

# New Input/Output Pairing Strategies based on Minimum Variance Control and Linear Quadratic Gaussian Control

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**Abstract**—In this paper a new input/output pairing strategy based on minimum variance control is proposed. A similar version based on linear quadratic Gaussian (LQG) control is also suggested. The strategies are used to compare the expected performance of decentralized control structures in some illustrative examples. The pairing suggestions are compared with the recommendations previously obtained using other interaction measures such as the Relative Gain Array (RGA). The new strategies give suitable pairing recommendations and are easy to interpret.

**Index Terms**—Decentralized control, interaction measures, multivariable systems.

## I. INTRODUCTION

A common problem for multiple-input multiple-output (MIMO) systems is channel interaction, i.e. when one input affects several outputs. Therefore, the expected level of coupling between the selected control loops is essential to establish prior to the control structure selection. There are today several interaction measures that assist in this selection. The perhaps most commonly used is the Relative Gain Array (RGA) introduced by [3]. Later, several extensions and modifications to the RGA have been introduced, see for instance [18] for a review. Other, more recent interaction measures include the Effective Relative Gain Array (ERGA) [25] and the Gramian-based measures Hankel Interaction Index Array (HIIA) [23] and the Participation Matrix [4], [20]. A  $\mathcal{H}_2$ -norm based interaction measure was introduced by [2] and further analyzed by [7]. One advantage of these measures is that they are not restricted to decentralized control structures. This is a main restriction of the RGA.

In control performance assessment, the lowest possible output variance, the minimum variance, has been a key component in many benchmarks since the work of Harris [8] two decades ago. The main idea is to compare the calculated minimum variance with the actual achieved output variance and thereby get an indication of the current performance of the controller. [10] gives a review of the intense research that has followed in this field. Some of the extensions to the original idea of [8] are made by [9], [12], [14], [24] and [22]. [15] propose performance indices used for control structure selection and performance assessment for disturbance rejection in MIMO processes.

Inspired by the above mentioned work in the field of control performance assessment we here propose a new input/output pairing strategy based on minimum variance (MV)

control. The key idea is to design single-input single-output (SISO) MV controllers for each input/output pairing and thereafter form the closed loop MIMO system. The closed loop performance in terms of the output variance is computed for each control structure and the pairing corresponding to the lowest output variance is selected. As an alternative, we also propose a version based on linear quadratic Gaussian (LQG) control.

The structure of this paper is as follows: In Section II the general idea for the new input/output pairing strategies is outlined. The two-by-two system that is later considered in the theoretical derivations is described in Section III. In Section IV the controllers are designed and the expressions for the full closed loop MIMO systems for decentralized control are derived. The proposed pairing strategies are presented in Section V. In Section VI different interaction measures are compared in three illustrative examples. Finally, the conclusions are drawn in Section VII.

## II. THE GENERAL IDEA

For each elementary SISO subsystem of the full MIMO plant, the corresponding MV controller is derived. The SISO controllers are thereafter combined to form a closed loop expression for the full MIMO system. The loops in the full MIMO system that are not part of the controller design are the cross-couplings.

In the present paper, we restrict the theoretical derivation to two-by-two systems. However, the extension to larger systems is straight-forward and is exemplified in the concluding example where a plant with three inputs and three outputs is analyzed. Note also that the proposed evaluation procedure is not limited to decentralized control structures.

For the design, white unit-variance measurement noise is assumed to be present at the outputs. The process noise is assumed to have the same characteristics. The reference signals are set to zero. Thus the noise sources are the only driving signals. The resulting output variances for the outputs are thereafter calculated and compared for each of the decentralized input-output pairings. The pairing selection that results in the lowest sum of the output variances is the most desirable control structure in a minimum variance sense and is recommended. The channel interaction in the system will manifest itself as an extra contribution to the output variances.

As an alternative, the same procedure is repeated but with the MV controllers replaced by LQG controllers. The minimum variance version can be seen as a version where the lower bound of what is possible to obtain is calculated,

whereas the LQG version can be seen as a more realistic design scenario where the user can select LQ weights of desire. Furthermore, to get stable controllers, the design of MV controllers requires minimum-phase systems. This is not a requirement in the LQG design.

