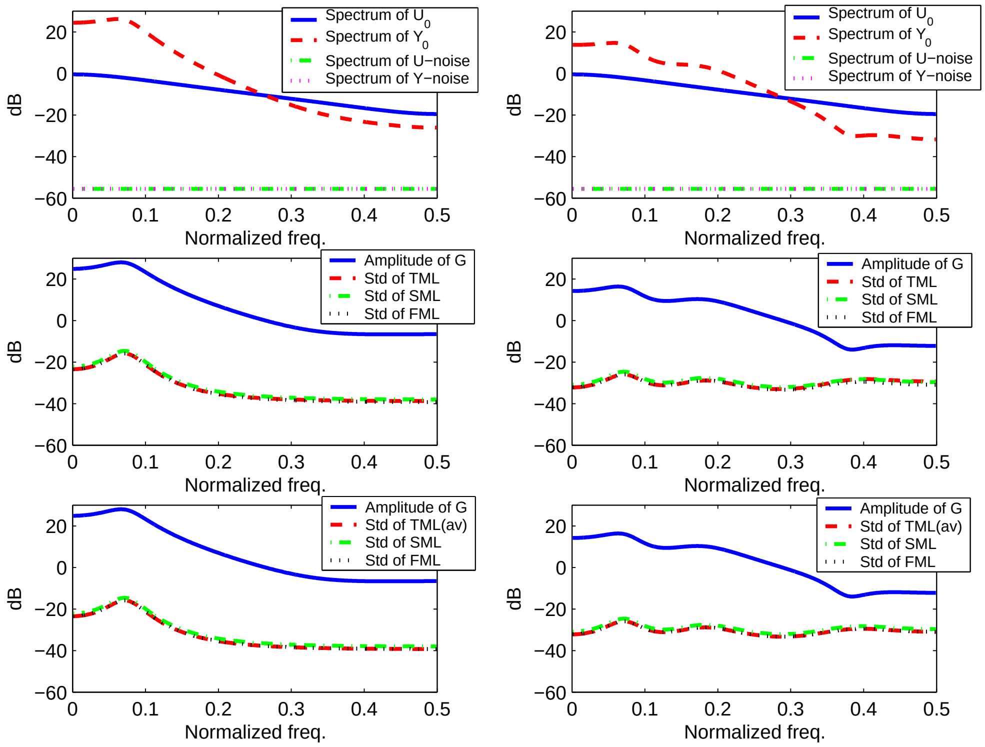

Figure 10: Comparison of the TML (without or with averaging of the data), SML and FML estimates for a second (left) and a sixth (right) order system with $\lambda_{\tilde{u}}^2 = 0.01$ and $\lambda_{\tilde{y}}^2 = 0.01$, i.e. case I.

