Discontinuous Galerkin multiscale methods for convection dominated problems

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Abstract

We propose an extension of the discontinuous Galerkin multiscale method, presented in [11], to convection dominated problems with rough, heterogeneous, and highly varying coefficients. The properties of the multiscale method and the discontinuous Galerkin method allows us to better cope with multiscale features as well as boundary layers in the solution. In the proposed method the trail and test spaces are spanned by a corrected basis calculated on localized patches of size $\mathcal{O}(H\log(H^{-1}))$, where H is the mesh size. We prove convergence rates independent of the variation in the coefficients and present numerical experiments which verify the analytical findings.

1 Introduction

In this paper we consider numerical approximation of convection dominated problems with rough, heterogeneous, and highly varying coefficients, without assumption on scale separation or periodicity. This class of problems, normally refereed to as multiscale problem, are know to be very computational demanding and arise in many different areas of the engineering sciences, e.g., porous media flow and in composite materials. More precisely, we consider the following convection-diffusion-reaction equation: find the weak solution $u \in H^1_0$ such that

$$\begin{aligned} -\nabla\cdot A\nabla u + \mathbf{b}\cdot\nabla u + cu &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \Gamma, \end{aligned} \tag{1.1}$$

with multiscale coefficients $A, \mathbf{b}, c$, which will be specified later.

There are two key issues which make classical conforming finite element methods perform badly for these kind of problems.

- First, the multiscale features of the coefficient need to be resolved by the finite element mesh.

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- Second, strong convection leads to boundary layers in the solution which also need to be resolved.

To overcome the lack of performance using classical finite element methods many different so called multiscale methods have been proposed. Some important contributions are: the multiscale finite element method (MsFEM) [14, 4, 10, 9]), the heterogeneous multiscale method (HMM) [7, 8]), and the variational multiscale method (VMS) [16, 17, 22, 25, 26, 23]). Common to all mentioned approaches is that fine scale problems are solved on localized patches and the result is used to construct a different basis or a modified coarse scale operator. The analysis for most of these methods rely strongly on scale separation or periodicity. Recently there was a leap forward in the analysis of multiscale methods. In [26] a new technique for proving convergence for a class of multiscale methods without any assumptions on scale separation or periodicity, was proposed. The method proposed in [26] uses a trail and test space spanned by a corrected basis function computed on patches of size $\mathcal{O}(H \log(H^{-1}))$. Textbook convergence with respect the mesh size H was proven. This technique was furthered generalized to a class of non-conforming multiscale methods based on the discontinuous Galerkin multiscale method in [11].

There is a vast literature on numerical methods for convection dominated problems. Two such examples are, the streamline diffusion/Petrov Galerkin (SUPG) method [19, 15] and Galerkin least square method (GLS) [18]. There has also been a lot of work on discontinuous Galerkin (dG) methods, we refer to [27, 24, 2, 20] for some early work and to [5, 13, 28, 6] and references therein for recent development and a literature review. DG methods exhibit attractive properties for convection dominated problems, e.g., they have enhanced stability properties, good conservation property of the state variable, and the use of complex and/or irregular meshes are admissible. DG multiscale methods has also been considered, see e.g. [1, 29].

To better coop with convection dominated multiscale problems we extend the discontinuous Galerkin multiscale method in [11] to convection dominated problems, in the sense that the convective term is included when calculating the corrected basis. For problems with weak convection it is not necessary to include the convective part, see e.g. [12].

The outline of this paper is as follows. In section 2 the discrete setting and underlying dG method is presented. In section 3 the multiscale decomposition and the dG multiscale method and the corresponding convergence result are stated. The proofs for the theoretical results are given in Section 4. Finally, in Section 5 numerical experiments are presented.

2 Preliminaries

In this section we present some notations and properties frequently used in the paper. Throughout this paper standard notations of Lebesgue and Sobolev spaces are used.