### III. SYSTEM DESCRIPTION

Consider a stable linear discrete-time MIMO system with two inputs,  $u_1$  and  $u_2$ , and two outputs,  $y_1$  and  $y_2$ . The transfer function matrix  $G(q)$  of the system can be partitioned as

$$G(q) = \begin{bmatrix} G_{11}(q) & G_{12}(q) \\ G_{21}(q) & G_{22}(q) \end{bmatrix}. \quad (1)$$

Figure 1 gives a graphical representation of the considered system and its cross-couplings. The subsystems  $G_{12}(q)$  and  $G_{21}(q)$  represents the cross-couplings if  $y_1$  is controlled by  $u_1$  and  $y_2$  by  $u_2$ . The subsystems  $G_{ij}(q)$  are assumed to be strictly proper and can hence be equivalently expressed in state space form represented by the system matrices  $(A_{ij}, B_{ij}, C_{ij}, 0)$ . In the design of the controllers, it is assumed that process noise and measurement noise are present. A state-space representation of the full system is thus given by

$$\begin{aligned} x(t+1) &= A_f x(t) + B_f u(t) + N_f w(t), \\ y(t) &= C_f x(t) + e(t). \end{aligned} \quad (2)$$

where

$$\begin{aligned} x(t) &= \begin{bmatrix} x_{11}(t) \\ x_{12}(t) \\ x_{21}(t) \\ x_{22}(t) \end{bmatrix}, A_f = \begin{bmatrix} A_{11} & 0 & 0 & 0 \\ 0 & A_{12} & 0 & 0 \\ 0 & 0 & A_{21} & 0 \\ 0 & 0 & 0 & A_{22} \end{bmatrix}, \\ B_f &= \begin{bmatrix} B_{11} & 0 \\ 0 & B_{12} \\ B_{21} & 0 \\ 0 & B_{22} \end{bmatrix}, C_f = \begin{bmatrix} C_{11} & C_{12} & 0 & 0 \\ 0 & 0 & C_{21} & C_{22} \end{bmatrix}, \\ y(t) &= [y_1(t) \ y_2(t)]^T. \end{aligned}$$

$\{w(t)\}$  and  $\{e(t)\}$  are white noise sequences with diagonal covariance matrices, i.e. the noise components are assumed to be uncorrelated. Observe that all system matrices may be block matrices. Therefore, the zeros may also be matrices (with zeros in all elements) of appropriate dimensions.

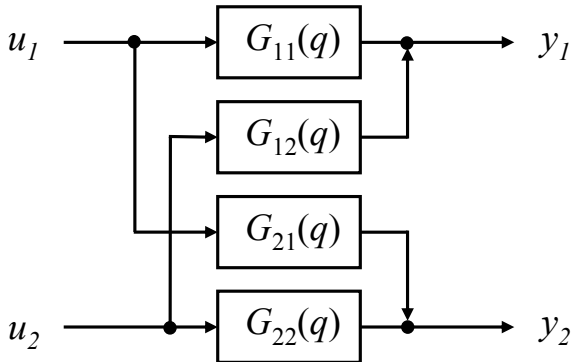


Fig. 1. Block diagram of the considered system.

### IV. CONTROL DESIGN

For each SISO subsystem, both a MV controller and a LQG controller were designed. The subsystems were then connected to form the full closed loop system. Different control structures, i.e. different pairings of the inputs and outputs, give different closed loop expressions for the full system. These are derived in this section.

#### A. Minimum variance (MV) control

The state-space version of minimum variance control for a general SISO system  $(A, B, C, 0)$  is obtained by requiring the  $d$ -step prediction of the output to be zero [17]:

$$\begin{aligned} \hat{y}(t+d|t) &= CA^{d-1}\hat{x}(t+1) \\ &= CA^{d-1}((A-KC)\hat{x}(t) + Bu(t) + Ky(t)) \\ &= 0 \end{aligned} \quad (3)$$

where  $d$  is the number of delay-steps in the considered system,  $\hat{x}(t)$  is the estimate of the state vector  $x(t)$  using measurements up to time instant  $t-1$ :

$$\hat{x}(t+1) = A\hat{x}(t) + Bu(t) + K(y(t) - C\hat{x}(t)). \quad (4)$$

$K$  is the Kalman gain given by

$$K = APC^T(CPC^T + R_e)^{-1} \quad (5)$$

where  $P$  is the positive semidefinite solution to the discrete-time Riccati equation

$$P = APA^T + NR_vN^T - APC^T(CPC^T + R_e)^{-1}CPA^T \quad (6)$$

where  $R_v$  and  $R_e$  are the intensities of the noise  $w$  and  $e$ , respectively. This gives the control law

