Combinatorial and Simultaneous Auction:

A Pragmatic Approach to Tighter Bounds on Expected Revenue

Jim Wilenius
Computing Science Division
Dept. of Information Technology
Uppsala University
Box 337, SE-75105 Uppsala, Sweden
jim.wilenius@it.uu.se

May 25, 2009

Abstract

It is a common belief that combinatorial auctions provide good solutions to resource-allocation in multiple-object markets with synergies. In this work we adopt a pragmatic approach to examining the revenue bounds on combinatorial and simultaneous auctions. The theoretical bounds from our previous work utilize a large number of bidders in order to show that combinatorial auctions yield a higher expected revenue. It is reasonable to believe that the true bounds are much tighter. We argue that this is the indeed the case and that the first-price combinatorial auction is revenue superior even when a relatively small number of bidders participate. The argument is based on three methods. (i) heuristic equilibrium-strategy search, (ii) sampling of the expected revenue in the first-price sealed-bid combinatorial auction, and (iii) a tightened theoretical upper bound on the sealed-bid simultaneous auction in the case of few bidders.

1 Introduction

We study two different multiple-item auctions where the auctioneer has a choice between allowing only individual bids for single items, or bids for indivisible bundles of items (combinatorial bids). Allowing bids only on individual items will often produce an inefficient outcome and will probably also result in lower revenue (certainly this is the case as shown in our previous work [1]), but the auction format is extremely simple and winners are easily determined and verified.

Allowing combinatorial bids enables bidders to express their preferences more accurately and with less risk, which can lead to increased efficiency [13, 15, 18]. Apart from efficiency, the question of revenue in multiple objects auctions has been studied broadly in the literature, work been done in deriving optimal auctions given two items, see for example Levin [9] and Armstrong [2]. Even though this work is important, it does not include nor apply to a model with many items where bidders are interested in randomly chosen subsets of items. Ledyard [8] investigates the optimal combinatorial auction for a model with single-minded bidders, but where the auctioneer knows which subset each bidder is interested in,. Work has also been done on approximating revenue maximizing combinatorial auctions [10]. For background on combinatorial auctions, see for example [5, 12, 11].

In our previous work we prove in a model with bidders interested in a specific combination of items, that the first-price sealed-bid combinatorial auction is revenue superior to the sealed-bid simultaneous auction in many cases [1]. Our proof is conceptual in nature and provides new theoretical evidence in favor of the combinatorial auction. The proof techniques require a rather large number of bidders to achieve the relevant results. However, it is natural to believe that the revenue superiority of the combinatorial auction holds even when a relatively small number of bidders participate in the auction. Here we test this hypothesis in the case of few bidders by using three methods, (i) a heuristic for determining equilibrium strategies, (ii) sampling of the expected revenue of the first-price sealed-bid combinatorial auction, and (iii) a tightening of the theoretical upper bound on the sealed-bid simultaneous auction.

In order to produce an estimate of the expected revenue of the first-price sealed-bid combinatorial action, a simulation based approach is taken by using the equilibrium strategy search heuristic presented in [17], and subsequent sampling of the expected revenue.

In this work we provide empirical results showing that the combinatorial auction yields higher revenue than the simultaneous auction for realistic bidding scenarios with medium sized auctions both in the number of bidders and items. This indicates that the results from our previous work are likely to hold more generally than theoretically proven.

The next section defines the common model which is studied, and also describes the combinatorial and simultaneous auctions. Section 3 investigates the expected revenue of the combinatorial auction using simulation and sampling. Section 4 revisits the upper bound on the simultaneous auction for cases with few bidders and finally in Section 5 results regarding the two auctions are combined followed by a short discussion.

[1] The proof is done for infinitely many bidders and several actual scenarios with many items and combinations of varying size. The latter are derived from a parameterized bound.

2 Model – Bidders and Valuations

We consider a standard private values model. That is, bidders have independent private valuations. The complete model is specified below, in Definition 2.1.

Our analysis is limited to two types of bidders, *synergy-bidders* and *single-bidders*. Synergy-bidders are weakly single-minded and have a synergy on one specific set of k randomly chosen items. A single-bidder is interested in one particular item.

Definition 2.1. The common model:

(a) Bidders are rational, risk neutral, and symmetric. Only pure symmetric equilibrium are considered.

(b) Valuations are private and independently drawn from a continuous uniform distribution on the interval [0,1].

(c) There are M items for sale.

(d) A single-bidder is interested in one item.