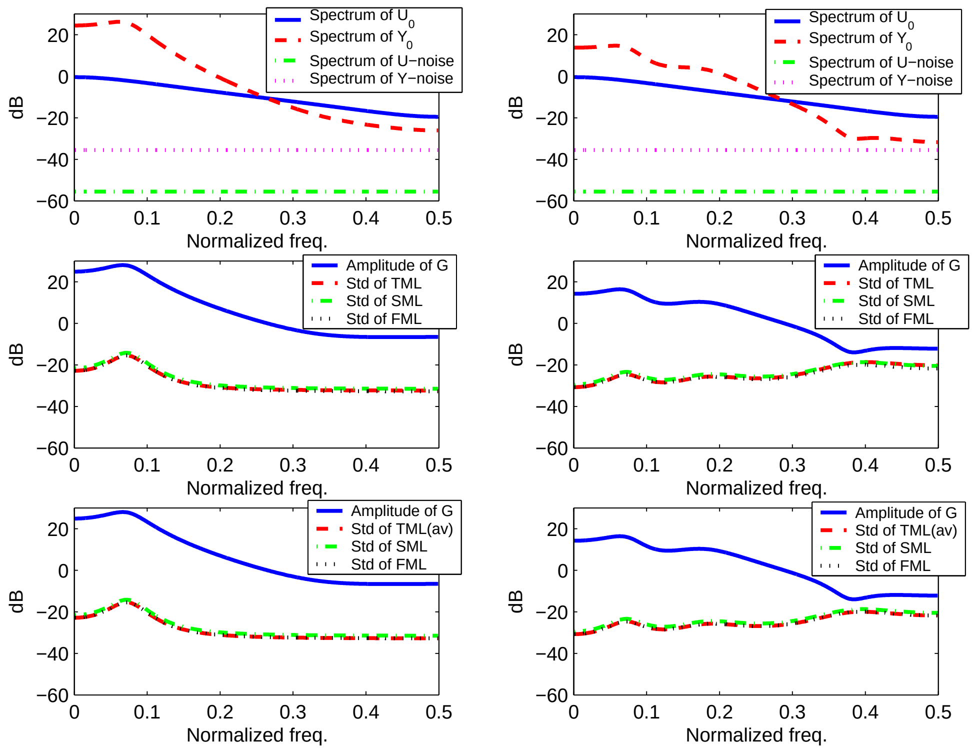

Figure 11: Comparison of the TML (without or with averaging of the data), SML and FML estimates for a second (left) and a sixth (right) order system with $\lambda_{\tilde{u}}^2 = 0.01$ and $\lambda_{\tilde{y}}^2 = 0.1$, i.e. case II.

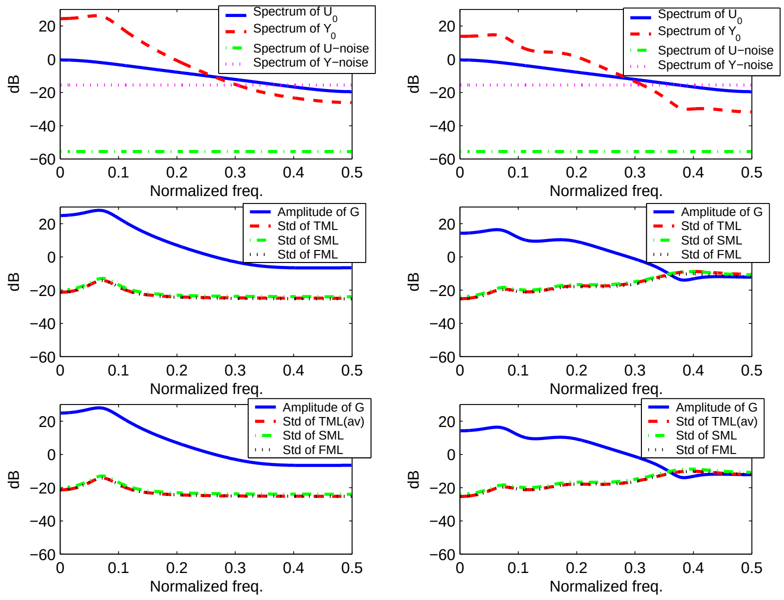

Figure 12: Comparison of the TML (without or with averaging of the data), SML and FML estimates for a second (left) and a sixth (right) order system with $\lambda_{\tilde{u}}^2 = 0.01$ and $\lambda_{\tilde{y}}^2 = 1$, i.e. case III.

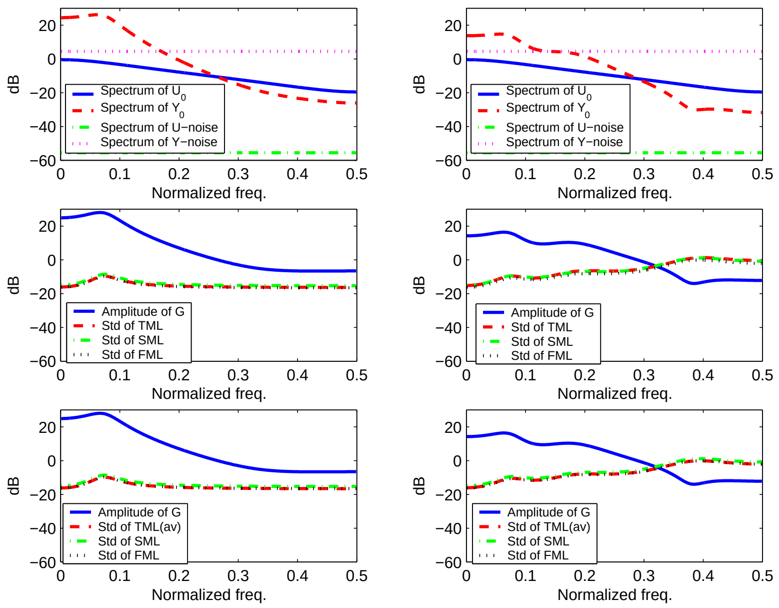

Figure 13: Comparison of the TML (without or with averaging of the data), SML and FML estimates for a second (left) and a sixth (right) order system with $\lambda_{\tilde{u}}^2 = 0.01$ and $\lambda_{\tilde{y}}^2 = 10$, i.e. case IV.