$$u(t) = -(CA^{d-1}B)^{-1}CA^{d-1}((A-KC)\hat{x}(t) + Ky(t)). \quad (7)$$

#### B. Linear quadratic Gaussian (LQG) control

Optimal linear control for a general SISO system  $(A, B, C, 0)$  is obtained by minimizing the criterion [5]

$$V = E \sum_t x^T(t)Q_1x(t) + u^T(t)Q_2u(t) \quad (8)$$

where  $E$  is the expectation operator. In the following, the first weight in criterion (8) is chosen as  $Q_1 = C^T\tilde{Q}_1C$  in order to penalize the outputs rather than the states. This gives the criterion

$$\tilde{V} = E \sum_t y^T(t)\tilde{Q}_1y(t) + u^T(t)\tilde{Q}_2u(t). \quad (9)$$

The following LQG control law is used:

$$\begin{aligned} u(t) &= -L\hat{x}(t|t) \\ &= -L(I-MC)\hat{x}(t) - LM y(t), \end{aligned} \quad (10)$$

where  $\hat{x}(t)$  is given in (4) (with the Kalman gain in (5)),  $\hat{x}(t|t)$  is the estimate of  $x(t)$  that uses measurements up to time instant  $t$ ,  $M$  is the innovation update gain obtained as

$$M = PC^T(CPC^T + R_e)^{-1} \quad (11)$$

where  $P$  is the positive semidefinite solution to the discrete-time Riccati equation given in (6).  $L$  is the optimal gain given by

$$L = (B^T S B + Q_2)^{-1} B^T S A \quad (12)$$

where  $S$  is the positive semidefinite symmetric solution to the discrete-time Riccati equation

$$S = A^T S A + Q_1 - A^T S B (B^T S B + Q_2)^{-1} B^T S A. \quad (13)$$

### C. Closed loop systems

The closed loop system can generally be expressed as

$$\begin{bmatrix} x(t+1) \\ \hat{x}(t+1) \end{bmatrix} = F \begin{bmatrix} x(t) \\ \hat{x}(t) \end{bmatrix} + J \begin{bmatrix} w(t) \\ e(t) \end{bmatrix},$$

$$y(t) = \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ \hat{x}(t) \end{bmatrix} + \begin{bmatrix} 0 & I \end{bmatrix} \begin{bmatrix} w(t) \\ e(t) \end{bmatrix}. \quad (14)$$

The closed-loop system with the MV controller in (7), is given by (14) with  $F = F^{MV}$  and  $J = J^{MV}$  where

$$\begin{aligned} [F^{MV}]_{11} &= A - B(CA^{d-1}B)^{-1}CA^{d-1}KC, \\ [F^{MV}]_{12} &= -B(CA^{d-1}B)^{-1}CA^{d-1}(A - KC), \\ [F^{MV}]_{21} &= (K - B(CA^{d-1}B)^{-1}CA^{d-1}K)C, \\ [F^{MV}]_{22} &= A - B(CA^{d-1}B)^{-1}CA^{d-1}(A - KC) - KC, \\ J^{MV} &= \begin{bmatrix} N & -B(CA^{d-1}B)^{-1}CA^{d-1}K \\ 0 & K - B(CA^{d-1}B)^{-1}CA^{d-1}K \end{bmatrix}. \end{aligned} \quad (15)$$

If MV controllers are designed for each SISO subsystem of the MIMO system in (2), then the full closed loop system can be expressed as

$$\begin{aligned} X(t+1) &= \text{diag}(F_{11}^{MV}, A_{12}^e, A_{21}^e, F_{22}^{MV})X(t) \\ &+ \begin{bmatrix} N_{11} & 0 & [J_{11}^{MV}]_{12} & 0 \\ 0 & 0 & [J_{11}^{MV}]_{22} & 0 \\ 0 & N_{12} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ N_{21} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & N_{22} & 0 & [J_{22}^{MV}]_{12} \\ 0 & 0 & 0 & [J_{22}^{MV}]_{22} \end{bmatrix} \begin{bmatrix} w_1(t) \\ w_2(t) \\ e_1(t) \\ e_2(t) \end{bmatrix}, \end{aligned} \quad (16)$$

