Constant is optimal – a strategy for resource efficient control of a class of processes

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Abstract In this technical report we consider the problem on how to control a class of processes in a resource/energy optimal way. The optimization is done under the constraint that the mean value of the output signal (typically an effluent discharge) should be below a certain threshold. A key assumption is that the magnitude of the control signal is proportional to the quantity that should be minimized (for example, consumption of energy or chemicals) and that the process efficiency decreases with increasing magnitude of the control signal. We show that for a wide class of processes it is optimal to keep the input signal constant. The result is illustrated in a simple numerical example.

Keywords: optimal control, energy and resource efficiency.

INTRODUCTION

In a traditional control problem the task of the controller is to keep the output sufficiently close to a give set-point despite that the process is subjected external disturbances. In this paper we consider a particular optimal control problem where the task is to keep the arithmetic mean of the output signal lower than a given threshold. Such type of problems arise, for example, when the output signal corresponds to an effluent discharge whose daily mean value should be less than or equal to a given threshold. We want to find a control input which minimizes the resource consumption given the above requirement. The solution to this problem is believed to be of great interest in operation of wastewater treatment plants but other applications may also exist. The application to wastewater treatment plants is a topic for current research as well as a thorough literature review. The goal of this technical report is to present the general problem and give two alternative proofs of the corresponding optimization problem.

METHOD

Statement of the problem

Consider the unit process depicted in Figure 1. The control signal (control handle) is denoted $u(k)$, the output signal (typically an effluent discharge component) is $y(k)$. The disturbance to the plant $d(k)$ may consist of changes in flow rates and influent concentrations. The argument k is a discrete time index.

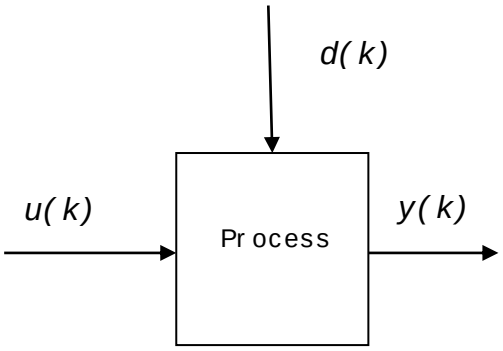

Figure 1: A general unit process**.**

The classical way to control the output *y(k)* is to use feedback and/or feedforward control. See Figure 2 for the archetypical approach. The goal of the controller is to keep the output close to the set point. In practice constraints on the control signal often appear which gives a trade-off between set point tracking and the magnitude of control signal.

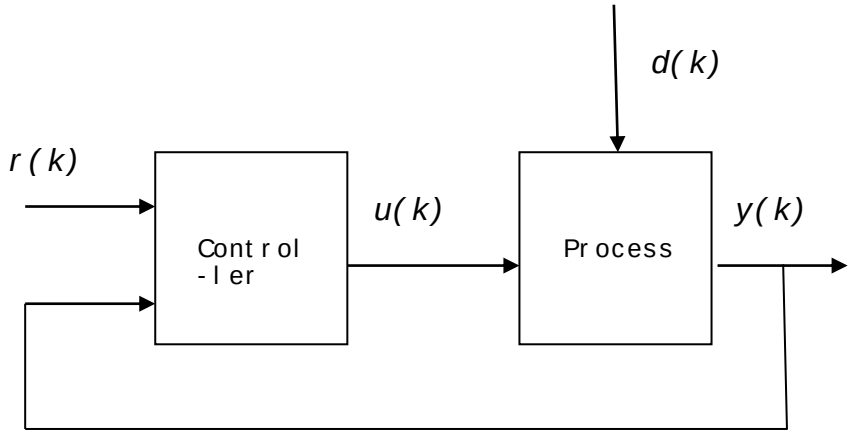

Figure 2: Feedback control. The controller should adjust the control signal *u(k)* so that *y(k*) is kept close to the set point *r(k)* despite the disturbances *d(k)*.