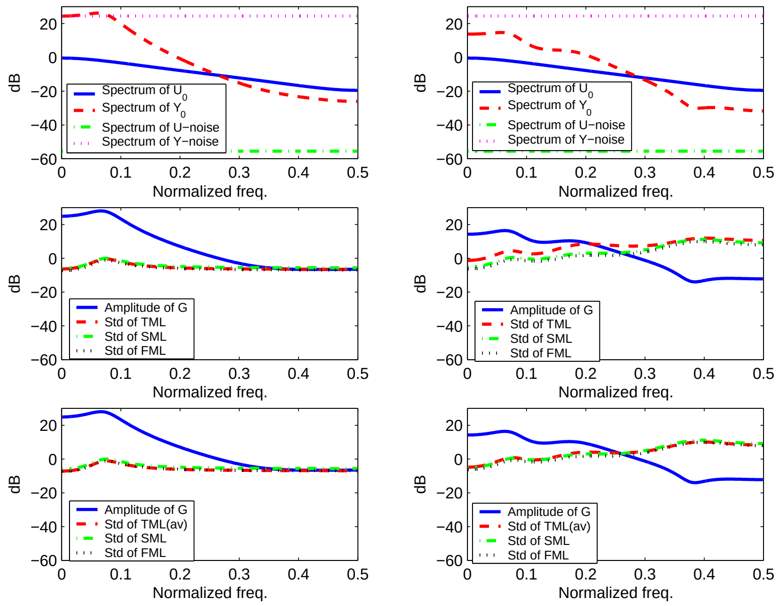

Figure 14: Comparison of the TML (without or with averaging of the data), SML and FML estimates for a second (left) and a sixth (right) order system with $\lambda_{\tilde{u}}^2 = 0.01$ and $\lambda_{\tilde{y}}^2 = 100$, i.e. case V.

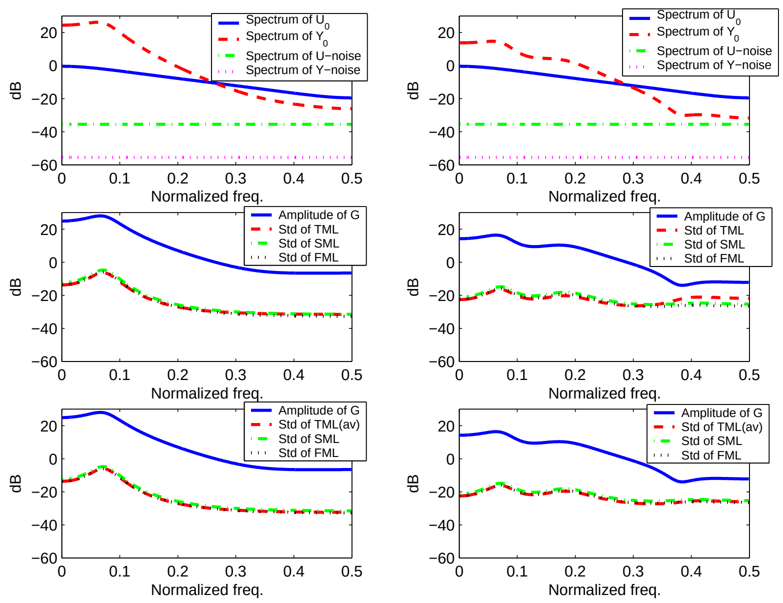

Figure 15: Comparison of the TML (without or with averaging of the data), SML and FML estimates for a second (left) and a sixth (right) order system with $\lambda_{\tilde{u}}^2 = 0.1$ and $\lambda_{\tilde{y}}^2 = 0.01$, i.e. case VI.