$$\begin{aligned} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} &= \begin{bmatrix} C_{11} & 0 & C_{12} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{21} & 0 & C_{22} & 0 \end{bmatrix} X(t) \\ &+ \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} w_1(t) \\ w_2(t) \\ e_1(t) \\ e_2(t) \end{bmatrix} \end{aligned} \quad (17)$$

for the decentralized pairing choice  $y_1 - u_1$  and  $y_2 - u_2$ , where

$$\begin{aligned} X(t) &= [x_{11}^T(t) \ x_{11}^T(t) \ x_{12}^T(t) \ \hat{x}_{12}^T(t) \ \dots \\ &\dots \ x_{21}^T(t) \ \hat{x}_{21}^T(t) \ x_{22}^T(t) \ \hat{x}_{22}^T(t)]^T, \end{aligned}$$

$\text{diag}(A_1, A_2, A_3, A_4)$  denotes a block diagonal matrix with the matrices  $A_1, A_2, A_3$  and  $A_4$  along the diagonal,  $F_{ij}^{MV}$  is the corresponding closed loop system for subsystem  $ij$ ,

$$A_{ij}^e = \begin{bmatrix} A_{ij} & 0 \\ 0 & 0 \end{bmatrix}$$

for  $i, j \in \{1, 2\}$ , and, finally,  $[J_{ij}^{MV}]_{kl}$  is element  $kl$  in the corresponding matrix  $J^{MV}$  (defined in Equation (15)) for subsystem  $ij$ .

For the decentralized pairing choice  $y_1 - u_2$  and  $y_2 - u_1$ , the full closed loop system with MV control is

$$\begin{aligned} X(t+1) &= \text{diag}(A_{11}^e, F_{12}^{MV}, F_{21}^{MV}, A_{22}^e)X(t) \\ &+ \begin{bmatrix} N_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & N_{12} & [J_{12}^{MV}]_{12} & 0 \\ 0 & 0 & [J_{12}^{MV}]_{22} & 0 \\ N_{21} & 0 & 0 & [J_{21}^{MV}]_{12} \\ 0 & 0 & 0 & [J_{21}^{MV}]_{22} \\ 0 & N_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} w_1(t) \\ w_2(t) \\ e_1(t) \\ e_2(t) \end{bmatrix}. \end{aligned} \quad (18)$$

with outputs given in (17).

The LQG controller corresponds to  $F = F^{LQG}$  and  $J = J^{LQG}$  where

$$\begin{aligned} F^{LQG} &= \begin{bmatrix} A - BLMC & -BL(I - MC) \\ KC - BLMC & A - KC - BL(I - MC) \end{bmatrix}, \\ J^{LQG} &= \begin{bmatrix} N & -BLM \\ 0 & K - BLM \end{bmatrix}. \end{aligned} \quad (19)$$

The full closed-loop system is derived for the two decentralized pairings  $y_1 - u_1$  and  $y_2 - u_2$ , and  $y_1 - u_2$  and  $y_2 - u_1$ , respectively. The full closed loop system for the first pairing choice  $y_1 - u_1$  and  $y_2 - u_2$  is given by

$$\begin{aligned} X(t+1) &= \text{diag}(F_{11}^{LQG}, A_{12}^e, A_{21}^e, F_{22}^{LQG})X(t) \\ &+ \begin{bmatrix} N_{11} & 0 & [J_{11}^{LQG}]_{12} & 0 \\ 0 & 0 & [J_{11}^{LQG}]_{22} & 0 \\ 0 & N_{12} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ N_{21} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & N_{22} & 0 & [J_{22}^{LQG}]_{12} \\ 0 & 0 & 0 & [J_{22}^{LQG}]_{22} \end{bmatrix} \begin{bmatrix} w_1(t) \\ w_2(t) \\ e_1(t) \\ e_2(t) \end{bmatrix} \end{aligned} \quad (20)$$

$[J_{ij}^{LQG}]_{kl}$  is element  $kl$  in the corresponding matrix  $J^{LQG}$  (defined in Equation (19)) for subsystem  $ij$ .