(e) A synergy-bidder is *weakly single-minded*, that is, interested in one uniformly drawn random combination of size k. He has the same valuation on all k items, and receives a synergy of α per item if all k items are won.

(f) There are a total of N synergy-bidders, and for each item there are N_s single-bidders.

Note that the above model differs in a minor way from that used in our previous work, in which the number of single-bidders is identical to the number of synergy-bidders ($N_s = N$).

2.1 The Combinatorial and Simultaneous Auction

This work is concerned with the most straightforward of the combinatorial auctions, the first-price sealed-bid combinatorial auction. Bidders may submit bids for combinations of items, and if they win, pay the value of the bid. Winners are determined by solving the (generally) NP-hard maximization problem known as the winner determination problem (WDP), where the auctioneer typically wants to maximize the revenue by selecting the set of non-overlapping bids with the maximum sum. This combinatorial puzzle is one of the factors that fundamentally separates the combinatorial auction from single item auctions in general.

Another distinguishing property of a combinatorial auction is that bidders have no need to speculate on the number of items he will win, since a bid on a combination of items either wins in its entirety or not at all.

This allows bidders to express preferences on combinations of items without the risk of winning only part of the combination, a problem bidders face in the simultaneous auction. The combinatorial auction gives rise to a *threshold problem* (free-riding), the fact that a bidder could potentially under-bid and still be a part of the optimal allocation. The bound presented in our previous work (Andersson and Wilenius [1]) implies that, in our model, the threshold problem does not have a dramatic effect on revenue.

The question of optimal bidding strategy, which in the case of one item for sale is well studied, is still an open problem in the first-price sealed-bid combinatorial auction, although some work towards approximating strategies has been done [17, 16]. Bernheim and Whinston provide an analysis of a first-price menu-auction in a complete information setting [3].

In contrast to the combinatorial auction, another way of selling multiple items simultaneously is by using a sealed-bid *simultaneous auction*. This auction is made up of multiple single-item auctions taking place at the same time. The winner of each of these auctions is determined separately by the highest bidder in each auction, and the item is sold for an amount specified by some payment rule.

Clearly the winner determination problem in this auction is trivial. However, the issue of bidding when synergies between items exist is not trivial. Although some work has been done in this area [7, 6, 4, 14], we have found no work that analyze the simultaneous auction for a general model with many items and several bidders interested random in combinations.

Bidding strategies naturally depend on the synergies and the bidders view of the competition. Furthermore, a bidder in a simultaneous auction is faced with an uncertainty regarding which auctions he will win and lose. This implies an uncertainty regarding whether or not to bid above the single item value, utilizing the potential gain of the synergy in the gamble that all items are won. If he bids above the single item value and not all items are won, he loses money. This dilemma is commonly known as the *exposure problem*.

3 Expected Revenue of the Combinatorial Auction

In our previous work, proofs for several cases where the combinatorial auction expected revenue exceeds that of the simultaneous auction are provided. However, in the presented scenarios a large number of bidders were required for technical reasons due to the nature of the proof. Given these results it is only natural to ask what a tighter estimation of the expected revenue of the combinatorial auction might be, and if it is possible to achieve a higher expected revenue than the simultaneous auction even with a small number of bidders.

Table 1: Simulation parameters

Ψ_p	41	No. of points used when estimating $P(\cdot)$.
Ψ_s	1000	sample per point when estimating $P(\cdot)$.
η_r	0.07	max allowed squared error in regression of $\overline{P}(\cdot)$.
η_e	0.01	max average absolute difference between strategies in equilibrium.
Θ	19	No. points to measure difference in strategies $\beta(\cdot)$.
Φ_r	30000	No. of samples when estimating the expected revenue.
Φ_s	70	No. of trials to run the simulation.

Proposed here is a less strict approach that possibly provides a more realistic estimate of the expected revenue. By using the equilibrium strategy search heuristic described in a previous work (Wilenius and Andersson [17]), we estimate the expected revenue of the combinatorial auction by sampling.

3.1 Revenue Simulation

In this section the simulation procedure is described and we present the revenue results of the simulations. Most simulations have been performed with the parameters found in Table 1. Any deviations from the parameters stated in Table 1 are explicitly mentioned in conjunction with the presented results. The simulation procedure can be summarized in three steps.

1. Choose, for every bidder, a random straight line as the initial strategy, and run the strategy search as detailed in the previous work [17].
2. Sample the expected revenue using the discovered equilibrium strategy and Φ_r samples.
3. Repeat the above steps Φ_s times.