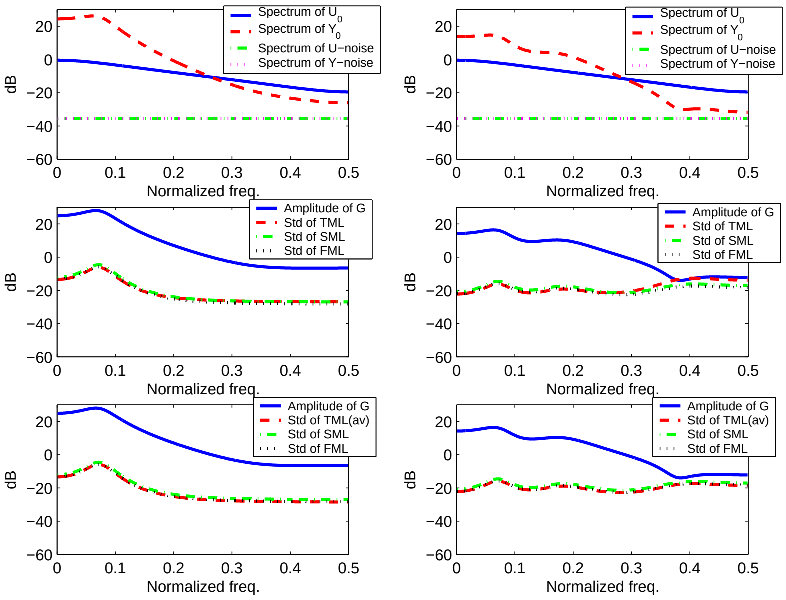

Figure 16: Comparison of the TML (without or with averaging of the data), SML and FML estimates for a second (left) and a sixth (right) order system with $\lambda_{\tilde{u}}^2 = 0.1$ and $\lambda_{\tilde{y}}^2 = 0.1$, i.e. case VII.

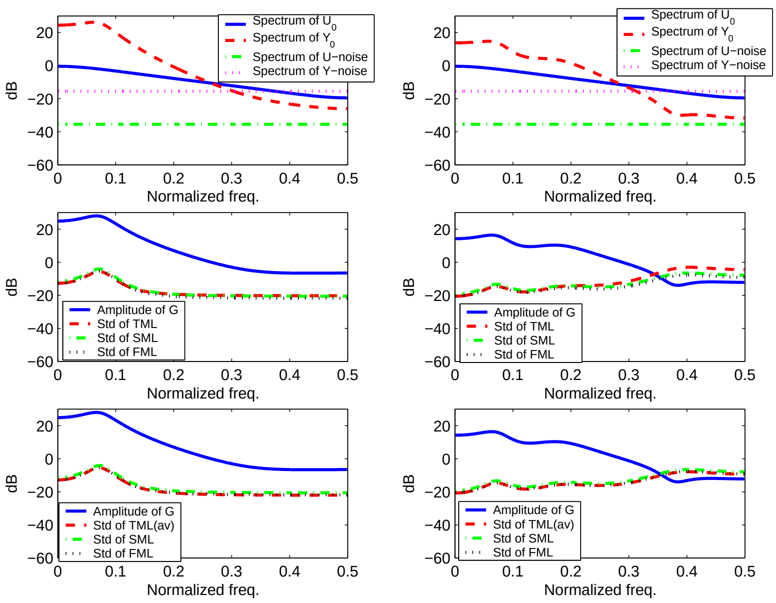

Figure 17: Comparison of the TML (without or with averaging of the data), SML and FML estimates for a second (left) and a sixth (right) order system with $\lambda_{\tilde{u}}^2 = 0.1$ and $\lambda_{\tilde{y}}^2 = 1$, i.e. case VIII.