It is quite common that the control signal is proportional to resource/energy consumption. Typical examples include when the control signal is an air flow rate for aeration, a water flow (pumping energy), or a flow rate of chemical precipitation.

Often the set point is set to the required effluent standard even though the effluent requirement may be based on a flow proportional daily (weekly/monthly/yearly) average. The problem we want to solve is as follows. ***What sequence of control signals minimises the resource consumption but keeps the (daily) mean value of the effluent discharge equal to a certain threshold?***

In the next section, we present a solution to the above problem statement given some additional assumptions.

Main result

In this section the main results are presented. Consider the following plant model

$$y(k) = f(u(k)) + d(k) \tag{1}$$

where (as above) the control signal is *u(k)*, the output signal *y(k*) and the disturbance to the plant is

d(k). For further use, let *u=[u(1),u(2),..,u(N)*] and *y=[y(1),y(2),..,y(N)]***.** We assume that for the chosen time scale, the plant can be modelled by a static nonlinear function *f*. It is natural to set *f(0)=0* and in the following we let *d(k)* be a positive time varying disturbance. We assume that the function *f*

(i) is smooth,

(ii) has a derivative which is strictly negative ($\frac{df}{du} < 0$),

(iii) has a second derivative which is strictly positive ($\frac{d^2f}{du^2} > 0$).

Assumption (ii) *s*imply means that *y* is decreasing when *u* is increasing. It also guarantees that a well defined inverse exists (a continuous function *f* is invertible if and only if it is either strictly increasing or decreasing). Assumption (iii) means that the function is strictly convex and that (relative) efficiency decreases when *u* is increased. This is a common property in many unit processes in, for example, biological wastewater treatment where the growth rate of microorganisms (and treatment results) is described by Monod functions. A typical form of *f is*

$$f\big(u(k)\big) = -K_M \frac{u(k)}{u(k) + K_S}$$

In figure 3, this function is depicted for the case K_M=40 and K_S=1.