2.1 Setting

Let $\Omega \subset \mathbb{R}^d$ be a polygonal domain with Lipschitz boundary Γ. We assume that: the diffusion coefficients, $A \in L^\infty(\Omega, \mathbb{R}^{d\times d}_{sym})$, has uniform spectral bounds $0 < \alpha, \beta < \infty$, defined by

$$0 < \alpha := \operatorname*{ess\,inf}_{x\in\Omega} \inf_{v\in\mathbb{R}^d\setminus\{0\}} \frac{(A(x)v)\cdot v}{v\cdot v} \leq \operatorname*{ess\,sup}_{x\in\Omega} \sup_{v\in\mathbb{R}^d\setminus\{0\}} \frac{(A(x)v)\cdot v}{v\cdot v} =: \beta < \infty, \quad (2.1)$$

the convective, $\mathbf{b} \in [W^1_\infty(\Omega)]^d$, and reactive, $c \in L^\infty(\Omega)$, coefficient fulfill the condition

$$(c_0(x))^2 = c(x) - \frac{1}{2}\nabla\cdot\mathbf{b}(x) \geq \mu_0 \text{ a.e. } x \in \Omega, \quad (2.2)$$

where $\mu_0 \in \mathbb{R} > 0$ is constant. Finally we assume $f \in L^2(\Omega)$.

In the rest of the paper we will consider two different meshes, one coarse and one fine with mesh function h and H, respectively. Let $\mathcal{T}_k$, for $k = \{h, H\}$, denote a shape-regular subdivision of Ω into (closed) regular simplexes or into quadrilaterals/hexahedra ($d = 2/d = 3$), given a mesh function $k : \mathcal{T}_k \to \mathbb{R}$ defined as $k := \operatorname{diam}(T) \in P_0(\mathcal{T}_k)$ for all $T \in \mathcal{T}_k$. Also, let $\nabla_k v$ denote the $\mathcal{T}_k$-broken gradient defined as $(\nabla v)|_T = \nabla v|_T$ for all $T \in \mathcal{T}_k$. For simplicity we will also assume that $\mathcal{T}_k$ is conforming in the sense that no hanging nodes are allowed, but the analysis can easily be extend to non-conforming meshes with a finite number of hanging nodes on each edge. Let $\mathcal{E}_k$ denote the set of edges in $\mathcal{T}_k$, where $\mathcal{E}_k(\Omega)$ is the set of interior edges and $\mathcal{E}_k(\Gamma)$ is the set of boundary edges, such that $\mathcal{E}_k = \mathcal{E}_k(\Omega) \cup \mathcal{E}_k(\Gamma)$. Let $\hat{T}$ be the reference simplex or (hyper)cube. We define $\mathcal{P}_p(\hat{T})$ to be the space of polynomials of degree less than or equal to p if $\hat{T}$ is a simplex, or the space of polynomials of degree less than or equal to p, in each variable, if $\hat{T}$ is a (hyper)cube. The space of discontinuous piecewise polynomial function is defined by

$$P_p(\mathcal{T}_k) := \{v : \Omega \to \mathbb{R} \mid \forall T \in \mathcal{T}_k,\ v|_T \circ F_T \in \mathcal{P}_p(\hat{T})\}, \quad (2.3)$$

where $F_T : \hat{T} \to T$, $T \in \mathcal{T}_k$ is a family of element maps. Also, let $\Pi_p(\mathcal{T}_k) : L^2(\Omega) \to P_p(\mathcal{T}_k)$ denote the L^2-projection onto $P_p(\mathcal{T}_k)$. Let T^+ and T^- be two adjacent elements in $\mathcal{T}_k$ sharing an edge $e = T^+ \cap T^- \in \mathcal{E}_k(\Omega)$, and let ν_e be the outer normal pointing from T^- to T^+, and for $e \in \mathcal{E}_k(\Gamma)$ let ν_e be outward unit normal of Ω. For any $v \in P_p(\mathcal{T}_k)$ we denote the value on edge $e \in \mathcal{E}(\Omega)$ as $v^\pm = v|_{e\cap T^\pm}$. The jump and average of $v \in P_p(\mathcal{T}_k)$ is defined as, $[v] = v^- - v^+$ and $\{v\} = (v^- + v^+)/2$ respectively for $e \in \mathcal{E}_k(\Omega)$, and $[v] = \{v\} = v|_e$ for $e \in \mathcal{E}_k(\Gamma)$.

Let $0 \leq C < \infty$ denote any generic constant that neither depends on the mesh size or the variables A, $\mathbf{b}$, and c; then $a \lesssim b$ abbreviates the inequality $a \leq Cb$.