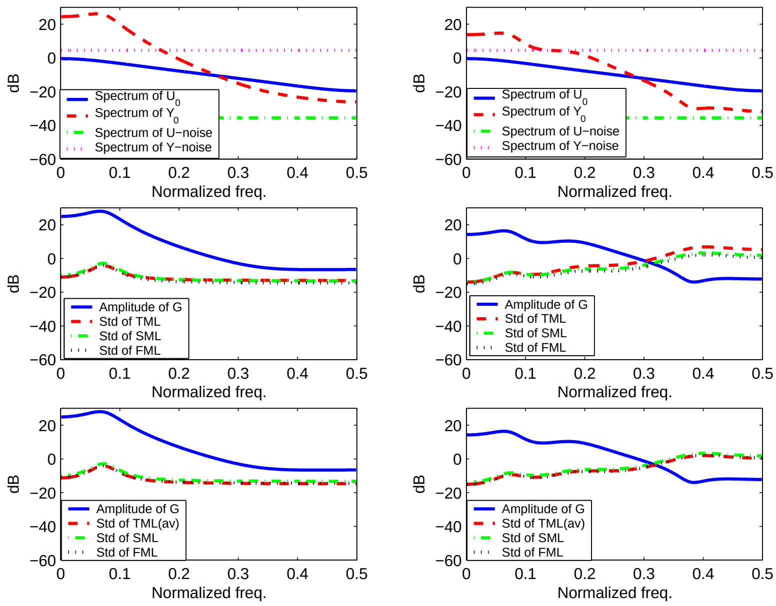

Figure 18: Comparison of the TML (without or with averaging of the data), SML and FML estimates for a second (left) and a sixth (right) order system with $\lambda_{\tilde{u}}^2 = 0.1$ and $\lambda_{\tilde{y}}^2 = 10$, i.e. case IX.

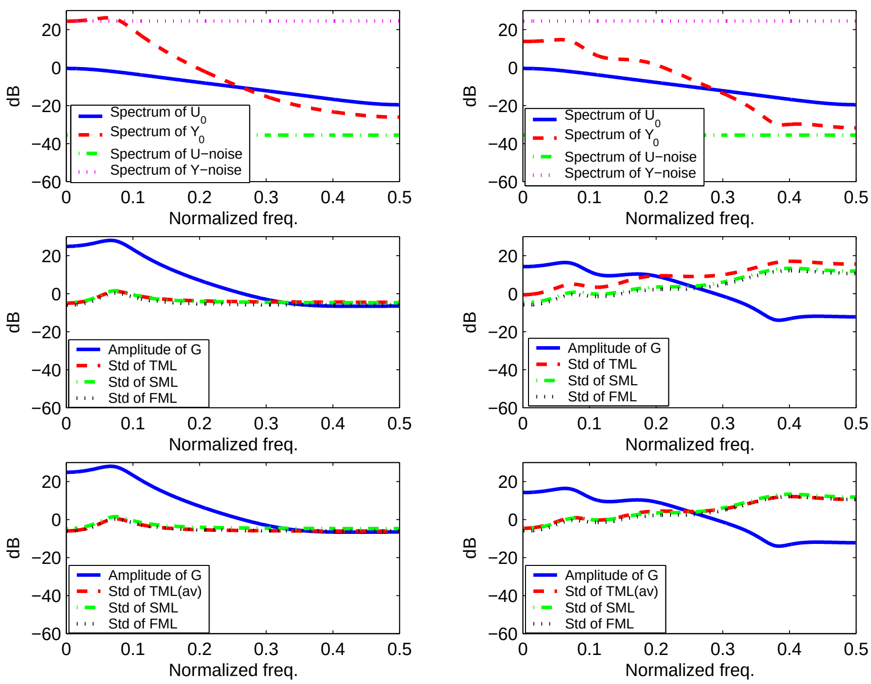

Figure 19: Comparison of the TML (without or with averaging of the data), SML and FML estimates for a second (left) and a sixth (right) order system with $\lambda_{\tilde{u}}^2 = 0.1$ and $\lambda_{\tilde{y}}^2 = 100$, i.e. case X.