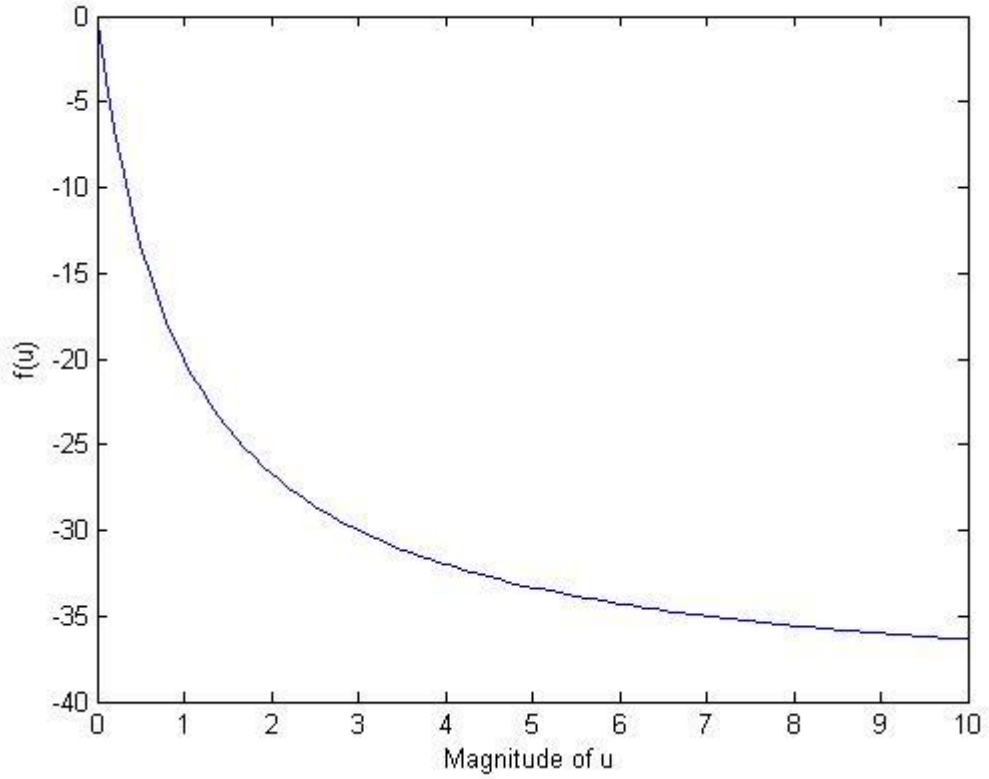

Figure 3: Example of a function fulfilling the conditions (i)-(iii).

Other examples of functions fulfilling the conditions include $f\big(u(k)\big) = -K\log(u(k)+1))$ and $(k) = K(e^{-bu(k)} - 1)$.

Next consider the following optimisation problem for the plant (1):

Minimize

$$c(u) = \sum_{k=1}^{N} u(k) \tag{2}$$

subject to the constraint

$$g(y) = \frac{1}{N}\sum_{k=1}^{N} y(k) = \alpha \tag{3}$$

The criteria (2) is of interest to minimize when the amplitude of the control signal is proportional to consumption of resources, a very common example is that *u* is proportional to energy consumption. The constraint (3) means that the mean value of the output should be equal to a given threshold α. In practice, the constraint is $g(y) \leq \alpha$ but it is clear that (2) is minimized if we use an equality constraint. The output signal is typical an effluent concentration (possible flow weighted) and the choice of α is then a mean value which should be fulfilled. The time window *N* and α may be obtained from legal regulations. However, it can be both easier to tune the control signal (see below) and increase the robustness by choosing N lower than the official regulation. A natural choice may be to let N corresponds to 24 hours.

The solution to the optimization problem is presented in the following theorem:

Theorem 1:
Consider the optimisation problem (2)-(3) for the plant (1) under the assumptions (i)-(iii). The optimal control input sequence is then

$u(k)=u^* = f^{-1}(H),\ k=1,2...\ N$

where f^{-1} is the inverse of f and $H = \alpha - \frac{1}{N}\sum_{k=1}^{N} d(k) = \alpha - \bar{d}$. Hence, the criterion is minimized by a constant input signal.

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Proof of Theorem 1:
We will give two alternative proofs. In Appendix A, a proof based on Jensen's inequality is given and in Appendix B the standard Lagrange function is used.

Comments
We will here give some comments of the result:

- In practice, the optimal value of the control signal is seldom known a priori. Often, however, the disturbances to the plant have a periodic pattern. It is then possible to

iteratively estimate u^* by a recursive algorithm. A simple example is

$$\hat{u}^*(d) = \hat{u}^*(d-1) + K(\bar{y}(d-1) - \alpha)$$

where $\overline{y}$ (d-1) is the mean value obtained during the previous evaluation time, say last day, when the control signal was chosen as $\hat{u}^*(d-1)$. The new control signal (to be applied during the next N samples) is denoted $\hat{u}^*(d)$. The gain K is a user choice which determines the tracking speed vs noise sensitivity.

- In a practical implementation more constraints than (3) may occur. For example, there may be a limit y_{max} on the maximum value of *y(k)* so that $y(k) < y_{max}$ should be fulfilled for all *k*. A solution is then to use a feed back control if y is getting close to y_{max}. Another example is if *u(k)* represents the air flow rate into an activated sludge process. We may then want to impose maximum and minimum of the resulting dissolved oxygen concentrations.

- For the simple process model (1), there is no guarantee that *y(k)* takes only positive values or that the optimal value of u^* is positive. This is not an issue if the strategy is used on a real process but when using the model (1) for illustrations some care must be taken. In order for *y(k)* to be positive we must require $d(k) > \overline{d} - \alpha$. In case we have a maximum value of *u*, say u_{max} , we must require $f(u_{max}) > \alpha - \overline{d}$, which means that we have enough input power to keep the mean value of the output equal to the constraint. Finally, for obvious reasons it is natural to assume $\alpha < \overline{d}$.

