Preconditioning the incompressible Navier-Stokes equations with variable viscosity

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Abstract

This paper deals with preconditioners for the iterative solution of the discrete Oseen's problem with variable viscosity. The motivation of this work originates from numerical simulations of multiphase flow, governed by the coupled Cahn-Hilliard and incompressible Navier-Stokes equations. The impact of variable viscosity on some known preconditioning technique is analyzed. Numerical experiments show that the preconditioning technique for the Oseen's problem with constant viscosity is also efficient when the viscosity is varying.

1 Introduction

In this paper we consider preconditioned iterative solution methods for the stationary incompressible Navier-Stokes (N-S) equations with variable viscosity. Here we assume that the kinematic viscosity coefficient is a smooth function, such that

$$0 \leq \nu_{\min} \leq \nu(\mathbf{x}) \leq \nu_{\max},$$

where $\nu_{\min}$ and $\nu_{\max}$ denote its minimal and maximal value. Many mathematical models in fluid dynamics involve nonconstant viscosity. For example, viscosity is a function of the temperature in convection flows (see e.g. [13, 31]); it is a function of pressure and the rate-of-strain tensor in non-Newtonian flows (see e.g. [8, 30]). In some non-Newtonian flows the variable viscosity may also depend on pressure and shear (see e.g. [22, 25, 29]). In this paper the motivation to consider models with variable viscosity arises from numerical simulations of multiphase flow, which is often simulated by the so-called phase-field model. The phase-field model is used to model two immiscible and incompressible fluids and is described by the Cahn-Hilliard (C-H) equation [9, 10]. By taking into account the convective effect of the fluid's motion, a convective form of the time-dependent C-H equation is derived (see e.g. [11])

$$\frac{\partial C}{\partial t} + (\mathbf{u} \cdot \nabla)C = \nabla \cdot [\kappa(C)\nabla(\beta\Psi'(C) - \alpha\Delta C)], \quad \text{in } \Omega \times (0, T] \tag{1}$$

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with suitable boundary and initial conditions for the primal variable C. Here C represents the different phases and is referred to as the *phase field* or the *concentration*. It takes distinct values in each of the phases (for instance $+1$ and -1 for a binary fluid), with a smooth and rapid change between both values in the interface zone. The coefficient $\kappa(C)$ is the so-called *mobility*, assumed to be a function of the *concentration* C. The coefficients α and β are constants. The function $\Psi(C)$ is a double well potential with minimal value at $+1$ and -1 (under the assumption that the *concentration* C varies between $+1$ and -1). For instance, $\Psi(C) = \frac{1}{4}(C^2-1)^2$. One can refer to [32, 11, 12, 23] for more details of the C-H equation. In equation (1), the vector $\mathbf{u}$ denotes the velocity. The term $\mathbf{u}\cdot\nabla$ presents the convective effect of the fluid's motion, governed by the time-dependent incompressible Navier-Stokes (N-S) equations

$$\rho(\frac{\partial \mathbf{u}}{\partial \mathrm{t}} + \mathbf{u}\cdot\nabla\mathbf{u}) - \nabla\cdot(2\mu\mathbf{D}\mathbf{u}) + \nabla p = \mathbf{f} - (\beta\Psi'(C) - \alpha\Delta C)\nabla C, \quad \text{in } \Omega\times(0,T] \tag{2}$$

$$\frac{\partial \rho}{\partial \mathrm{t}} + \nabla\cdot(\rho\mathbf{u}) = 0, \quad \text{in } \Omega\times(0,T] \tag{3}$$

$$\nabla\cdot\mathbf{u} = 0, \quad \text{in } \Omega\times(0,T] \tag{4}$$

with some given boundary and initial conditions for $\mathbf{u}$. Here $\Omega\times(0,T]\subset\mathbb{R}^d$ $(d=2,3)$ is a bounded, connected domain with boundary $\partial\Omega$ and $\mathbf{f}:\Omega\to\mathbb{R}^d$ is a given force field. The operator $\mathbf{D}\mathbf{u} = (\nabla\mathbf{u} + \nabla^T\mathbf{u})/2$ denotes the rate-of-strain tensor for Newtonian fluids and the force term $(\beta\Psi'(C) - \alpha\Delta C)\nabla C$ denotes the surface tension force and constitutes the coupling with the C-H equation (1) (cf, e.g., [15]). The coefficient μ denotes the dynamic viscosity and ρ denotes the density.

We point out that a numerical simulation of a multiphase flow problem requires to solve the coupled system, consisting of the time-dependent C-H and incompressible N-S equations, where the N-S equations are formulated in their full complexity, including the time-dependence, variable density and variable viscosity. Note, that density and viscosity remain constant within each phase, however they vary in the interface region, which evolves with time (cf, e.g., [15]). Therefore, these can be seen as smooth functions of space position and time in the whole domain.

In this paper we limit ourselves to the stationary incompressible N-S equations with constant density while allowing viscosity to vary. The main task in this paper is to analyse the effect of variable viscosity on some of the established preconditioning techniques, used for the system matrix arising form the finite element (FEM) discretization of the stationary incompressible N-S equations with constant viscosity. The structure of the paper is as follows. In Section 2 we state the problem setting. In Section 3 we recall the augmented Lagrangian (AL) method and analyse the impact of variable viscosity on the AL preconditioner. In Section 4 we briefly recall the element-by-element sparse approximate inverse technique which constructs an approximate inverse of the finite element mass matrix, which is involved in the AL preconditioner. Section 5 contains numerical illustrations. Some conclusions are given in Section 6.

2 Problem setting and preliminaries

As mentioned above, in this paper we focus on preconditioners for the iterative solution of the stationary incompressible N-S problems with variable viscosity and constant density ($\rho = 1$ for simplicity). The governing equations read as follows.

$$\begin{aligned} -\nabla \cdot (2\nu(\mathbf{x})\mathbf{D}\mathbf{u}) + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = \mathbf{f}, \quad & \text{in } \Omega \times (0, T] \\ \nabla \cdot \mathbf{u} = 0, \quad & \text{in } \Omega \times (0, T] \end{aligned} \tag{5}$$

where the coefficient ν denotes the kinematic viscosity which is defined as $\nu = \frac{\mu}{\rho}$.
Remark.

(1) In the case of constant viscosity and with the condition $\nabla \cdot \mathbf{u} = 0$, we have

$$\begin{aligned} \nabla \cdot (2\nu\mathbf{D}\mathbf{u}) &= \nabla \cdot (\nu\nabla\mathbf{u}) + \nabla \cdot (\nu\nabla^T\mathbf{u}) \\ &= \nu\nabla^2\mathbf{u} + \nu\nabla(\nabla \cdot \mathbf{u}) \\ &= \nu\nabla^2\mathbf{u}. \end{aligned}$$

(2) There are several ways to treat the nonlinear convection term $(\mathbf{u}\cdot\nabla)\mathbf{u}$ in (5). (i) One option is to treat it explicitly and move it into the right hand side. This leads to a Stokes-type problem with variable viscosity and the matrix in the linear systems to be solved during each nonlinear iteration does not change. For this formulation, a block preconditioner involving a mass-type matrix for the pressure is proposed and analyzed in [19]. However, the above treatment of the nonlinear term may be unsuitable for more convection-dominated problems. (ii) Another option is to linearize and discretize the convection term, and incorporate the corresponding matrix term into the system matrix. Thus, an Oseen-type problem with variable viscosity arises. In this paper, we focus on the latter approach.