The corresponding system for the other decentralized pairing choice,  $y_1 - u_2$  and  $y_2 - u_1$ , is

$$\begin{aligned} X(t+1) &= \text{diag}(A_{11}^e, F_{12}^{LQG}, F_{21}^{LQG}, A_{22}^e)X(t) \\ &+ \begin{bmatrix} N_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & N_{11} & [J_{12}^{LQG}]_{12} & 0 \\ 0 & 0 & [J_{12}^{LQG}]_{22} & 0 \\ N_{11} & 0 & 0 & [J_{21}^{LQG}]_{12} \\ 0 & 0 & 0 & [J_{21}^{LQG}]_{22} \\ 0 & N_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} w_1(t) \\ w_2(t) \\ e_1(t) \\ e_2(t) \end{bmatrix} \end{aligned} \quad (21)$$

with outputs given by (17).

## V. CONTROL STRUCTURE SELECTION

As a measure of the performance of the considered control structure, the sum of the output variances,  $\text{tr}(\text{cov}\{y\})$ , is here used. This measure gives an indication of how appropriate the control structure is compared to other structures, and

will here be used as a criterion when pairing the inputs and outputs. The control structure that gives the smallest sum is the structure that has the most desirable pairing combination in a minimum variance sense.

The very same criterion is here used both with MV control and with LQG control. The latter version is useful if  $(CA^{d-1}B)^{-1}$  does not exist for some subsystem, when the system is not minimum-phase or when it is desired to have the possibility to specify the weights in a LQG design. To get a stable MV controller, it is required that the system is minimum-phase. Note again, that the pairing strategy is not limited to decentralized control structures. However, in the present paper, only decentralized structures are compared.

The output variances were calculated in the following way: The stationary state covariance matrix  $\Pi = Ex(t)x^T(t) = Ex(t+1)x^T(t+1)$  for a stable state space system  $(A, B, C, 0)$  with process noise intensity  $\lambda^2$ , was found by solving the discrete-time Lyapunov equation [21]

$$\Pi = A\Pi A^T + \lambda^2 B B^T. \quad (22)$$

The stationary output covariance matrix was then obtained as

$$\text{cov}\{y\} = C\Pi C^T. \quad (23)$$

If the measurements of the outputs are noisy, the noise intensities have to be added to the output variances obtained in (23).

## VI. EXAMPLES

In this section three examples are presented where the proposed pairing strategies are used in order to decide appropriate decentralized control structures. The suggestions are also compared with the ones previously obtained using some other interaction measures (see [6] and [7]).

In the examples it is assumed that the process noise is additive on the inputs, i.e.  $N = B$ , and for this reason that  $w_{ij} = w_j$ . All noise sequences are assumed to be white and have unit variance. Prior to the control design, all of the considered systems were sampled with a sampling period of 1 s. In the LQG control design the LQ weights in the used criterion (9) were selected in two ways. First,  $\tilde{Q}_i = 1$ ,  $i, j \in \{1, 2\}$ . This is a reasonable choice that has the possibility of giving controllers with output variances only marginally higher than with minimum variance control (but with substantially lower variances of the control signals), see [1]. This is confirmed in the presented examples below. Secondly,  $\tilde{Q}_1 = 1$  and  $\tilde{Q}_2 = 1 \cdot 10^{-9}$ . This choice corresponds to LQG control that will be very close to minimum variance control (for minimum-phase systems), but with the advantage of not being restricted to minimum-phase systems to get stable controllers.

### A. Example 1

In the first example the interactions present in a quadruple-tank system will be examined (see [16] for a general description of this process). The considered continuous-time linear

TABLE I

OUTPUT VARIANCES FOR MV AND LQG CONTROL IN EXAMPLE 1. THE PAIRING IS SPECIFIED AS  $i - j$  WHERE  $i$  IS THE OUTPUT INDEX AND  $j$  THE INPUT INDEX. THE CONTROLLER TYPE DENOTED LQG1 IS DESIGNED WITH  $\tilde{Q}_2 = 1$  AND LQG2 WITH  $\tilde{Q}_2 = 1 \cdot 10^{-9}$ .