- Even though the proof is valid for static nonlinear processes we have empirically found that the results (ie that a constant input signal is optimal) can give a good guideline for controlling dynamic processes which have a steady state behavior which fulfils the conditions imposed on the function *f*.

Illustration of Theorem 1

In order to illustrate the basics of Theorem 1, the following simple example will be used.
Let

$$f\big(u(k)\big) = -K_M \frac{u(k)}{u(k) + K_S} \qquad k = 1,2,\ldots,24$$

where, initially, $K_M=40$ and $K_S=1$. The control signal hence has a Monod-type effect on the process, see also Figure 3. The disturbance *d(k)* is shown in Figure 4.

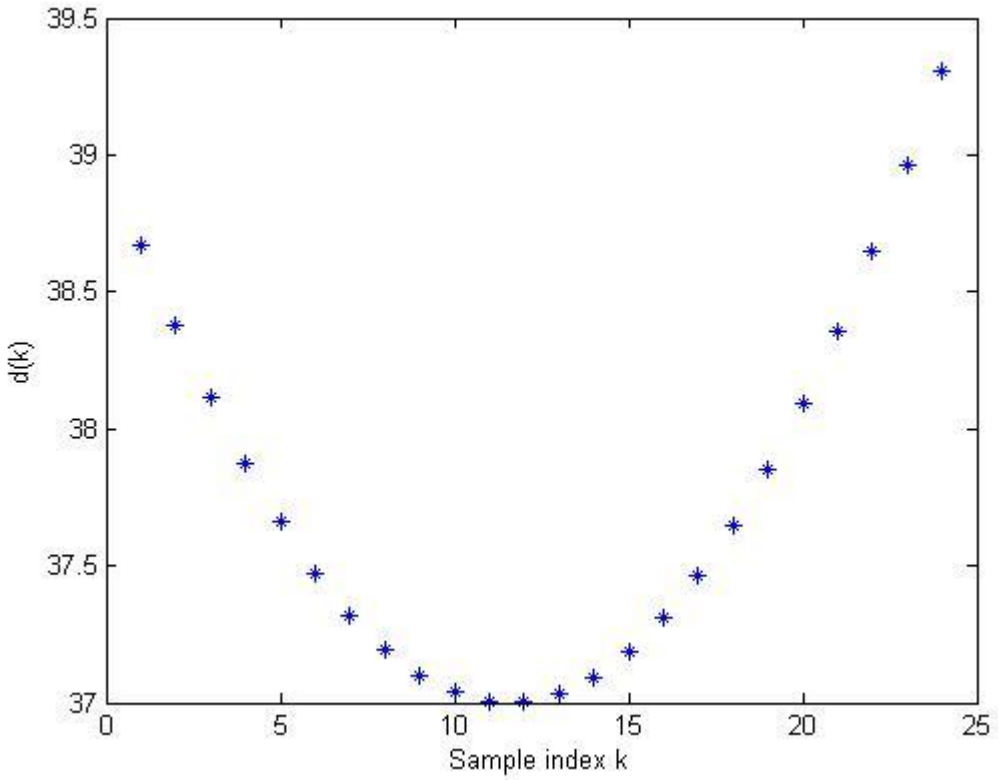

Figure 4: The disturbance *d(k).*

The objective is to find the control signal which minimizes (2) subject to the constraint (3) where N=24 and α=1. The minimization was done using the function fmincon (a function for constrained nonlinear optimization) in Matlab. Hence, the result in Theorem 1 was not used to find the minima. The result of the optimization is shown in Figure 5.