Here, when solving (5) we choose to apply Picard iterations. The technique requires to solve a sequence of approximate solutions of the linear Oseen's problem (see e.g. [17]), which reads as follows:
At each Picard iteration, find $\mathbf{u} : \Omega \times (0, T] \to \mathbb{R}^d$ and $p : \Omega \times (0, T] \to \mathbb{R}$ satisfying

$$\begin{aligned} -\nabla \cdot (2\nu(\mathbf{x})\mathbf{D}\mathbf{u}) + (\mathbf{w} \cdot \nabla)\mathbf{u} + \nabla p = \mathbf{f}, \quad & \text{in } \Omega \\ \nabla \cdot \mathbf{u} = 0, \quad & \text{in } \Omega \end{aligned} \tag{6}$$

subject to suitable boundary conditions for $\mathbf{u}$ on $\partial\Omega$. Here $\mathbf{w} = \mathbf{u}^{(k-1)}$ is the velocity, which has been computed in the previous Picard iteration, and is updated at every Picard iteration.

Let $\mathbf{H}^1_E = \{\mathbf{u} \in \mathcal{H}^1(\Omega)^d |\ \mathbf{u} = \mathbf{w} \text{ on } \partial\Omega_D\}$ and $H^1_{E_0} = \{\mathbf{v} \in \mathcal{H}^1(\Omega)^d |\ \mathbf{v} = \mathbf{0} \text{ on } \partial\Omega_D\}$. The weak formulation of (6) reads as follows:
Find $\mathbf{u} \in \mathbf{H}^1_E$ and $\mathbf{p} \in L_2(\Omega)$ such that

$$\begin{aligned} (2\nu(\mathbf{x})\mathbf{D}\mathbf{u}, \mathbf{D}\mathbf{v}) + ((\mathbf{w} \cdot \nabla)\mathbf{u}, \mathbf{v}) - (\nabla \cdot \mathbf{v}, \mathbf{p}) &= (\mathbf{f}, \mathbf{v}), \\ (\nabla \cdot \mathbf{u}, \mathbf{q}) &= 0, \end{aligned} \tag{7}$$

for all $\mathbf{v} \in H^1_{E_0}$ and all $\mathbf{q} \in L_2(\Omega)$.

For clarity, we consider the term $(2\nu(\mathbf{x}), \mathbf{Du}, \mathbf{Dv})$. If we perform integration by parts of the term $-\int_\Omega \nabla \cdot (2\nu(\mathbf{x})\mathbf{Du})\mathbf{v}d\Omega$, its equivalence to the term $(2\nu(\mathbf{x})\mathbf{Du}, \nabla\mathbf{v})$ appears straightforwardly. Indeed,

$$\begin{aligned}
(2\nu(\mathbf{x})\mathbf{Du}, \mathbf{Dv}) &= 2 \times \frac{1}{4} \times \int_\Omega \nu(\mathbf{x}) \sum_{i,j}^{d} (\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}) \\
&= 2 \times \frac{1}{4} \times \{\int_\Omega \nu(\mathbf{x}) \sum_{i,j}^{d} \frac{\partial v_i}{\partial x_j}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}) + \int_\Omega \nu(\mathbf{x}) \sum_{i,j}^{d} \frac{\partial v_j}{\partial x_i}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})\} \\
&= 2 \times \frac{1}{2} \times \int_\Omega \nu(\mathbf{x}) \sum_{i,j}^{d} \frac{\partial v_i}{\partial x_j}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}) \\
&= (2\nu(\mathbf{x})\mathbf{Du}, \nabla\mathbf{v}),
\end{aligned}$$

where in two dimensions $d = 2$ and in two dimensions $d = 3$. It is readily seen, that the discrete matrix operator, corresponding to the form $(2\nu(\mathbf{x})\mathbf{Du}, \mathbf{Dv})$, is a symmetric and positive definite.

Let $\mathbf{X}^h_E$ and P^h be the finite dimensional subspaces of $\mathbf{H}^1_E$ and $L_2(\Omega)$. The discrete formulation of the weak form (6) defined using finite-dimensional spaces reads:
Find $\mathbf{u}_h \in \mathbf{X}^h_E \subset \mathbf{H}^1_E$ and $\mathbf{p}_h \in P^h \subset L_2(\Omega)$ such that

$$\begin{aligned}
(2\nu(\mathbf{x})\mathbf{Du}_h, \mathbf{Dv}_h) + ((\mathbf{w} \cdot \nabla)\mathbf{u}_h, \mathbf{v}_h) - (\nabla \cdot \mathbf{v}_h, \mathbf{p}_h) &= (\mathbf{f}_h, \mathbf{v}_h), \\
(\nabla \cdot \mathbf{u}_h, \mathbf{q}_h) &= 0,
\end{aligned} \tag{8}$$

for all $\mathbf{v}_h \in \mathbf{X}^h_{E_0} \subset H^1_{E_0}$ and all $\mathbf{q}_h \in P^h$.
Let $\{\vec{\varphi}_i\}_{1 \le i \le n_u}$ be the nodal basis of $\mathbf{X}^h_E$ and $\{\phi_i\}_{1 \le i \le n_p}$ be the nodal basis of P^h such that,

$$\mathbf{u}_h = \sum_{i=1}^{n_u} \mathbf{u}_i \vec{\varphi}_i, \quad \mathbf{p}_h = \sum_{i=1}^{n_p} p_i \phi_i,$$

where n_u and n_p are the total number of unknowns for the velocity and pressure. The linear systems arising from the discrete weak formulation (8) are of the form

$$\begin{bmatrix} F & B^T \\ B & O \end{bmatrix} \begin{bmatrix} \mathbf{u}_h \\ \mathbf{p}_h \end{bmatrix} = \begin{bmatrix} \mathbf{f} \\ \mathbf{g} \end{bmatrix} \quad \text{or} \quad \mathcal{A}\mathbf{x} = \mathbf{b}, \tag{9}$$

where the system matrix $\mathcal{A} = \begin{bmatrix} F & B^T \\ B & O \end{bmatrix}$ is nonsymmetric of saddle point form. We also assume that the discretization is done using a stable pair of FEM spaces, satisfying the Ladyzenskaya-Babuška-Brezzi (LBB) condition (cf, e.g., [17]). The unknown vector $\mathbf{u}_h$ is the discrete velocity vector and $\mathbf{p}_h$ is the discrete pressure vector. Combing them together we set $\mathbf{x}^T = [\mathbf{u}_h^T \quad \mathbf{p}_h^T]$. The matrix $B \in \mathbb{R}^{n_u \times n_p}$ corresponds to the discrete (negative)

divergence operator and B^T corresponds to the discrete gradient operator (see e.g. [17]). Clearly, when considering variable viscosity, the difference, compared to the stationary incompressible N-S equations with constant viscosity, can be observed in the pivot block $F \in \mathbb{R}^{n_u \times n_u}$, which, in the case of variable viscosity, has the form $F = A_\nu + N$. We discuss A_ν and N below.