2.2 Discontinuous Galerkin discretization

For simplicity let the bilinear form $a_k(\cdot,\cdot) : \mathcal{V}_k \times \mathcal{V}_k \to \mathbb{R}$, given any mesh function $k : \Omega \to P_0(\mathcal{T}_k)$, be split into two parts

$$a_k(u,v) := a_k^{\mathrm{d}}(u,v) + a_k^{\mathrm{c\text{-}r}}(u,v), \tag{2.4}$$

where $a_k^{\mathrm{d}}(\cdot,\cdot)$ represents the diffusion part and $a_k^{\mathrm{c\text{-}r}}(\cdot,\cdot)$ represents the convection-reaction part. The diffusion part is approximated using a symmetric interior penalty method, i.e.,

$$\begin{aligned} a_k^{\mathrm{d}}(u,v) := (A\nabla_k u, \nabla_k v)_{L^2(\Omega)} + \sum_{e\in\mathcal{E}_k} \Big(\frac{\sigma_e}{h_e}([u],[v])_{L^2(e)} \\ - (\{\nu_e \cdot A\nabla u\},[v])_{L^2(e)} - (\{\nu_e \cdot A\nabla v\},[u]_{L^2(e)})\Big), \end{aligned} \tag{2.5}$$

where σ_e is a constant, depending on the diffusion, large enough to make $a_k^{\mathrm{d}}(\cdot,\cdot)$ coercive. The convection-reaction part is approximated by

$$\begin{aligned} a_k^{\mathrm{c\text{-}r}}(u,v) := (\mathbf{b}\cdot\nabla_k u + cu, v)_{L^2(\Omega)} + \sum_{e\in\mathcal{E}_k} (b_e[u],[v])_{L^2(e)} \\ - \sum_{e\in\mathcal{E}_k(\Omega)} (\nu_e\cdot\mathbf{b}\{u\},[v])_{L^2(e)} - \sum_{e\in\mathcal{E}_k(\Gamma)} \frac{1}{2}((\nu_e\cdot\mathbf{b})u,v)_{L^2(e)}, \end{aligned} \tag{2.6}$$

or equivalently

$$\begin{aligned} a_k^{\mathrm{c\text{-}r}}(u,v) &:= ((c-\nabla\cdot\mathbf{b})u,v)_{L^2(\Omega)} - (u,\mathbf{b}\cdot\nabla_k v)_{L^2(T)} \\ &+ \sum_{e\in\mathcal{E}_k} (b_e[u],[v])_{L^2(e)} + \sum_{e\in\mathcal{E}_k(\Omega)} (\nu_e\cdot\mathbf{b}\{u\},[v])_{L^2(e)} \\ &+ \sum_{e\in\mathcal{E}_k(\Gamma)} \frac{1}{2}((\nu_e\cdot\mathbf{b})u,v)_{L^2(e)}, \end{aligned} \tag{2.7}$$

where upwind is imposed choosing the stabilization term as $b_e = |\mathbf{b}\cdot\nu_e|/2$, see e.g. [3]. The energy norm on $\mathcal{V}_k$ is given by

$$\begin{aligned} |||v|||_{k,d}^2 &= \|A^{1/2}\nabla_k v\|_{L^2}^2 + \sum_{e\in\mathcal{E}_k} \frac{\sigma_e}{k}\|[v]\|_{L^2(e)}^2, \\ |||v|||_{k,\mathrm{c\text{-}r}}^2 &= \|c_o v\|_{L^2(\Omega)}^2 + \sum_{e\in\mathcal{E}_k} \|b_e^{1/2}[v]\|_{L^2(e)}^2, \\ |||v|||_k^2 &= |||v|||_{k,d}^2 + |||v|||_{k,\mathrm{c\text{-}r}}^2. \end{aligned} \tag{2.8}$$

From Theorem 2.2 in [21] we have that for each $v \in \mathcal{V}_k$, there exist an averaging operator $\mathcal{I}_k^c : \mathcal{V}_k \to \mathcal{V}_k \cap H^1(\Omega)$ with the following property

$$\beta^{-1/2}\|A^{1/2}\nabla_k(v-\mathcal{I}_k^c v)\|_{L^2(\Omega)} + \|k^{-1}(v-\mathcal{I}_k^c v)\|_{L^2(\Omega)} \lesssim \alpha^{-1/2}|||v|||_k. \tag{2.9}$$

3 Multiscale method

In this section we present the multiscale decomposition, the multiscale methods, and the main convergence results.