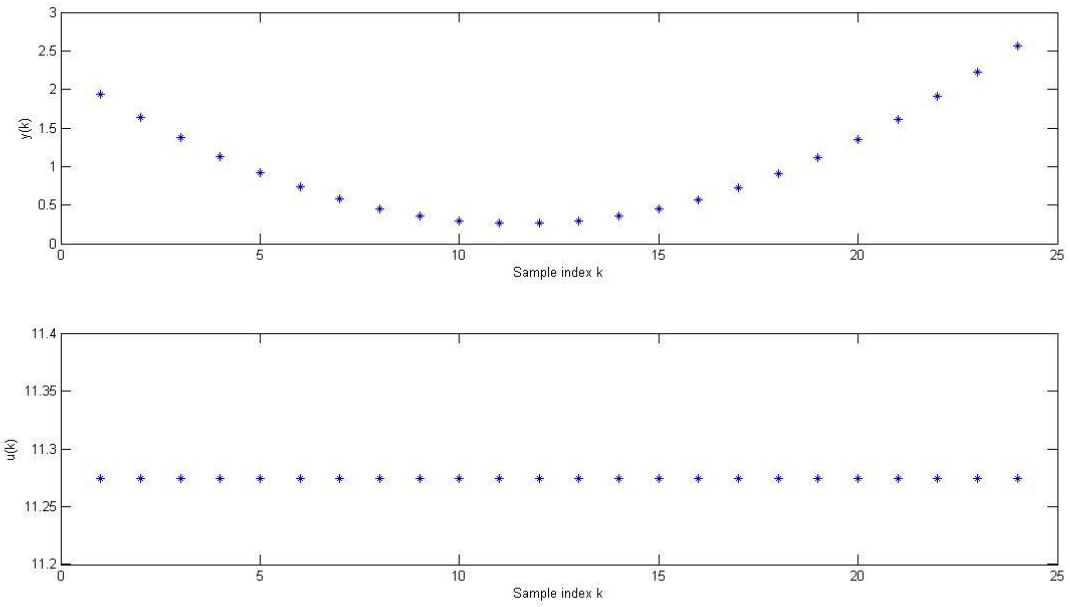

Figure 5: Upper plot - the output signal *y(k)*, Lower plot – the optimal input signal *u(k)*.

As expected from Theorem 1, the minimization done in Matlab produces a constant control signal (yielding a mean value of y(k) equal to one). From Theorem 1 we can calculate the constant value of the optimal input signal. The mean value of the disturbance is $\overline{d} = 37.74$ giving $H = -36.74$. The inverse of the chosen function is:

$$f^{-1}(z) = -\frac{K_S z}{K_M + z} = -\frac{z}{40 + z}$$

We now get $u^* = f^{-1}(-36.74) = 11.27$ which agrees with the numerical optimization shown in Figure 5.

Finally we will briefly illustrate the gain which is achieved by the optimal control signal compared to when the constraint is fulfilled for all *k*, that is y(k) = α =1, k=1,2...,24. In the latter case, the control signal is given by $u(k) = f^{-1}(\alpha - d(k)),\ k=1,2...24$. This corresponds to an ideal feedforward (FF) compensator. We denote the loss function (2) L_{opt} when the optimal input signal is used and L_{FF} when the feedforward strategy is used. Table 2 presents the results for a few trials.

Table 2 Comparison between the optimal strategy (constant input signal) and an ideal feedforward control.

K_m	L_{opt}	L_{FF}
80	*20.38*	*20.39*
40	*270.6*	*286.9*
38	*700.4*	*1771.7*

When K_M is large, the loss function is practically equal for the two strategies. The reason is that for this case, the required mean value of *y* is achieved by a low magnitude of the control signal. The process is then practically linear and the two strategies are equal. On the other hand when *Km* is decreased, higher control signals are needed and due to the saturation effect of *f* it is increasingly costly to fulfil the constraint at all times. Note that similar results will also be obtained by varying the amplitude of the disturbance.