Pairing	Controller type	$\text{var}\{y_1\}$	$\text{var}\{y_2\}$	$\sum_{i=1}^2 \text{var}\{y_i\}$
1-1, 2-2	MV	1.0513	1.0864	2.1376
	LQG1	1.0652	1.1077	2.1729
	LQG2	1.0513	1.0864	2.1376
1-2, 2-1	MV	1.1011	1.1796	2.2807
	LQG1	1.1026	1.1865	2.2891
	LQG2	1.1011	1.1796	2.2807

minimum-phase model is given by the following state space matrices:

$$A = \begin{bmatrix} -0.0159 & 0 & 0.159 & 0 \\ 0 & -0.0159 & 0 & 0.02651 \\ 0 & 0 & -0.159 & 0 \\ 0 & 0 & 0 & -0.02651 \end{bmatrix},$$

$$B = \begin{bmatrix} 0.05459 & 0 \\ 0 & 0.07279 \\ 0 & 0.0182 \\ 0.03639 & 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \quad (24)$$

The obtained (theoretical) output variances are given in Table 1. Clearly, the pairing combination  $u_1 - y_1$  and  $u_2 - y_2$  results in the smallest output variances for both of the LQG control settings as well as the MV control. Hence, in a minimum variance sense, this is the recommended pairing selection. Note that, in this case the system is minimum phase, and LQG control with  $\tilde{Q}_2 = 1 \cdot 10^{-9}$  results in the same output variances as the MV control. In [6], RGA, HIIA, PM and  $\Sigma_2$  are used in the study of this process. All of these interaction measures recommend the very same diagonal pairing for decentralized control.

### B. Example 2

Now consider the continuous-time  $2 \times 2$  process given by

$$G(s) = \begin{bmatrix} \frac{5e^{-40s}}{100s+1} & \frac{e^{-4s}}{10s+1} \\ \frac{-5e^{-4s}}{10s+1} & \frac{5e^{-40s}}{100s+1} \end{bmatrix}. \quad (25)$$

$s$  is the Laplace variable. This process has been extensively analyzed by [19] and [25] with the conclusion that the anti-diagonal pairing combination  $y_1 - u_2$  and  $y_2 - u_1$  is preferred for decentralized control. One reason for this is that the anti-diagonal pairing combination corresponds to faster elements in  $G$ . [19] came to this conclusion using the Dynamic Relative Gain Array (DRGA) and verified it in a simulation study involving optimal decentralized PI controllers. [25] used the Effective Relative Gain Array (ERGA) with the same result. In [6], [7] the  $\Sigma_2$  was used to find the very same pairing recommendation.

TABLE II

OUTPUT VARIANCES FOR LQG CONTROL IN EXAMPLE 2. THE PAIRING IS SPECIFIED AS  $i - j$  WHERE  $i$  IS THE OUTPUT INDEX AND  $j$  THE INPUT INDEX. THE CONTROLLER TYPE DENOTED LQG1 IS DESIGNED WITH  $\tilde{Q}_2 = 1$  AND LQG2 WITH  $\tilde{Q}_2 = 1 \cdot 10^{-9}$ .

Pairing	Controller type	$\text{var}\{y_1\}$	$\text{var}\{y_2\}$	$\sum_{i=1}^2 \text{var}\{y_i\}$
1-1, 2-2	MV	1.1376	2.3366	3.4742
	LQG1	1.1446	2.3436	3.4882
	LQG2	1.1376	2.3366	3.4742
1-2, 2-1	MV	1.1715	2.0369	3.2084
	LQG1	1.1738	2.0797	3.2536
	LQG2	1.1715	2.0369	3.2084

The output variances for MV control and for both of the LQG control settings indicate that the anti-diagonal pairing combination is the most suitable since this pairing gives the lowest output variances, see Table II. This is in contrast to the static RGA, the HIIA and the PM, but in agreement with for instance  $\Sigma_2$ , the RGA evaluated at frequencies  $\gtrsim 10^{-1.7}$  rad/s, and with other findings made by e.g. [19] and [25].

### C. Example 3

As a concluding example, consider the  $3 \times 3$  process given by

$$G(s) = \frac{1-s}{(1+5s)^2} \begin{bmatrix} 1 & -4.19 & -25.96 \\ 6.19 & 1 & -25.96 \\ 1 & 1 & 1 \end{bmatrix} \quad (26)$$

This process is used by [13] as an example of when the static RGA does not recommend the most desirable pairing. Note that the system is not minimum-phase. Hence, the MV controllers will give unstable closed loop systems where the input signals  $u_i(t)$ ,  $i \in \{1, 2\}$ , go to infinity. This is the reason for the difference in variance between MV control and LQG control with  $\tilde{Q}_2 = 1 \cdot 10^{-9}$  in this particular example, see Table 3.