Based on the discrete weak formulation (8) and combined with the the nodal basis $\{\vec{\varphi}_i\}_{1 \le i \le n_u}$, the matrix A_ν is the discrete operator, corresponding to the term $(2\nu(\mathbf{x})\mathbf{Du}, \mathbf{Dv})$, i.e.,

$$A_\nu \in \mathbb{R}^{n_u \times n_u}, \quad [A_\nu]_{i,j} = (2\nu(\mathbf{x})\mathbf{D}\vec{\varphi}_i, \mathbf{D}\vec{\varphi}_j). \tag{10}$$

And the matrix N is the discrete operator, corresponding to the term $((\mathbf{w} \cdot \nabla \mathbf{u}), \mathbf{v})$, i.e.,

$$N \in \mathbb{R}^{n_u \times n_u}, \quad [N]_{i,j} = ((\mathbf{w} \cdot \nabla \vec{\varphi}_i), \vec{\varphi}_j). \tag{11}$$

The system matrix $\mathcal{A}$ in (9), for both constant and variable viscosity, is nonsymmetric of saddle point form. As already mentioned, the matrix A_ν, defined in (10), is symmetric and positive definite. Indeed, since $(2\nu(\mathbf{x})\mathbf{Du}, \mathbf{Dv}) = (2\nu(\mathbf{x})\mathbf{Dv}, \mathbf{Du})$ for any $\mathbf{u}, \mathbf{v} \in \mathbf{H}^1$, the matrix A_ν is symmetric. Further, $[A_\nu]_{i,j} = ((2\nu(\mathbf{x})\mathbf{D}\vec{\varphi}_i, \mathbf{D}\vec{\varphi}_j)$ and for any $\mathbf{y} \in \mathbb{R}^{n_u}$

$$\mathbf{y}^T A_\nu \mathbf{y} = \int_\Omega 2\nu(\mathbf{x})(\mathbf{D}\mathbf{y}_h)^2,$$

where $\mathbf{y}_h = \sum_{i=1}^{n_u} \mathbf{y}_i \vec{\varphi}_i$. With the condition $\nu(\mathbf{x}) > 0$ we see that $\mathbf{y}^T A_\nu \mathbf{y} \ge 0$ for any $\mathbf{y} \in \mathbb{R}^{n_u}$. The equality holds true if and only if $\mathbf{D}\mathbf{y}_h = 0 \Rightarrow \mathbf{y} = 0$.

For the purpose of the analysis in Section 3, we consider the following modified form of the weak formulation (7)

$$\begin{aligned} (2\nu(\mathbf{x})\mathbf{Du}, \mathbf{Dv}) + \frac{1}{2}(((\mathbf{w} \cdot \nabla)\mathbf{u}, \mathbf{v}) - ((\mathbf{w} \cdot \nabla)\mathbf{v}, \mathbf{u})) - (\nabla \cdot \mathbf{v}, \mathbf{p}) &= (\mathbf{f}, \mathbf{v}), \\ (\nabla \cdot \mathbf{u}, \mathbf{q}) &= 0. \end{aligned} \tag{12}$$

This formulation is discussed in [20]. The difference between formulations (8) and (12) exhibits itself in the matrix, corresponding to the convection part. Let $\tilde{c}(\mathbf{w}, \mathbf{u}, \mathbf{v}) = \frac{1}{2}(c(\mathbf{w}, \mathbf{u}, \mathbf{v}) - c(\mathbf{w}, \mathbf{v}, \mathbf{u}))$, where $c(\mathbf{w}, \mathbf{u}, \mathbf{v}) = ((\mathbf{w} \cdot \nabla)\mathbf{u}, \mathbf{v})$. It is straightforward to see that $\tilde{c}(\mathbf{w}, \mathbf{u}, \mathbf{v}) = -\tilde{c}(\mathbf{w}, \mathbf{v}, \mathbf{u})$ for any $\mathbf{w}, \mathbf{u}, \mathbf{v} \in \mathbf{H}^1_E$. Thus, the matrix N_s, corresponding to the discrete operator $\tilde{c}(\mathbf{w}, \mathbf{u}, \mathbf{v})$, is skew-symmetric, and $N_s = \frac{1}{2}(N - N^T)$ (N is the same as in (11)).

Consider the form $\tilde{c}(\mathbf{w}, \mathbf{u}, \mathbf{v})$ and the operator $l(\mathbf{u}, \mathbf{v}) = (2\nu(\mathbf{x})\mathbf{Du}, \mathbf{Dv})$, where $\mathbf{Du} = \frac{1}{2}(\nabla \mathbf{u} + \nabla^T \mathbf{u})$. The following results holds.

Proposition 2.1 *Consider the above operators l and $\tilde{c}$. Assume that $\boldsymbol{w}$ and $\nu(\boldsymbol{x})$ are bounded in Ω and that the part of $\partial\Omega$, where Dirichlet conditions for velocity $\boldsymbol{u}$ are imposed, is not empty ($\partial\Omega_D \neq \emptyset$). Then there holds,*

$$\| \, l^{-1}\tilde{c}(\boldsymbol{w}, \boldsymbol{u}, \boldsymbol{u}) \, \|_{L_2} \le \frac{C}{\sqrt{\nu_{\min}}}. \tag{13}$$

Proof. We have

$$\| l(\mathbf{u},\mathbf{u}) \|_{L_2}^2 = \int_\Omega 2\nu \mid \mathbf{Du} \mid^2 \geq 2\nu_{\min} \int_\Omega \mid \mathbf{Du} \mid^2 \geq C_1 \nu_{\min} \| \nabla \mathbf{u} \|_{L_2}^2 . \tag{14}$$

For the last part of relation (14) we use Korn's inequality.
Further,

$$\begin{aligned} \| \tilde{c}(\mathbf{w},\mathbf{u},\mathbf{u}) \|_{L_2}^2 &= \int_\Omega \mid \tilde{c}(\mathbf{w},\mathbf{u},\mathbf{u}) \mid \\ &\leq C_2 \| \mathbf{w} \|_\infty \| \nabla \mathbf{u} \|_{L_2} \| \mathbf{u} \|_{L_2} \\ &\leq C_3 \| \mathbf{w} \|_\infty \| \nabla \mathbf{u} \|_{L_2}^2 . \end{aligned} \tag{15}$$

For the last part of relation (15) we use the assumption that $\partial\Omega_D \neq \emptyset$.

Combing (14) and (15), we obtain,

$$\| \tilde{c}(\mathbf{w},\mathbf{u},\mathbf{u}) \|_{L_2}^2 \leq \frac{C}{\nu_{\min}} \| l(\mathbf{u},\mathbf{u}) \|_{L_2}^2 . \tag{16}$$

Thus, $\| l^{-1}\tilde{c}(\mathbf{w},\mathbf{u},\mathbf{u}) \|_{L_2} \leq \frac{C}{\sqrt{\nu_{\min}}}$. ∎

Proposition 2.1 shows, that the absolute values of the eigenvalues of the discrete operator, corresponding to $l^{-1}\tilde{c}$ are bounded from above by a factor $\frac{C}{\sqrt{\nu_{\min}}}$, independently of the discretization parameter h.

Remark. The same result can be shown for the discrete matrix operators, namely, that the modulus of the eigenvalues of the skew-symmetric matrix $\widetilde{N} = A_\nu^{-1/2} N_s A_\nu^{-1/2}$, is bounded independently of the mesh size h. This result is used in [16] without a rigorous proof and is used for the analysis in Section 3 of this paper.

In this paper, formulation (12) is used to derive some spectrum bounds in Section 3. The numerical experiments in Section 5 are based on the original formulation (7).

3 Preconditioning strategy

Now we turn to the task to construct an efficient preconditioner for the system matrix $\mathcal{A}$ in (9), which is nonsymmetric of saddle point form. Preconditioned iterative solution methods for saddle point problems have been studied intensively during the last 30 years (see e.g. [17, 4] and the references therein). The exact factorization of a general matrix of two-by-two block form is

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & 0 \\ A_{21} & S \end{bmatrix} \begin{bmatrix} I_1 & A_{11}^{-1}A_{12} \\ 0 & I_2 \end{bmatrix}, \tag{17}$$

where I_1 and I_2 are identity matrices of proper dimensions. The pivot block A_{11} is assumed to be nonsingular and $S = A_{22} - A_{21}A_{11}^{-1}A_{12}$ is the exact Schur complement matrix. In our case, $A_{11} = F$, $A_{12} = B^T$, $A_{21} = B$ and $A_{22} = O$. So, $S_{\mathcal{A}} \equiv S = -BF^{-1}B^T$.