The complete details of the strategy search in step 1 can be found in [17]. However, for the sake of readability a brief summary follows. The strategy search heuristic is mainly based on two concepts.

(i) Sampling the probability function $P(\cdot)$ of a bid winning as a function of the bid's value, and then fitting a model function $\overline{P}(\cdot)$ to the sampled probability function which allows us to derive the best response strategy directly.

(ii) Searching for equilibrium by use of best-response iteration.

In each iteration of the algorithm, the best-response strategy is derived by determining the probability of a bid winning given a certain value, and analytically deriving the payoff maximizing strategy $\beta(\cdot)$. When the difference in strategies reaches an acceptable level, strategies are considered to be in equilibrium.

The setting described in Definition 2.1 includes both synergy-bidders and single-bidders, however consider the following Lemma.

Lemma 3.1 (Lemma 5.5, (Andersson and Wilenius [1])). *Given the number of bidders N_s and an arbitrarily small constant $\Delta > 0$, then with probability q, a winning single-bid in the combinatorial auction is greater than or equal to:*

$$\left(\frac{1-q}{N_s}\right)^{\frac{1}{N_s-1}} - \frac{\left(\frac{1-q}{N_s}\right)^{\frac{N_s}{N_s-1}}}{1-q} - \Delta \ .$$

Using Lemma 3.1, we can easily prove that given as few as 20 single-bidders per item, the winning single-bid will be at least 0.5 with probability at least 0.9998989. This observation allows us to make the following important and necessary simplification to our simulation environment: the single-bidders are excluded from the strategy search and instead we assume that there are enough (at least 20) single-bidders to "guarantee" a certain minimum single-bid on each item. Thus, on each item we assume (pessimistically) that there is a single-bid of value 0.5.

Table 2: Sampled Expected Revenue in the Combinatorial Auction

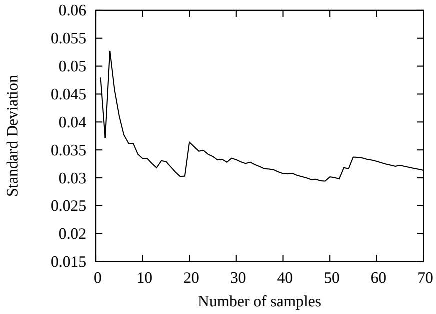
Items M	Combination Size k	No. Bidders	Combinatorial Auction Lower Bound	Sampled Expected Revenue	No. Synergy-Bidders	Sample Standard Deviation
8	3	197	11.658	11.810	26	0.0314
11	4	303	15.476	15.643	28	0.0232
14	5	502	19.341	19.510	33	0.0335
17	6	857	23.240	23.542	42	0.0446

Comparison of the sampled bound to the bound in Andersson and Wilenius [1]

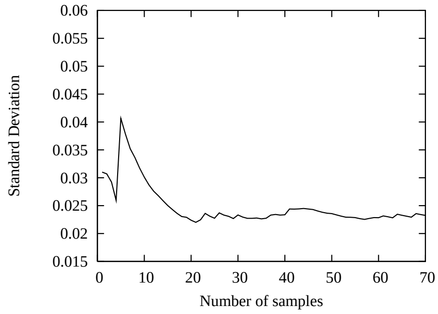
Table 2 compares the analytical lower bound of the combinatorial auction from our previous work [1] with the sampled expected revenue obtained here. Given a specific M and k the number of bidders have been chosen so that the expected revenues are comparable. The observation is made that significantly fewer bidders are needed in the simulations. The main reason for the large number of bidders in the analytical bound is due to the design of the proof, which is based on the probability of the existence of non-colliding bids of a certain minimum value.

The results in Table 2 were produced by simulation and sampling[2] and Figure 3.1 show the standard deviation convergence up to Φ_s trials. To get a rough idea of how the expected revenue grows as the number of bidders increase see Figure 3.1. This figure plots the expected revenue average of 5 samples together with the minimum and maximum values for the problem

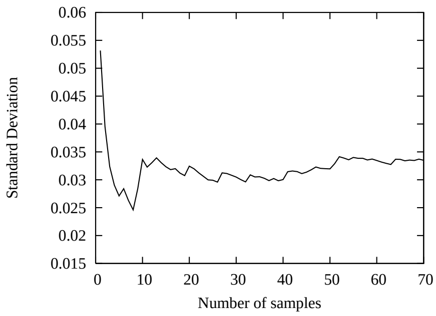
[2]The simulations required more than 388 hours to complete on an AMD Athlon X2 2GHz with 2GB of memory and more than 100 million combinatorial auctions were solved using ILOG CPLEX.