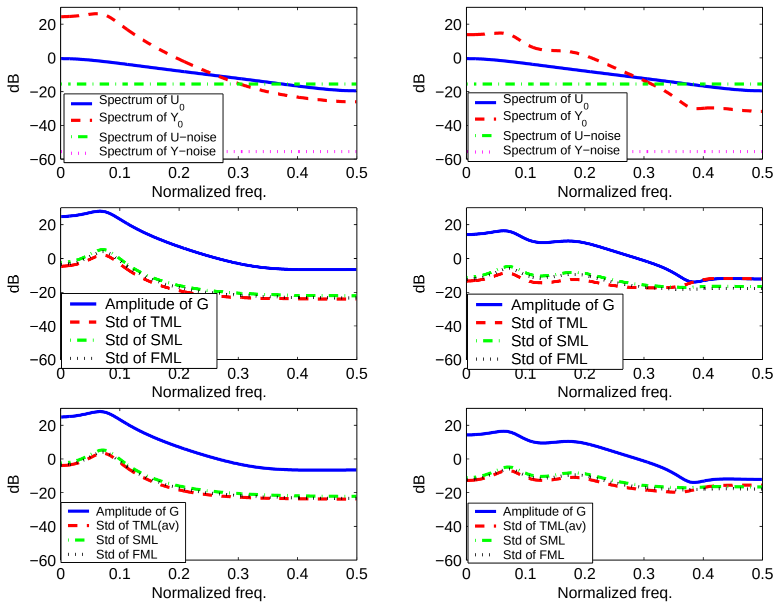

Figure 20: Comparison of the TML (without or with averaging of the data), SML and FML estimates for a second (left) and a sixth (right) order system with $\lambda_{\tilde{u}}^2 = 1$ and $\lambda_{\tilde{y}}^2 = 0.01$, i.e. case XI.

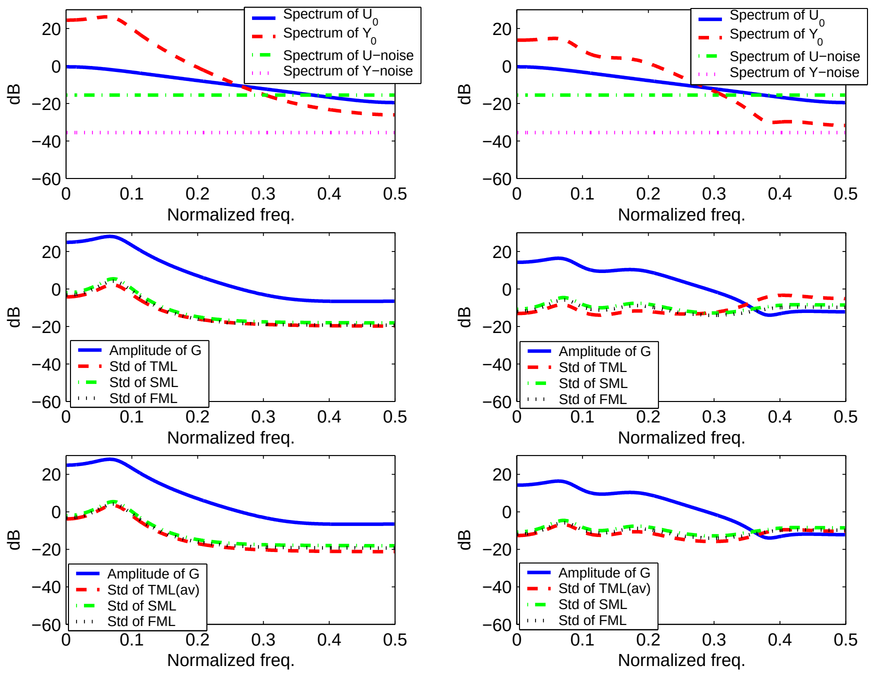

Figure 21: Comparison of the TML (without or with averaging of the data), SML and FML estimates for a second (left) and a sixth (right) order system with $\lambda_{\tilde{u}}^2 = 1$ and $\lambda_{\tilde{y}}^2 = 0.1$, i.e. case XII.