Some of the best known preconditioners for such two-by-two matrices are based on the exact factorization (17), where some of the blocks are approximated or neglected. Most often, block lower- or upper-triangular preconditioners are used, of the form

$$\mathcal{M}_L = \begin{bmatrix} \widetilde{A}_{11} & O \\ A_{21} & \widetilde{S} \end{bmatrix}, \qquad \mathcal{M}_U = \begin{bmatrix} \widetilde{A}_{11} & A_{12} \\ 0 & \widetilde{S} \end{bmatrix}. \tag{18}$$

Here, $\widetilde{A}_{11}$ denotes some approximation of A_{11}, given on explicit form or implicitly defined via iterative solution methods. The matrix $\widetilde{S}$ is some approximation of the exact Schur complement S.

The results in [3] show that the quality of the preconditioners in (18) depends on how well the pivot block A_{11} and the exact Schur complement matrix S are approximated. Compared to the approximations of A_{11}, the most challenging task, however, is how to construct numerically and computationally efficient approximations for the Schur complement. The research on Schur complement approximations for the Oseen's problem with constant viscosity is an active field during the past years (see e.g. [16, 17, 28] and the references therein, also [26]). Since the system matrices, arising from the Oseen's problem, share the same structure for both constant and variable viscosity, except for the difference in the pivot block, it is a nature idea to extend the known approximation techniques presented in [26, 28] to the Oseen's problem with variable viscosity. For example, a scaled pressure mass matrix $M_\nu = \{M_{\nu ij}\} \in \mathbb{R}^{n_p \times n_p}$ with $[M_\nu]_{ij} = (\nu^{-1}\phi_i, \phi_j)$ is proposed in [19] as an approximation of the negative Schur complement of the original system matrix. As mentioned already, the method in [19] suggests, when solving the stationary incompressible Navier-Stokes equations, at each nonlinear step to treat the convection term $(\mathbf{u} \cdot \nabla)\mathbf{u}$ as a force term and to move it to the right hand side. This approach is simple but may not be efficient for more convection-dominated problems.

In this paper we utilize the same AL preconditioning strategy as shown in [7, 6, 21] for the Oseen's problem with constant viscosity and extend it to the variable viscosity's case. Following the standard AL technique (for an earlier reference, see [18]), firstly, we algebraically transform the system (9) into an equivalent one

$$\begin{bmatrix} F + \gamma B^T W^{-1} B & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u}_h \\ \mathbf{p}_h \end{bmatrix} = \begin{bmatrix} \widehat{\mathbf{f}} \\ \mathbf{g} \end{bmatrix} \quad \text{or} \quad \widetilde{\mathcal{A}}\mathbf{x} = \widehat{\mathbf{b}}, \tag{19}$$

where $\widehat{\mathbf{f}} = \mathbf{f} + \gamma B^T W^{-1}\ \mathbf{g}$, and $\gamma > 0$ and W are suitable scalar and matrix parameters. Clearly, the transformation (19) holds true for any value of γ, including $\gamma = 1$ or $\gamma \ll 1$, and any nonsingular matrix W.

The equivalent system (19) is what we intent to solve and the extension of some known efficient preconditioner to the system matrix $\widetilde{\mathcal{A}}$ in the case of variable viscosity is the main issue considered in this section. Here, we again choose the AL type preconditioner, which is originally proposed in [6] and used in [7, 21] for the Oseen's problem with constant viscosity.

The AL type preconditioner for the transformed system matrix $\widetilde{\mathcal{A}}$ in (19) is of the form

$$\mathcal{M}_L = \begin{bmatrix} F + \gamma B^T W^{-1} B & 0 \\ B & -\frac{1}{\gamma} W \end{bmatrix}. \tag{20}$$

Comparison between the lower block-triangular preconditioner $\mathcal{M}_L$ in (18) and (20) shows that the exact Schur complement $S_{\widetilde{\mathcal{A}}} = -B(F + \gamma B^T W^{-1} B)^{-1} B^T$ of $\widetilde{\mathcal{A}}$ is approximated by $-\frac{1}{\gamma} W$. In previous studies, i.e., [2] and [14], W is chosen to be the identity matrix. In [6] a good choice for W is shown to be the pressure mass matrix M. For clarity reasons, we repeat here the analysis in [21] and we consider the following generalized eigenproblem

$$\widetilde{\mathcal{A}} \mathbf{v} = \lambda \mathcal{M}_L \mathbf{v}. \tag{21}$$

We see that

$$\mathcal{M}_L^{-1} \widetilde{\mathcal{A}} = \begin{bmatrix} I & (F + \gamma B^T W^{-1} B)^{-1} B^T \\ 0 & \gamma W^{-1} B (F + \gamma B^T W^{-1} B)^{-1} B^T \end{bmatrix}.$$

Thus, we know that the eigenvalues λ form (21) consist of two parts. The first part is equal to 1 (with multiplicity equal to the dimension of F) and the other part coincides with the eigenvalues of the matrix $\gamma W^{-1} B (F + \gamma B^T W^{-1} B)^{-1} B^T$. Applying the Sherman-Morrison-Woodbury's formula to $(F + \gamma B^T W^{-1} B)^{-1}$, we have

$$\gamma W^{-1} B (F + \gamma B^T W^{-1} B)^{-1} B^T = \gamma Q - \gamma Q (I + \gamma Q)^{-1} \gamma Q,$$

where $Q = W^{-1} B F^{-1} B^T$ and $B F^{-1} B^T$ is the the negative Schur complement of the original system matrix $\mathcal{A}$ in (9). The following results hold.

Theorem 3.1 *Let $\mu = a + i\,b$ be an eigenvalue of $Q = W^{-1} B F^{-1} B^T$. Let δ be an eigenvalue of the matrix $\widetilde{Q} = \gamma W^{-1} B (F + \gamma B^T W^{-1} B)^{-1} B^T$. Then the following holds:*

(1) The matrices Q and $\widetilde{Q}$ have the same eigenvectors and the eigenvalues of $\widetilde{Q}$ are equal to

$$\delta = \frac{\gamma \mu}{1 + \gamma \mu} = \frac{1}{1 + \dfrac{1}{\gamma \mu}}. \tag{22}$$

When $\gamma \to \infty$ all nonzero eigenvalues of the eigenproblem (21) converge to 1.

(2) If $\mu = a + i\,b$ are bounded independent on h, then the same holds for λ. Furthermore, if $a > 0$ then there holds

$$\mathcal{R}(\lambda) \leq 1 \quad \textit{and} \quad |\mathcal{I}(\lambda)| \leq \frac{\gamma |b|}{(1 + \gamma a)^2 + (\gamma b)^2}, \tag{23}$$

where $\mathcal{R}(\lambda)$ and $\mathcal{I}(\lambda)$ are the real and imaginary part of the eigenvalues λ of the generalized eigenvalue (21).

Remark. The theorem and its proof can be found in [21], where it is used in the context of Oseen's problem with constant viscosity. In fact, Theorem 3.1 holds true no matter whether the viscosity is constant or variable.

We see from the construction and Theorem 3.1 that on one side large values of γ ensure a better preconditioner. However, the value of γ should not be very large, because the pivot block $F + \gamma B^T W^{-1} B$ becomes increasingly ill-conditioned with $\gamma \to \infty$.

Next, we show that the condition (2) from Theorem 3.1 holds true for the matrix $Q = M^{-1} B F^{-1} B^T$, where M is the pressure mass matrix, which is the main contribution in this paper.