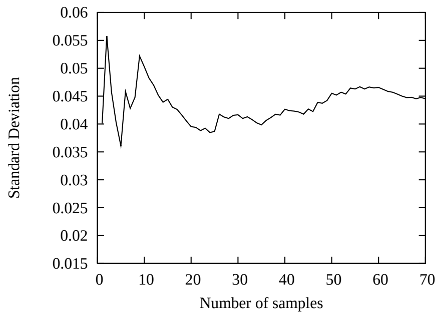
(a) M=8, k=3



(b) M=11, k=4



(c) M=14, k=5



(d) M=17, k=6

Figure 1: Convergence of the standard deviation.

instance with 8 items and combinations of size 3. This is only meant to show the tendency of the expected revenue as a function of the number of bidders.

4 Revisiting the Simultaneous Auction

The simulation of the expected revenue of the combinatorial auction produced results comparable with the theoretical lower bound but with few bidders. In this section the simultaneous auction upper bound is revisited and a new tighter bound is provided for the case of few bidders.

Consider the bound as stated in Corollary 6.1 in our previous work (Andersson and Wilenius [1]),

$$M + \frac{\alpha k}{1 - \prod_{i=0}^{k-1} \frac{M-k-i}{M-i}} \quad .$$

The bound is derived as the sum expected realized synergy, which is bounded from above by a geometric series given an infinite number of synergy-bidders and single-bidders, and finally adding 1 per item which is the maximum possible per-item valuation. However, with few bidders the upper bound on the per-item valuation can be improved. Given N_s single-bidders per item

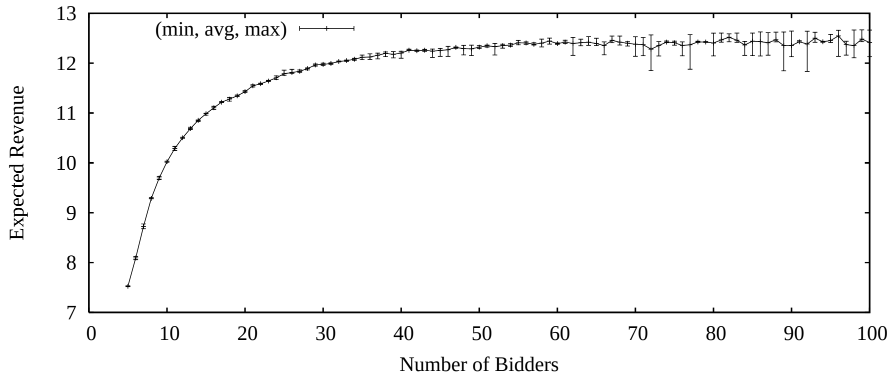


Figure 2: A rough estimation of expected revenue given an increasing number of bidders. Problem size: 8 items and combinations of size 3. $\Phi_s = 5$ and $\Phi_r = 10000$. The theoretical maximum revenue given fixed single-bids of 0.5 is 13.

and N synergy-bidders in total, the expected highest per-item valuation on any item can be explicitly calculated which will provide more precise upper bound.

Basically we are interested in the expected value of the highest order statistic of a set of samples, the size of which is determined by a stochastic variable. Let X_i be a stochastic variable corresponding to a per-item valuation drawn from some known distribution $F_X(x)$. Let Y be a stochastic variable denoting the number of bidders on a particular item, $N_s \leq Y \leq N + N_s$. A finite set of independent draws $\mathbf{X} = \{X_1, .., X_{Y=n}\}$ has the highest order stochastic $Z = \max(\mathbf{X})$, with distribution $F_{Z|Y=n}(x) = F_X(x)^n$ and density $f_{Z|Y=n}(x) = nF_X(x)^{n-1} f_X(x)$. The expected highest per-item valuation given some fixed number of bidders n is

$$E[Z|Y = n] = \int_{-\infty}^{\infty} x f_{Z|Y=n}(x)\, dx. = \int_{-\infty}^{\infty} xnF_X(x)^{n-1} f_X(x)\, dx.$$