CONCLUSIONS

The problem how to control a class of processes in a resource optimal way have been treated. We have shown for a class of processes it is optimal to keep the input signal constant. The constant optimal value of the input signal depends on the function *f*, the required mean value of the output signal, and the mean value of the disturbance. If the disturbance has a periodic pattern the optimal inputs signal may be estimated by a simple recursive algorithm.

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APPENDIX A – Proof of Theorem 1 using Jensen's inequality

The proof of Theorem 1 is based on two technical lemmas which are presented first.

Lemma 1 (Jensen's inequlity):
For a convex function *g*, numbers x(1), x(2), ..., x(N) in its domain, and positive numbers a(1), a(2), ..., a(N), Jensen's inequality can be stated as:

$$g\left(\frac{\sum_{k=1}^{N} a(k)x(k)}{\sum_{k=1}^{N} a(k)}\right) \leq \frac{\sum_{k=1}^{N} a(k)g(x(k))}{\sum_{k=1}^{N} a(k)}$$

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Proof of Lemma 1:
See, for example, Lohwater (1982)

Lemma 2 (Convexity of the inverse):
Consider a function f which fulfil Assumptions (i)-(iii). Let *z=f(u*), and *u=g(z)=g(f(u))*, where $g=f^{-1}$. The function *g* is then convex that is $\frac{d^2g}{dz^2} > 0$.

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Proof of Lemma 2:
Applying the chain rule on *u=g(z)=g(f(u))* gives

$$1 = \frac{dg}{dz}\frac{df}{du}$$

$$0 = \frac{d^2g}{dz^2}\left(\frac{df}{du}\right)^2 + \frac{dg}{dz}\frac{d^2f}{du^2}$$

The first relation gives $\frac{dg}{dz} < 0$ since $\frac{df}{du} < 0$ (Assumption (ii)). Next, use $\frac{d^2f}{du^2} > 0$ (Assumption (iii) and $\frac{dg}{dz} < 0$ in the second relation which shows the lemma.

■

We are now ready to present the main result:

Proof of Theorem 1:

By inserting (1) in (3) we get the constraint $\frac{1}{N}\sum_{k=1}^{N}(f(u(k))+d(k))=\alpha$. We can now write the optimization problem form as

$$\text{minimize } \sum_{k=1}^{N} u(k)$$

$$\text{subject to } \sum_{k=1}^{N} f(u(k)) = NH$$

where $H=\alpha-\frac{1}{N}\sum_{k=1}^{N} d(k)$

In order to proceed we define a new variable $z=f(u)$, Since f is strictly decreasing it has a unique inverse and we can write $u=f^{-1}(z)$ or $u=g(z)$ where $g=f^{-1}$ We can now rephrase the optimization problem as

$$\text{minimize } \sum_{k=1}^{N} g(z(k))$$

$$\text{subject to } \sum_{k=1}^{N} z(k) = NH$$

By applying Jensen's inequality, using Lemma 1 with $a(k)=1$, to $g(z(k))$ (which is convex according to Lemma 2) we get

$$\frac{1}{N}\sum_{k=1}^{N} g(z(k)) \geq g\left(\frac{\sum_{k=1}^{N} z(k)}{N}\right) = g(H)$$

In the last equality the constraint has be used. Hence,

$$\sum_{k=1}^{N} g(z(k)) \geq N\, g(H)$$

It is immediately seen that the minima is achieved by choosing $z(k)=H,\ k=1,2,...N$ (which fulfils the constraint). This means that the optimal solution for the original problem is the constant input signal $u(k)=u^*=g(H)=f^{-1}(H),\ k=1,2,...N$.