Theorem 3.2 *For $W = M$ (where M is the pressure mass matrix), the eigenvalues of the matrix $Q = M^{-1} B F^{-1} B^T$ are contained in a rectangular box in the right half plane, with boundaries independent of the mesh size h.*

Proof. Let $S = BF^{-1}B^T$, $C = B(\dfrac{F^{-1} + F^{-T}}{2})B^T$ denote the symmetric part of S, and $R = B(\dfrac{F^{-1} - F^{-T}}{2})B^T$ denotes the skew-symmetric part of S. So we have $S = C + R$. By Bendixson's theorem, the eigenvalues μ satisfy

$$\min_{\mathbf{p}} \frac{(\mathbf{p}, C\mathbf{p})}{(\mathbf{p}, M\mathbf{p})} \leq \mathrm{Re}(\mu) \leq \max_{\mathbf{p}} \frac{(\mathbf{p}, C\mathbf{p})}{(\mathbf{p}, M\mathbf{p})} \quad \text{and} \quad |\ \mathrm{Im}(\mu)\ | \leq \max_{\mathbf{p}} \frac{|\ (\mathbf{p}, R\mathbf{p})\ |}{(\mathbf{p}, M\mathbf{p})}.$$

We let $S' = BA_\nu^{-1}B^T$ and denote

$$\nu_{\min} = \inf_{\Omega} \nu(\mathbf{x}), \qquad \nu_{\max} = \sup_{\Omega} \nu(\mathbf{x}).$$

For the symmetric part C, for an arbitrary nonzero vector $\mathbf{p}$ we have

$$\frac{(\mathbf{p}, C\mathbf{p})}{(\mathbf{p}, M\mathbf{p})} = \frac{(\mathbf{p}, C\mathbf{p})}{(\mathbf{p}, S'\mathbf{p})} \frac{(\mathbf{p}, S'\mathbf{p})}{(\mathbf{p}, M\mathbf{p})}.$$

It has been proved in [19] that

$$c_0^2/\nu_{\max} \leq \frac{(\mathbf{p}, S'\mathbf{p})}{(\mathbf{p}, M\mathbf{p})} \leq 1/\nu_{\min}, \tag{24}$$

where the coefficient c_0 comes from the LBB condition.
Next, we consider

$$\begin{aligned} \frac{F^{-1} + F^{-T}}{2} &= F^{-1} \frac{F + F^T}{2} F^{-T} \\ &= (A_\nu + N_s)^{-1} A_\nu (A_\nu - N_s)^{-1} \\ &= A_\nu^{-1/2} (I - \widetilde{N}^2)^{-1} A_\nu^{-1/2}, \end{aligned}$$

where $\widetilde{N} = A_\nu^{-1/2} N_s A_\nu^{-1/2}$. Thus

$$\frac{(\mathbf{p}, C\mathbf{p})}{(\mathbf{p}, S'\mathbf{p})} = \frac{(\mathbf{p}, BA_\nu^{-1/2}(I - \widetilde{N}^2)^{-1} A_\nu^{-1/2} B^T \mathbf{p})}{(\mathbf{p}, BA_\nu^{-1} B^T \mathbf{p})} = \frac{(\mathbf{q}, (I - \widetilde{N}^2)^{-1}\mathbf{q})}{(\mathbf{q}, \mathbf{q})},$$

where $\mathbf{q} = A_\nu^{-1/2} B^T \mathbf{p}$.

The matrix N_s is skew-symmetric, so is $\widetilde{N}$. Thus, the eigenvalues of $-\widetilde{N}^2$ are real and nonnegative. We apply the result from Proposition 2.1, namely, that the modulus of the eigenvalues of $\widetilde{N}$ is bounded by $\frac{\beta}{\nu_{\min}}$, where β is a constant independent of h. Therefore, the spectrum of $I - \widetilde{N}^2$ lies in the interval $[1, 1 + \frac{\beta^2}{\nu_{\min}^2}]$. Thus, we have

$$\frac{\nu_{\min}^2}{\nu_{\min}^2 + \beta^2} \leq \frac{(\mathbf{p}, C\mathbf{p})}{(\mathbf{p}, S'\mathbf{p})} \leq 1.$$

The latter result, combined with (24), gives us the bound as follows

$$\frac{c_0^2 \nu_{\min}^2}{\nu_{\max}(\nu_{\min}^2 + \beta^2)} \leq \frac{(\mathbf{p}, C\mathbf{p})}{(\mathbf{p}, M\mathbf{p})} \leq \frac{1}{\nu_{\min}},$$

where the coefficients c_0, β, $\nu_{\min}$ and $\nu_{\max}$ do not depend on the mesh size h.

For the skew-symmetric part R, an analogous reasoning holds. First,

$$\frac{(\mathbf{p}, R\mathbf{p})}{(\mathbf{p}, M\mathbf{p})} = \frac{(\mathbf{p}, R\mathbf{p})}{(\mathbf{p}, S'\mathbf{p})} \frac{(\mathbf{p}, S'\mathbf{p})}{(\mathbf{p}, M\mathbf{p})}.$$

Then, we have

$$\begin{aligned} \frac{F^{-1} - F^{-T}}{2} &= F^{-1} \frac{F^T - F}{2} F^{-T} \\ &= -(A_\nu + N_s)^{-1} N_s (A_\nu - N_s)^{-1} \\ &= -A_\nu^{-1/2} (I + \widetilde{N})^{-1} \widetilde{N}^{-1} (I - \widetilde{N})^{-1} A_\nu^{-1/2}, \end{aligned}$$

where $\widetilde{N} = A_\nu^{-1/2} N_s A_\nu^{-1/2}$. Therefore

$$\begin{aligned} \frac{(\mathbf{p}, R\mathbf{p})}{(\mathbf{p}, S'\mathbf{p})} &= -\frac{(\mathbf{p}, BA_\nu^{-1/2}(I + \widetilde{N})^{-1} \widetilde{N} (I - \widetilde{N})^{-1} A_\nu^{-1/2} B^T \mathbf{p})}{(\mathbf{p}, BA_\nu^{-1} B^T \mathbf{p})} \\ &= -\frac{(\mathbf{w}, \widetilde{N}\mathbf{w})}{(\mathbf{w}, (I - \widetilde{N}^2)\mathbf{w})}, \end{aligned}$$

where $\mathbf{w} = (I - \widetilde{N})^{-1} A_\nu^{-1/2} B^T \mathbf{p}$. It is easy to prove that $\widetilde{N}$ is a normal matrix, so that it can be represented by a diagonal matrix Λ and unitary matrix U by the formula

$$\widetilde{N} = iU\Lambda U^*,$$

where $\Lambda = \mathrm{diag}(-\sigma_1 i, -\sigma_2 i, \cdots)$ and σ is an eigenvalue of $\widetilde{N}$. Therefore, $\widetilde{N}^2 = -U\Lambda^2 U^*$, and the modulus of $-\dfrac{(\mathbf{w}, \widetilde{N}\mathbf{w})}{(\mathbf{w}, (I - \widetilde{N}^2)\mathbf{w})}$ can be presented as

$$\frac{|\ (\mathbf{v}, \Lambda\mathbf{v})\ |}{(\mathbf{v}, (I + \Lambda^2)\mathbf{v})},$$

where $\mathbf{v} = U^*\mathbf{w}$. For any $\mathbf{v}$, there exists a positive constant σ so that the following relation holds

$$\frac{|\ (\mathbf{v}, \Lambda\mathbf{v})\ |}{(\mathbf{v}, (I + \Lambda^2)\mathbf{v})} = \frac{\sigma}{1 + \sigma^2} \quad (\sigma > 0).$$

Therefore, the modulus is bounded and the maximal upper bound is 1/2, attained for $\sigma = 1$.
Thus, we have

$$\frac{|\ (\mathbf{p}, R\mathbf{p})\ |}{(\mathbf{p}, S'\mathbf{p})} \leq \frac{1}{2}.$$