The probability that exactly j synergy-bidders are bidding on some particular item is

$$P(j) = \binom{N}{j} \left(\frac{k}{M}\right)^j \left(1 - \frac{k}{M}\right)^{N-j}$$

and since there are N_s single-bidders bidding on each item, the probability that exactly n bidders are bidding on a particular item is therefore $P(n-N_s)$. Finally, the expected value of Z is

$$E[Z] = E[E[Z|Y]] = \sum_{n=N_s}^{N_s+N} \Big(P(n - N_s) \cdot E[Z|Y = n]\Big).$$

Table 3: Simultaneous Auction Upper Bounds

Items M	Combination Size k	Old Upper Bound	New Upper Bound	No. Synergy-Bidders	No. Single-Bidders
8	3	11.652	11.390	26	20
11	4	15.475	15.119	28	20
14	5	19.336	18.906	33	20
17	6	23.233	22.755	42	20

Since we are interested in the uniform distribution on $[0, 1]$, that is $F_X(x) = x$ and $f_X(x) = 1$, we explicitly have

$$E[Z] = \sum_{n=N_s}^{N_s+N} \Big(P(n - N_s) \cdot \int_0^1 xnx^{n-1}\, dx \Big) = \sum_{n=N_s}^{N_s+N} \Big(P(n - N_s) \cdot \frac{n}{n+1} \Big).$$

Theorem 1. *In a sealed-bid simultaneous auction with M items, N_s single-bidders, and N synergy-bidders each bidding on one set of k random items, each realizing a synergy of αk when all k items are won; the sum of all bidders' expected realized valuations is less than*

$$M \cdot \sum_{n=N_s}^{N_s+N} \Big(P(n - N_s) \cdot \frac{n}{n+1} \Big) + \frac{\alpha k}{1 - \prod_{i=0}^{k-1} \frac{M-k-i}{M-i}}$$

where

$$P(n) = \binom{N}{n} \left(\frac{k}{M} \right)^n \left(1 - \frac{k}{M} \right)^{N-n}$$

is the probability that exactly n synergy-bidders bid on a specific item.

Proof. The proof follows directly from the proof of Theorem 1 in our previous work [1], replacing M with the expected highest per-item valuation times the number of items, that is

$$M \cdot \sum_{n=N_s}^{N_s+N} \Big(P(n - N_s) \cdot \frac{n}{n+1} \Big).$$

□

In Table 3 the new tightened upper bound is compared to the upper bound in our previous work, on a subset of the problems presented in there. The number of bidders used for the tightened bound were selected to match the number of bidders for the simulated expected revenue of the combinatorial auction as presented in Table 2.

Table 4: Comparing Expected Revenue

Items M	Combination Size k	No. Synergy-Bidders	S.A. Upper Bound	Sampled Expected Revenue	Sample Standard Deviation
8	3	26	11.390	11.810	0.0314
11	4	28	15.119	15.643	0.0232
14	5	33	18.906	19.510	0.0335
17	6	42	22.755	23.542	0.0446

5 Combining the Results

The results from the expected revenue simulations performed in Section 3.1, together with the results provided by our new tighter upper bound on the simultaneous auction from Section 4, are presented in Table 4 to give an overview.

We achieve, given a relatively small number of bidders, a sampled expected revenue comparable to the lower bound on the combinatorial auction provided in our previous work [1].

The results in Table 4 clearly support the hypothesis that the expected revenue of the combinatorial auction exceeds that of the simultaneous auction even for settings with few bidders.

The revenue comparison in this work is based on the rather pessimistic assumption that a single-bid of at least 0.5 per item can be guaranteed with high probability in the combinatorial auction, while it is assumed that 20 single-bidders per item are participating in the simultaneous auction. This is likely to be unfavorable for the combinatorial auction.

The bounds could perhaps be improved further by relaxing the single-bidder assumption, and using as lower bound (quite reasonably) the same bid as a single-bidder would use in a normal first-price sealed-bid single-item auction.

The upper bound on the simultaneous auction is still fairly loose and a tighter bound could possibly be simulated in a way similar to that of the combinatorial auction. However, the search heuristic proposed in [17] is not suited for this scenario. The core principle that makes the equilibrium strategy search heuristic efficient, is the method by which the best-response strategy is derived. Instead of sampling the expected payoff given a strategy, the current environment of strategies is used to sample the probability of winning with a certain value of a bid. After this process is completed a suitable model function is fitted to the sampled probability data. The probability model in essence encodes the relevant information about the competing bidders' strategies. The payoff can then be expressed explicitly and the best response, or payoff maximizing, strategy function can be expressed analytically, and most importantly, defined beforehand given a specific probability model. The same principles do not necessarily apply to the simultaneous auction. The foremost reason would be that more than one probability function needs to be sampled [3] , and that the resulting payoff function may not have the same readily derivable closed form expression for the strategy. This is not to say that a good simulation method does not exist, however, this is left as an open problem.