APPENDIX B – Proof of Theorem 1 using the Lagrange function

By inserting (1) in (3) we get the constraint $\frac{1}{N}\sum_{k=1}^{N} f(u(k)) + d(k) = \alpha$. We can now write the optimization problem in standard form as

minimize *c(u)*

subject to *h(u)=0*

where

$$h(u) = \frac{1}{N}\sum_{k=1}^{N}(f(u(k)) + d(k)) - \alpha$$

The Lagrange function (Bryson and Ho, 1969) is defined as

$$L(u) = c(u) + \lambda h(u) = \sum_{k=1}^{N}(u(k) + \lambda\frac{1}{N}\sum_{k=1}^{N}(f(u(k)) + d(k)) - \alpha$$

The first-order necessary conditions for a solution to be a local minimum subject to the constraint is that

$$\nabla c(u) + \lambda\nabla h(u) = 0$$

We hence get

$$1 + \lambda\frac{1}{N}\frac{\partial f(u(k))}{\partial u(k)} = 0 \qquad k = 1,2...N \tag{B1}$$

The derivative of f with respect to u must hence be constant for the whole input sequence. Since we have a one-to-one correspondence between the input and the derivative of f (Assumption iii), the condition (B1) implies

$$u(k) = u^* \quad k = 1,2...N \tag{B2}$$

Hence, a constant input sequence is necessary for an optimal solution.

In order to proceed we need to show that (B2) also is a sufficient condition. A sufficient condition can be found by using the bordered Hessian matrix (Magnus and Neudecker, 1988)

$$B_r = \begin{bmatrix} 0 & \nabla h_r \\ \nabla h_r{}^T & \nabla_u^2 L_{rr} \end{bmatrix} \qquad r = 1,...,N$$

where

$$\nabla h_r = (\frac{\partial h(u)}{\partial u(1)}, \frac{\partial h(u)}{\partial u(2)}.... \frac{\partial h(u)}{\partial u(r)})$$

and

$$\nabla_u^2 L_{rr} = \begin{bmatrix} \dfrac{\partial^2 L(u)}{\partial u(1)\partial u(1)} & \cdots & \dfrac{\partial^2 L(u)}{\partial u(1)\partial u(r)} \\ \vdots & \vdots & \vdots \\ \dfrac{\partial^2 L(u)}{\partial u(r)\partial u(1)} & & \dfrac{\partial^2 L(u)}{\partial u(r)\partial u(r)} \end{bmatrix}$$

From Theorem 12, Chapter 7 in Magnus and Neudecker (1988), we have that if

$$\det B_r < 0 \quad r = 2,...,N$$

the stationary point (in our case $u(k)=u^*$) is a strict local minimum.

In our case (assumption (ii)) gives

$$\frac{\partial h(u)}{\partial u(1)} = \frac{\partial h(u)}{\partial u(2)} = ... = \frac{\partial h(u)}{\partial u(N)} = -b \quad (b > 0)$$

Furthermore

$$\nabla_u^2 L_{rr} = \begin{bmatrix} a & 0 & \cdots & 0 \\ 0 & a & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{bmatrix} \quad (a > 0)$$

This follows from assumption (iii) and that λ must be positive. We must hence prove that

$$\det B_r = \det \begin{bmatrix} 0 & -b & \cdots & -b \\ -b & a & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -b & 0 & \cdots & a \end{bmatrix} < 0, \quad r = 2,...,N$$

In order to proceed we use the following matrix partitioning

$$\det \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det D \det \left[A - BD^{-1}C \right]$$

with $A=0$, $B=\Delta h_r$, $C=\Delta h^T$ and $D=\Delta^2_u L_{rr}$ which gives

$$\det B_r = \det \begin{bmatrix} a & 0 & \cdots & 0 \\ 0 & a & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{bmatrix} \det \left[0 - \begin{bmatrix} -b & \cdots & -b \end{bmatrix} \begin{bmatrix} a & 0 & \cdots & 0 \\ 0 & a & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{bmatrix}^{-1} \begin{bmatrix} -b \\ \vdots \\ -b \end{bmatrix} \right] = -a^r r \frac{(-b)^2}{a} = -ra^{r-1}(-b)^2$$

Hence det $B_r<0$ and we have a sufficient condition for a local minimum.