The RGA recommends the diagonal pairing combination  $y_1 - u_1$ ,  $y_2 - u_2$  and  $y_3 - u_3$ . However, as found by [13] this pairing combination is not suitable due to instability issues. Instead, they recommend the pairing combination  $y_1 - u_2$ ,  $y_2 - u_3$  and  $y_3 - u_1$ . The same pairing suggestion is found by [11] when considering loop-by-loop interaction energy. The HIIA, the PM and  $\Sigma_2$  all recommend the pairing combination  $y_1 - u_3$ ,  $y_2 - u_1$  and  $y_3 - u_2$  for decentralized control. The RGA also indicates (by negative elements) that the pairing combination  $y_1 - u_3$ ,  $y_2 - u_1$  and  $y_3 - u_2$  should be avoided, see for instance [13]. If the HIIA, the PM and  $\Sigma_2$  are combined with the RGA rule of avoiding pairings corresponding to negative RGA elements (this is one component of the pairing rule used by [11]), the HIIA, the PM and  $\Sigma_2$  suggest the very same pairing combination as the one recommended by [13] and [11].

All six decentralized pairing combinations were evaluated. In Table III the sum of the output variances is presented for each combination. The proposed pairing criteria are minimized for the pairing combination  $y_1 - u_3$ ,  $y_2 - u_1$  and

TABLE III

THE SUM OF THE OUTPUT VARIANCES FOR LQG CONTROL IN EXAMPLE 3. THE PAIRING IS SPECIFIED AS  $i - j$  WHERE  $i$  IS THE OUTPUT INDEX AND  $j$  THE INPUT INDEX. THE CONTROLLER TYPE DENOTED LQG1 IS DESIGNED WITH  $\tilde{Q}_2 = 1$  AND LQG2 WITH  $\tilde{Q}_2 = 1 \cdot 10^{-9}$ .

Pairing	Controller type	$\sum_{i=1}^3 \text{var}\{y_i\}$
1-1, 2-2, 3-3	MV	75.7614
	LQG1	75.7860
	LQG2	75.7704
1-1, 2-3, 3-2	MV	43.6560
	LQG1	50.3364
	LQG2	49.8111
1-2, 2-1, 3-3	MV	73.6870
	LQG1	74.5408
	LQG2	74.2068
1-2, 2-3, 3-1	MV	43.0675
	LQG1	50.0170
	LQG2	49.3745
1-3, 2-2, 3-1	MV	43.6560
	LQG1	50.3364
	LQG2	49.8111
1-3, 2-1, 3-2	MV	42.1700
	LQG1	49.4106
	LQG2	48.6842

$y_3 - u_2$  for MV control and for both of the LQG control settings. This supports the recommendation made by the HIIA, the PM and  $\Sigma_2$ . However, the pairing combination  $y_1 - u_2$ ,  $y_2 - u_3$  and  $y_3 - u_1$  also gives variances that are very close to the variances of the suggested pairing. If the pairing combinations corresponding to negative RGA elements are avoided, the MV and the LQG control criteria give the very same pairing suggestion as the one recommended by [13] and [11]. It is further found (see Table III) that the diagonal pairing recommended by the RGA is not desirable in a minimum variance sense. In fact, this pairing results in the largest output variances. There are also two other pairings that give low output variances: pairing combination  $y_1 - u_1$ ,  $y_2 - u_3$  and  $y_3 - u_2$  and pairing combination  $y_1 - u_3$ ,  $y_2 - u_2$  and  $y_3 - u_1$ .

## VII. CONCLUSIONS

New input/output pairing strategies based on minimum variance control and LQG control have been presented. The obtained decentralized input-output pairing suggestions for three MIMO plants have been compared with those previously obtained with other interaction measures. It was found that the proposed pairing strategies give suitable decentralized pairing suggestions for the studied processes. This motivates a further study of the strategies. In the present paper only decentralized control structures have been considered. However, this is not an inherent limitation of the proposed pairing strategies since they are able to evaluate the performance of other control structures as well. This is an advantage compared to the RGA. Furthermore, since the strategies are based on what control performance that can be

expected to be achieved with the designed control structures, the proposed interaction measures are easy to interpret.

#### ACKNOWLEDGEMENT

The authors are grateful to Dr. Torsten Wik (Chalmers University of Technology, Gothenburg, Sweden) for valuable discussions.