Again, the latter result, combined with (24), shows that

$$\frac{|\ (\mathbf{p}, R\mathbf{p})\ |}{(\mathbf{p}, M\mathbf{p})} \leq \frac{1}{2\nu_{\min}}.$$

Thus, there holds

$$\frac{c_0^2\nu_{\min}^2}{\nu_{\max}(\nu_{\min}^2 + \beta^2)} \leq \mathrm{Re}(\mu) \leq \frac{1}{\nu_{\min}} \quad \text{and} \quad |\ \mathrm{Im}(\mu)\ | \leq \frac{1}{2\nu_{\min}}.$$

■

Remark. The proof follows that of Theorem 1 in [16], where Oseen's problem with constant viscosity is considered. Theorem 3.2 generalizes the result from Theorem 1 in [16], since for $\nu_{\min} = \nu_{\max} = \nu$, it is straightforward to obtain

$$\frac{c_0^2\nu}{(\nu^2 + \beta^2)} \leq \mathrm{Re}(\mu) \leq \frac{\Gamma}{\nu} \quad \text{and} \quad |\ \mathrm{Im}(\mu)\ | \leq \frac{\Gamma}{2\nu}.$$

Of course, the values of the constants c_0, β and Γ are different.

4 The choice $W = M$ and approximations of M^{-1}

We see that when solving systems with the AL type preconditioner $\mathcal{M}_L$ from (20), we need the action of W^{-1} twice. For $W = M$ (M denotes the pressure mass matrix), except some special cases, cf. e.g., piecewise-constant basis functions for the pressure, in general the pressure mass matrix is not of diagonal form. A widely used method is to replace M by its diagonal, D_M, which turns out to be a very good approximation of M. Spectral bounds

for $D_M^{-1}M$ are derived in [33]. Another method to compute the approximation of M^{-1} has been introduced in [21]. Here, we briefly recall the major ideas of the latter method.

Being a finite element matrix, M is obtained via a standard FEM assembly, namely,

$$M = \sum_{k=1}^{n_E} R_k^T M_k R_k,$$

where M_k are the local mass matrices. An approximation of M^{-1} can be constructed as follows:

$$\widehat{M}^{-1} = \sum_{k=1}^{m} R_k^T M_k^{-1} R_k.$$

Here R_k are the standard Boolean matrices which prescribe the local-to-global correspondence of the degrees of freedom and n_E denotes the number of finite elements. This method is referred to as the element-by-element sparse approximate inverse (EBE-SPAI) technique and can be applied to any finite element matrix, provided that the element matrices are invertible. For more details about the EBE-SPAI method applied to approximate various FEM matrices and the properties of the obtained approximations, see for instance, [24, 5, 27, 21, 26]. One can use the diagonal entries of $\widehat{M}^{-1}$, denoted by $D_{\widehat{M}^{-1}}$, to have a diagonal approximation of the inverse. The extreme eigenvalues and condition number of $D_M^{-1}M$ and $D_{\widehat{M}^{-1}}M$, under the condition of the Cartesian mesh and the bilinear basis function for the pressure, are computed and presented in Table 1.
Remark.

(1) The construction of $D_{\widehat{M}^{-1}}$ is rather cheap. The only additional task, compared to D_M^{-1}, is to compute M_k^{-1} (the exact inverse of element's pressure mass matrix) on each element, which can be fully parallelized. In the case of uniform meshes, we need to compute M_k^{-1} only for one element and then assemble it through all elements.

(2) From Table 1 we see that the condition number of $D_M^{-1}M$ is smaller than that of $D_{\widehat{M}^{-1}}M$. However, numerical evidence in Section 5 shows, that for $\gamma = 1$ the preconditioner $\mathcal{M}_L$ for $\widetilde{\mathcal{A}}$, constructed with $W^{-1} = D_{\widehat{M}^{-1}}$ works somewhat better than that, constructed with $W^{-1} = D_M^{-1}$.

5 Numerical illustrations

We choose as a benchmark the well-known two-dimensional lid-driven cavity problem, equipped with the boundary conditions $u_1 = u_2 = 0$ for $x = 0, x = 1$ and $y = 0$; $u_1 = 1, u_2 = 0$ for $y = 1$.

The problem is discretized using Cartesian meshes and the modified Taylor-Hood FE pair Q2isoQ2, i.e., piecewise bilinear basis functions on a mesh of size $2h$ for the pressure and piecewise bilinear basis functions for the velocity on a mesh of size h, obtained by one regular refinement of the discretization mesh for the pressure.

Table 1: Extreme eigenvalues and condition number of $D_M^{-1}M$ and $D_{\widehat{M}^{-1}}M$.

mesh size	min	max	cond.
$D_{\widehat{M}^{-1}}M$			
$h = 1/16$	1.2029	60.9601	51.6564
$h = 1/32$	1.2038	63.2049	53.4885
$h = 1/64$	1.2038	63.7975	53.9860
$D_M^{-1}M$			
$h = 1/16$	0.25	2.25	9.3098
$h = 1/32$	0.25	2.25	9.1791
$h = 1/64$	0.25	2.25	9.1244

To linearize the stationary incompressible N-S equations we use the Picard's method, and the nonlinear iterations are refereed to as Picard's iterations with stopping tolerance 10^{-6}. At each Picard iteration we solve the system (19), with the system matrix $\widetilde{\mathcal{A}}$ preconditioned by the AL type preconditioner $\mathcal{M}_L$ from (20) constructed with $W^{-1} = D_M^{-1}$ or $W^{-1} = D_{\widehat{M}^{-1}}$. To solve the systems with $\widetilde{\mathcal{A}}$ we use the Generalized Conjugate Gradient-Minimal Residual (GCG-MR) iterative method (cf, e.g., [1]) and the corresponding iterations are referred to as the inner iterations. When solving systems with the preconditioner $\mathcal{M}_L$, we use a direct method for all the blocks, e.g., the modified pivot block $\widetilde{F} = F + \gamma B^T W^{-1} B$.

The viscosity is chosen to be piecewise constant, namely, we divide the computational domain into two equal parts and, within each part, we set the viscosity to be a constant, namely, $\nu_{left} = \nu_{\max}$ on the left part and $\nu_{right} = \nu_{\min}$ on the right part, where $\nu_{\max} \neq \nu_{\min}$. This setting models the stationary situation of two phases with an interface of zero width between them. The setting is simple but the viscosity in the whole domain is discontinuous. The reason why we choose it can be explained by the results in Theorem 3.1 and Theorem 3.2, which show that the quality of the AL type preconditioner $\mathcal{M}_L$ only depends on the values of $\nu_{\max}$ and $\nu_{\min}$.

Table 2 shows the minimal nonzero and maximal eigenvalues of $M^{-1}S_{\mathcal{A}}$, where $S_{\mathcal{A}}$ is the negative the Schur complement of the system matrix $\mathcal{A}$ from (9), $S_{\mathcal{A}} = BF^{-1}B^T$ and M is the pressure mass matrix. We observe that

$$\mathrm{Re}(M^{-1}S_{\mathcal{A}})_{\min} = O(\nu_{\max}^{-1}), \quad \mathrm{Re}(M^{-1}S_{\mathcal{A}})_{\max} = O(\nu_{\min}^{-1})$$
$$| \mathrm{Im}(M^{-1}S_{\mathcal{A}}) |_{\max} \leq 0.5\nu_{\min}^{-1},$$

The bounds for the $\mathrm{Re}(M^{-1}S_{\mathcal{A}})_{\max}$ and $| \mathrm{Im}(M^{-1}S_{\mathcal{A}}) |_{\max}$ agree with the theoretical prediction given in Proposition 2.1. However, from Table 2, instead of the theoretical bound $\mathrm{Re}(M^{-1}S_{\mathcal{A}})_{\min} = O(\nu_{\max}^{-1}\frac{1}{1+\beta^2/\nu_{\min}^2})$, the relation $\mathrm{Re}(M^{-1}S_{\mathcal{A}})_{\min} = O(\nu_{\max}^{-1})$ is observed. A reason which could explain the observation is that the value of the constant β turns out to be much smaller than the chosen values of $\nu_{\min}$ (0.1, 0.01 and 10^{-3}), so that the ratio

Table 2: Extreme eigenvalues of $M^{-1}S_{\mathcal{A}}$ on the last Picard iteration.