6 Conclusions

In this work, the expected revenue of the first-price sealed-bid combinatorial auction was determined through simulation and sampling. The results suggest that a comparable expected revenue to the theoretical lower bound presented in our previous work (Andersson and Wilenius [1]) can be achieved also in scenarios with a small number of bidders, in contrast to the (for technical reasons) larger number of bidders needed for in the theoretical proofs in said work. Furthermore, a tightening of the upper bound on the simultaneous auction is also presented in the case of a small number of bidders.

The results presented here provide a strong indication that the first-price sealed-bid combinatorial auction may indeed be revenue superior to the sealed-bid simultaneous auction even when relatively few bidders participate in the auction.

7 Acknowledgment

I would like to thank Arne Andersson and Justin Pearson for their valuable comments.

References

[1] Arne Andersson and Jim Wilenius. A new analysis of revenue in the combinatorial and simultaneous auction. Technical Report 2009-001, Department of Information Technology, Uppsala University, Sweden, January 2009. Submitted for publication.

[2] Mark Armstrong. Optimal multi-object auctions. *The Review of Economic Studies*, 67(3):455–481, 2000.

[3] B. Douglas Bernheim and Michael D. Whinston. Menu auctions, resource allocation, and economic influence. *The Quarterly Journal of Economics*, 101(1):1–32, 1986.

[3] One probability function per outcome (winning 1 item, 2 items, etc.), since the probabilities of winning are not independent.

[4] Fernando Branco. On the superiority of the multiple round ascending bid auction. *Economics Letters*, 70(2):187–194, February 2001.

[5] Peter Cramton, Yoav Shoham, and Richard Steinberg. *Combinatorial Auctions*. The MIT Press, 2006.

[6] Fabrizio Germano Gian Luigi Albano and Stefano Lovo. A comparison of standard multi-unit auctions with synergies. *Economics Letters*, 71(1):55–60, 2001.

[7] Vijay Krishna and Robert W. Rosenthal. Simultaneous auctions with synergies. *Games and Economic Behavior*, 17(1):1–31, November 1996.

[8] John O. Ledyard. Optimal combinatoric auctions with single-minded bidders. In *EC '07: Proceedings of the 8th ACM conference on Electronic commerce*, pages 237–242, New York, NY, USA, 2007. ACM.

[9] Jonathan Levin. An optimal auction for complements. *Games and Economic Behavior*, 18(2):176–192, February 1997.

[10] Anton Likhodedov and Tuomas Sandholm. Approximating revenue-maximizing combinatorial auctions. In *Proceedings of the National Conference on Artificial Intelligence*, volume 20, pages 267–274, 2005.

[11] Paul Milgrom. *Putting Auction Theory to Work*. Cambridge University Press, 2004.

[12] Aleksandar Pekeč and Michael H. Rothkopf. Combinatorial auction design. *Manage. Sci.*, 49(11):1485–1503, 2003.

[13] S. J. Rassenti, V. L. Smith, and R. L. Bulfin. A combinatorial auction mechanism for airport time slot allocation. *Bell Journal of Economics*, 13:402–417, 1982.

[14] Robert W. Rosenthal and Ruqu Wang. Simultaneous auctions with synergies and common values. *Games and Economic Behavior*, 17(1):32–55, November 1996.

[15] M. H. Rothkopf, A. Pekeč, and R. M. Harstad. Computationally manageable combinatorial auctions. *Management Science*, 44(8):1131–1147, 1998.

[16] Yevgeniy Vorobeychik, Michael P. Wellman, and Satinder Singh. Learning payoff functions in infinite games. *Machine Learning*, 67(1-2):145–168, 2007.

[17] Jim Wilenius and Arne Andersson. Discovering equilibrium strategies for a combinatorial first price auction. *IEEE E-Commerce Technology*

and IEEE Enterprise Computing, E-Commerce, and E-Services, 2007. CEC/EEE 2007, pages 13–20, July 2007.

[18] P. Wurman. *Market Structure and Multidimensional Auction Design for Computational Economies.* PhD thesis, Department of Computer Science, University of Michigan, 1999. (Available from www.csc.ncsu.edu/faculty/wurman).