#### REFERENCES

- [1] K. J. Åström and B. Wittenmark. *Computer-Controlled Systems*. Prentice-Hall International Editions, Upper Saddle River, USA, 1990.
- [2] W. Birk and A. Medvedev. A note on gramian-based interaction measures. In *Proceedings of European Control Conference, Cambridge, UK, September 2003*, 2003.
- [3] E. H. Bristol. On a new measure of interaction for multivariable process control. *IEEE Trans. Automatic Control*, AC-11:133–134, 1966.
- [4] A. Conley and M. E. Salgado. Gramian based interaction measure. In *Proceedings of the 39th IEEE Conference on Decision and Control*, pages 5020–5022, Sydney, Australia, December 2000.
- [5] T. Glad and L. Ljung. *Reglerteori*. Studentlitteratur, 1997. In Swedish.
- [6] B. Halvarsson. *Interaction Analysis and Control of Bioreactors for Nitrogen Removal*. IT Licentiate thesis 2007-006, Systems and Control, Department of Information Technology, Uppsala University, 2007.
- [7] B. Halvarsson. Comparison of some gramian based interaction measures. In *2008 IEEE International Symposium on Computeraided Control System Design (CACSD 2008), Part of IEEE Multiconference on Systems and Control, San Antonio, Texas, USA*, pages 138–143, 2008.
- [8] T. Harris. Assessment of control loop performance. *The Canadian Journal of Chemical Engineering*, 67:856–861, 1989.
- [9] T. Harris, F. Boudreau, and J. Macgregor. Performance assessment of multivariable feedback controllers. *Automatica*, 32:1505–1518, 1996.
- [10] T. Harris, C. Seppala, and L. Desborough. A review of performance monitoring and assessment techniques for univariate and multivariate control systems. *Journal of Process Control*, 9:1–17, 1999.
- [11] M.-J. He and W.-J. Cai. New criterion for control-loop configuration of multivariable processes. *Ind. Eng. Chem. Res.*, 43:7057–7064, 2004.
- [12] A. Horch and A. Isaksson. A modified index for control performance assessment. *Journal of Process Control*, 9:475–483, 1999.
- [13] M. Hovd and S. Skogestad. Simple frequency-dependent tools for control system analysis, structure selection and design. *Automatica*, 28(5):989–996, 1992.
- [14] B. Huang, S. Ding, and N. Thornhill. Practical solutions to multivariate feedback control performance assessment problem: reduced a priori knowledge of interactor matrices. *Journal of Process Control*, 15:573–583, 2005.
- [15] H.-P. Huang, F.-Y. Lin, and J.-C. Jeng. Control Structure Selection and Performance Assessment for Disturbance Rejection in MIMO Processes. *Ind. Eng. Chem. Res.*, 46:9170–9178, 2007.
- [16] K.H. Johansson. The quadruple-tank process: A multivariable laboratory process with an adjustable zero. *IEEE Transactions on Control Systems Technology*, 8(3):456–465, May 2000.
- [17] M.R. Katebi and A.W. Ordys. *Minimum Variance Control*, chapter 62 The Control Handbook, pages 1089–1096. CRC Press, 1996.
- [18] M. Kinnaert. Interaction measures and pairing of controlled and manipulated variables for multiple-input multiple-output systems: A survey. *Journal A*, 36(4):15–23, 1995.
- [19] T. Mc Avoy, Y. Arkun, R. Chen, D. Robinson, and P. D. Schnelle. A new approach to defining a dynamic relative gain. *Control Engineering Practice*, 11(8):907–914, 2003.
- [20] M. E. Salgado and A. Conley. MIMO interaction measure and controller structure selection. *Int. J. Control*, 77(4):367–383, 2004.
- [21] T. Söderström. Computational methods for evaluating covariance functions. Systems and Control Group, Uppsala University, Uppsala, Sweden, 2003.
- [22] X. Wang, B. Huang, and T. Chen. Multirate minimum variance control design and control performance assessment: A data-driven subspace approach. *IEEE Transactions on Control Systems Technology*, 15:65–74, 2007.
- [23] B. Wittenmark and M. E. Salgado. Hankel-norm based interaction measure for input-output pairing. In *Proc. of the 2002 IFAC World Congress*, Barcelona, Spain, 2002.
- [24] H. Xia, P. Majecki, A. Ordys, and M. Grimble. Performance assessment of MIMO systems based on I/O delay information. *Journal of Process Control*, 16:373–383, 2006.
- [25] Q. Xiong, W.-J. Cai, and M.-J. He. A practical loop pairing criterion for multivariable processes. *Journal of Process Control*, 15:741–747, 2005.