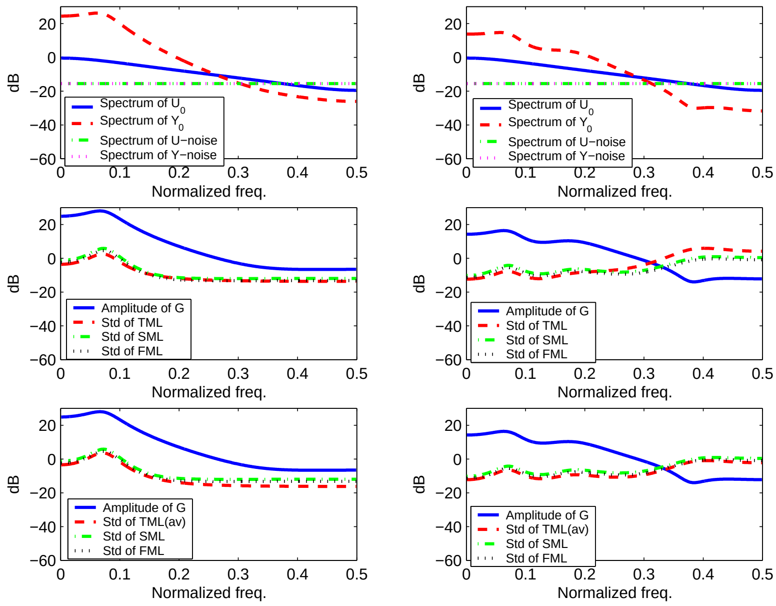

Figure 22: Comparison of the TML (without or with averaging of the data), SML and FML estimates for a second (left) and a sixth (right) order system with $\lambda_{\tilde{u}}^2 = 1$ and $\lambda_{\tilde{y}}^2 = 1$, i.e. case XIII.

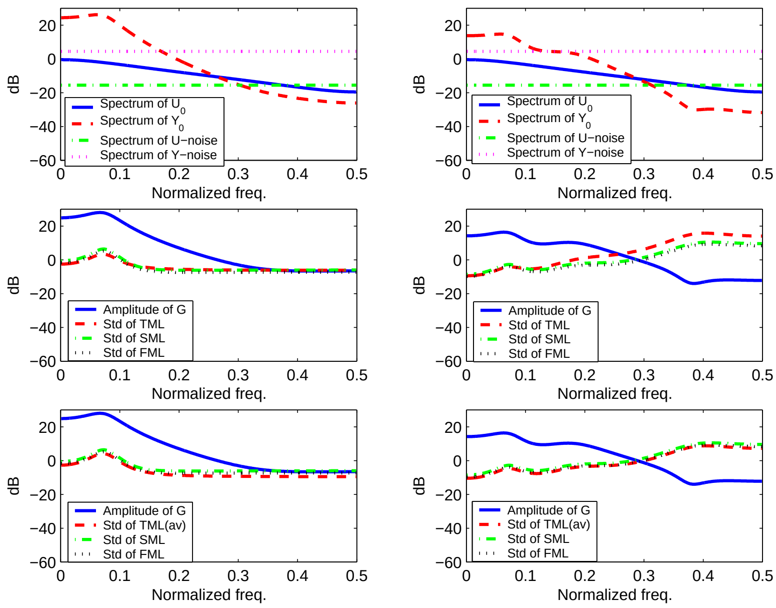

Figure 23: Comparison of the TML (without or with averaging of the data), SML and FML estimates for a second (left) and a sixth (right) order system with $\lambda_{\tilde{u}}^2 = 1$ and $\lambda_{\tilde{y}}^2 = 10$, i.e. case XIV.

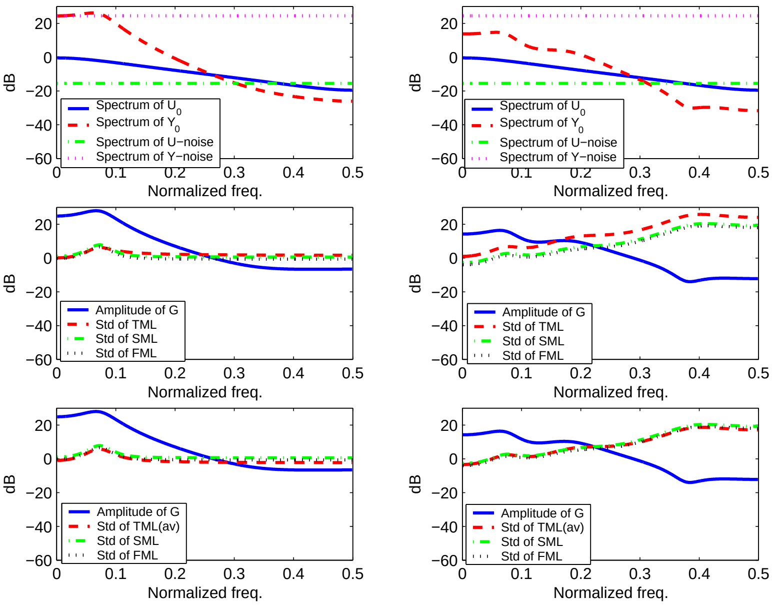

Figure 24: Comparison of the TML (without or with averaging of the data), SML and FML estimates for a second (left) and a sixth (right) order system with $\lambda_{\tilde{u}}^2 = 1$ and $\lambda_{\tilde{y}}^2 = 100$, i.e. case XV.