	$\nu_{\max}=1$			$\nu_{\max}=0.1$	
$\nu_{\min}$	0.1	0.01	10^{-3}	0.01	10^{-3}
$h=1/32$					
$\mathrm{Re}(M^{-1}S_{\mathcal{A}})_{\min}$	0.2521	0.2521	0.2522	2.5213	2.5211
$\mathrm{Re}(M^{-1}S_{\mathcal{A}})_{\max}$	9.5670	90.9124	825.83	90.6483	822.1309
$\mid \mathrm{Im}(M^{-1}S_{\mathcal{A}}) \mid_{\max}$	1.1137	41.2356	246.41	41.2727	263.65
$h=1/64$					
$\mathrm{Re}(M^{-1}S_{\mathcal{A}})_{\min}$	0.2515	0.2515	0.2515	2.5154	2.5152
$\mathrm{Re}(M^{-1}S_{\mathcal{A}})_{\max}$	9.8004	97.5956	910.56	97.3955	933.4744
$\mid \mathrm{Im}(M^{-1}S_{\mathcal{A}}) \mid_{\max}$	1.2119	45.0595	389.48	44.9165	355.5560

$\frac{1}{1+\beta^2/\nu_{\min}^2} \approx 1$ and $\mathrm{Re}(M^{-1}S_{\mathcal{A}})_{\min} = O(\nu_{\max}^{-1})$.

Recall that the eigenvalues of the preconditioned matrix $\mathcal{M}_L\widetilde{\mathcal{A}}$, denoted by λ, comprise two parts, namely, the first part is equal to 1 and the other part satisfies

$$\lambda = \frac{1}{1+\frac{1}{\gamma\mu}},$$

where μ are the eigenvalues of $M^{-1}S_{\mathcal{A}}$. We see that the maximal value of λ is 1 and the minimal value is proportional to the minimal value of μ. Because of $\mathrm{Re}(\mu)_{\min} = O(\nu_{\max}^{-1})$, relatively smaller values of $\nu_{\max}$ result in a better clustering of the spectrum of $\mathcal{M}_L\widetilde{\mathcal{A}}$ around 1 (the modulus of imaginary part of λ is relatively small, compared to the value of the real part as shown in Figures 1, 2 and 3) and faster convergence for inner solver. Table 3 present the inner and Picard's iterations, and the inner iterations there confirm the analysis that the value of $\nu_{\max}$ is the main factor determining the convergence rate of the inner solution. Independence on the mesh size of Picard's and inner iterations is also observed in Table 3.

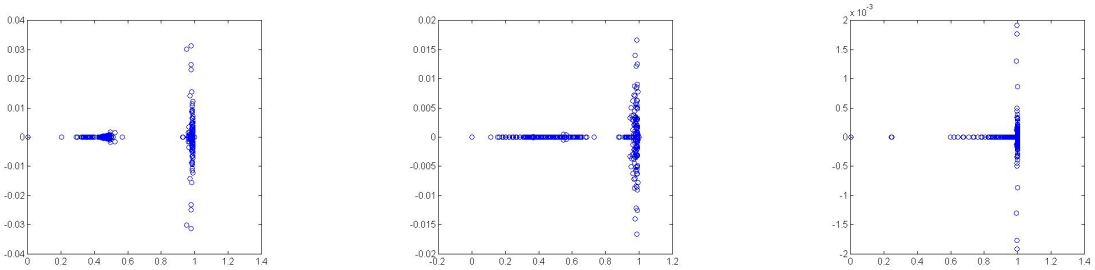
As presented in Section 4, there are two methods to construct the diagonal approximation of M^{-1}. $D_{\widehat{M}^{-1}}$ are constructed via EBE-SPAI technique and D_M^{-1} denotes the approximation by inverting the diagonal entries of M^{-1}. Figures 1, 2 and 3 plot the spectrum of $\mathcal{M}_L^{-1}\widetilde{\mathcal{A}}$ with $W^{-1} = D_M^{-1}$ and $W^{-1} = D_{\widehat{M}^{-1}}$. For different values of $\nu_{\min}$ and $\nu_{\max}$, we see that with $W^{-1} = D_{\widehat{M}^{-1}}$ the clustering of the eigenvalues around 1 is significantly improved compared to $W^{-1} = M^{-1}$ and even compared to $W^{-1} = D_M^{-1}$. The above observation also reflects the corresponding number of inner iterations in Table 4.

To check how much the inner tolerance can affect Picard's iterations, we choose a tighter inner tolerance, 10^{-5}. From Table 5 we see that with $W^{-1} = D_{\widehat{M}^{-1}}$ Picard iterations are not improved by decreasing the inner tolerance. Thus, if we solve with the modified pivot block exactly, or accurately enough, an inner stopping tolerance of 10^{-2} suffices.

Table 6 shows the inner and Picard's iterations with $W^{-1} = D_{\widehat{M}^{-1}}$ for varying mesh

Table 3: Iterations with AL preconditioner $\mathcal{M}_L$ to $\widetilde{\mathcal{A}}$, $\gamma = 1$ and inner tolerance is 10^{-2}.

	$\nu_{\max} = 1$			$\nu_{\max} = 0.1$	
$\nu_{\min}$	0.1	0.01	10^{-3}	0.01	10^{-3}
$h = 1/32$					
Picard iter.	4	8	18	10	25
inner iter. $W = M$	5	5	4	3	3
$h = 1/64$					
Picard iter.	4	8	18	10	25
inner iter. $W = M$	5	5	4	3	3



(a) $W = M$ (b) $W^{-1} = D_M^{-1}$ (c) $W^{-1} = D_{\widehat{M}^{-1}}$

Figure 1: The spectrum of $\mathcal{M}_L^{-1}\widetilde{\mathcal{A}}$ on the last Picard iteration. $\nu_{\max} = 1$, $\nu_{\min} = 10^{-2}$, $\gamma = 1$ and $h = \frac{1}{32}$

size, and, clearly, both are independent of the mesh size. Figure 4 shows some selected streamlines in the last Picard's iteration with different values of $\nu_{\min}$ and $\nu_{\max}$.

6 Conclusions

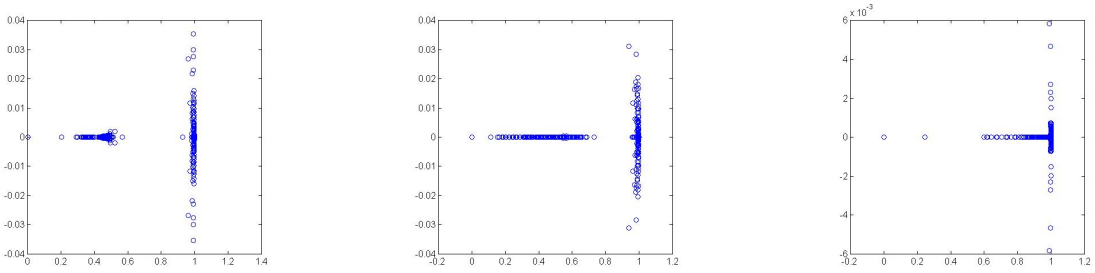
In this paper we consider using the augmented Lagrangian approach to precondition the matrices arising from the finite element discretization of the linearized stationary incompressible Navier-Stokes equations with variable viscosity. We prove that the AL preconditioner involving the finite element mass matrix for the pressure is independent of the mesh size, but depends on the maximal and minimal values of the viscosity, i.e., $\nu_{\max}$ and $\nu_{\min}$. The above result is derived in a general formulation and by taking $\nu_{\max} = \nu_{\min} = \nu$, the conclusion in [16] can be straightforwardly obtained. Numerical experiments in this paper show that the AL preconditioner works efficiently for a large range of the values of $\nu_{\max}$ and $\nu_{\min}$. However, some observation from the experiments, which is not the same as predicted, needs a further study. Another research direction in the future is the efficient solution methods for the modified pivot block matrix of the transformed system matrix.

Table 4: Iterations with AL preconditioner $\mathcal{M}_L$ to $\widetilde{\mathcal{A}}$, $\gamma = 1$ and inner tolerance is 10^{-2}.

	$\nu_{\max} = 1$			$\nu_{\max} = 0.1$	
$\nu_{\min}$	0.1	10^{-2}	10^{-3}	10^{-2}	10^{-3}
$h = 1/32$					
Picard iter.	4	8	18	10	25
inner iter. $W^{-1} = D_{\widehat{M}^{-1}}$	4	4	3	2	2
inner iter. $W^{-1} = D_M^{-1}$	7	5	4	3	3
$h = 1/64$					
Picard iter.	4	8	17	10	24
inner iter. $W^{-1} = D_{\widehat{M}^{-1}}$	4	4	3	2	2
inner iter. $W^{-1} = D_M^{-1}$	7	5	4	3	3

Table 5: Iterations for AL preconditioner $\mathcal{M}_L$ to $\widetilde{\mathcal{A}}$ with $W^{-1} = D_{\widehat{M}^{-1}}$ and $\gamma = 1$, inner tolerance is 10^{-5}.

	$\nu_{\max} = 1$			$\nu_{\max} = 0.1$	
$\nu_{\min}$	0.1	0.01	10^{-3}	0.01	10^{-3}
$h = 1/32$					
Picard iter.	4	8	18	10	25
inner iter.	8	8	7	5	5
$h = 1/64$					
Picard iter.	4	8	17	10	24
inner iter.	8	8	7	5	5

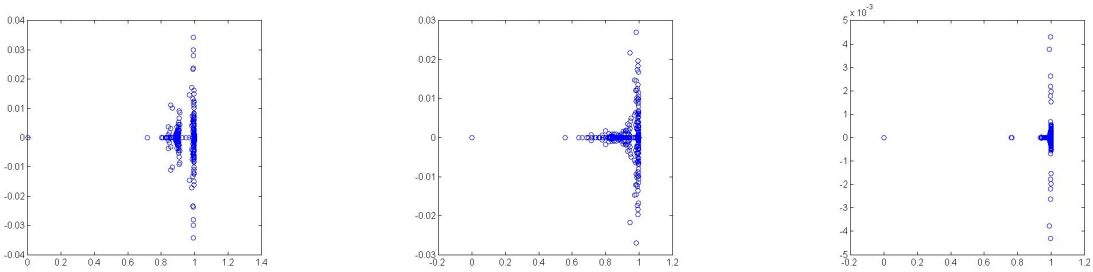


(a) $W = M$ (b) $W^{-1} = D_M^{-1}$ (c) $W^{-1} = D_{\widehat{M}^{-1}}$

Figure 2: The spectrum of $\mathcal{M}_L^{-1}\widetilde{\mathcal{A}}$ on the last Picard iteration. $\nu_{\max} = 1$, $\nu_{\min} = 10^{-3}$, $\gamma = 1$ and $h = \frac{1}{32}$

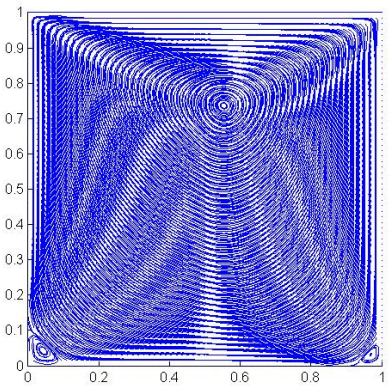
Table 6: Iterations for AL preconditioner $\mathcal{M}_L$ with $W^{-1} = D_{\widehat{M}^{-1}}$ and $\gamma = 1$, inner tolerance is 10^{-2}.

	$\nu_{\max} = 1$			$\nu_{\max} = 0.1$	
$\nu_{\min}$	0.1	0.01	10^{-3}	0.01	10^{-3}
$h = 1/32$					
Picard iter.	4	8	18	10	25
inner iter.	4	4	3	2	2
$h = 1/64$					
Picard iter.	4	8	17	10	24
inner iter.	4	4	3	2	2
$h = 1/128$					
Picard iter.	4	8	17	10	24
inner iter.	3	3	3	2	2
$h = 1/256$					
Picard iter.	4	8	17	10	24
inner iter.	3	3	2	2	2

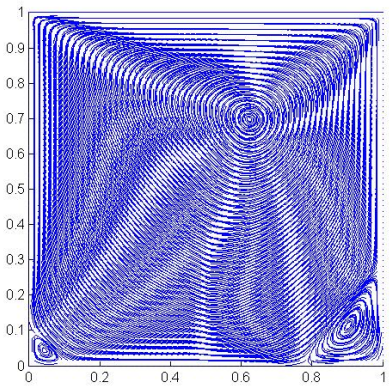


(a) $W = M$ (b) $W^{-1} = D_M^{-1}$ (c) $W^{-1} = D_{\widehat{M}^{-1}}$

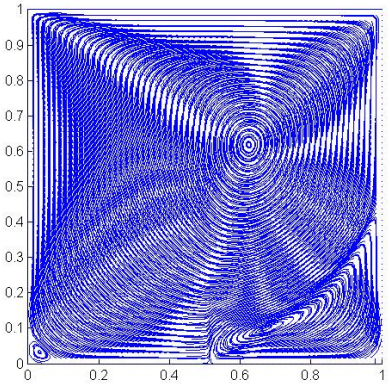
Figure 3: The spectrum of $\mathcal{M}_L^{-1}\widetilde{\mathcal{A}}$ on the last Picard iteration. $\nu_{\max} = 0.1$, $\nu_{\min} = 10^{-3}$, $\gamma = 1$ and $h = \frac{1}{32}$



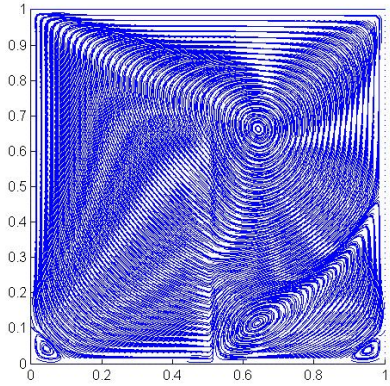
(a) $\nu_{\max} = 1$, $\nu_{\min} = 0.1$



(b) $\nu_{\max} = 1$, $\nu_{\min} = 10^{-2}$



(c) $\nu_{\max} = 0.1$, $\nu_{\min} = 10^{-3}$



(d) $\nu_{\max} = 1$, $\nu_{\min} = 10^{-3}$

Figure 4: Selected streamline, $h = \frac{1}{64}$

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