3.1 Multiscale decomposition

In order to do the multiscale decomposition the problem is divided into a coarse and a fine scale. To this end, let $\mathcal{T}_H$ and $\mathcal{T}_h$, with the respective mesh function H and h, denote the two different subdivisions, where $\mathcal{T}_h$ is constructed using one or more (possible adaptive) refinements of $\mathcal{T}_H$.

The aim of this section is to construct a coarse generalized finite element space based on $\mathcal{T}_H$, which takes the fine scale behavior of the data into account. That is, we assume that the solution given by: find $u_h \in \mathcal{V}_h := P_1(\mathcal{T}_h)$ such that

$$a_h(u_h, v) = F(v) \quad \text{for all } v \in \mathcal{V}_h, \tag{3.1}$$

gives a sufficiently good approximation of the weak solution u to (1.1). However, u_h never has to be computed in practice, it only acts as a reference solution. We introduce a coarse projection operator $\Pi_H := \Pi_1(\mathcal{T}_H)$ and let the fine scale space be defined by the kernel of Π_H, i.e.,

$$\mathcal{V}^{\mathrm{f}} := \{v \in \mathcal{V}_h \mid \Pi_H v = 0\} \subset \mathcal{V}_h. \tag{3.2}$$

The next step is to split any $v \in \mathcal{V}_h$ into some coarse part based on $\mathcal{T}_H$, such that the fine scale reminder in the space $\mathcal{V}^{\mathrm{f}}$ is sufficiently small. The naive way to this splitting is to use a L^2-orthogonal split. Then the coarse space is defined by $\mathcal{V}_H := \Pi_H \mathcal{V}_h = P_1(\mathcal{T}_H)$ and is the standard dG space on the coarse scale. A given basis of $\mathcal{V}_H$ is the element-wise Lagrange basis functions $\{\lambda_{T,j} \mid T \in \mathcal{T}_H, \, j = 1, \dots, r\}$ where $r = (1 + d)$ for simplexes or $r = 2^d$ for quadrilaterals/hexahedra. The space $\mathcal{V}_H$ is known to give poor approximation properties if $\mathcal{T}_H$ does not resolve the variable coefficients in (1.1). We will use another choice, see [26, 11], based on $a_h(\cdot, \cdot)$, to construct a space of corrected basis functions. To this end, we define a fine scale projection operator $\mathfrak{F} : \mathcal{V}_h \to \mathcal{V}^{\mathrm{f}}$ by

$$a_h(\mathfrak{F}v, w) = a_h(v, w) \quad \text{for all } v \in \mathcal{V}^{\mathrm{f}}, \tag{3.3}$$

and let the corrected coarse space be defined as

$$\mathcal{V}_H^{ms} := (1 - \mathfrak{F})\mathcal{V}_H. \tag{3.4}$$

The correctors for the coarse basis are computed as follows: find $\phi_{T,j} \in \mathcal{V}^{\mathrm{f}}$ such that

$$a_h(\phi_{T,j}, v) = a_h(\lambda_{T,j}, v) \quad \text{for all } v \in \mathcal{V}^{\mathrm{f}}. \tag{3.5}$$

That is, the space of corrected basis functions is defined by $\mathcal{V}_H^{ms} := \{\lambda_{T,j} - \phi_{T,j} \mid T \in \mathcal{T}_H, \, j = 1, \dots, r\}$. Note that, $\dim(\mathcal{V}_H^{ms}) = \dim(\mathcal{V}_H)$. From (3.4) we have that any $v_h \in \mathcal{V}_h$ can be decomposed into a coarse, $v_H^{ms} \in \mathcal{V}_H^{ms}$, and a fine, $v^f \in \mathcal{V}^{\mathrm{f}}$, scale contribution, i.e., $v_h = v_H^{ms} + v^f$.

3.2 Methods and convergence results

In this section the main results in [11] is extended to convection dominated problem. For the convenience of the reader a short recap of the different constants used in the error estimate are stated below:

- $C_{\beta/\alpha} = \beta/\alpha$, where α and β is the lower respectively upper spectral bound of the diffusion matrix A defined in (2.1),
- $C_s = \left(C^2_{\beta/\alpha} + \|c_0\|^2_{L^\infty(\Omega)}\mu_0^{-2} + \|H\mathbf{b}\|_{L^\infty(\Omega)}\alpha^{-1}\right)^{1/2}$, appear in Lemma 7 which proves stability estimate in energy norm for Π_H and c_0, μ_0 are defined in (2.2),
- $C_b = C_{\beta/\alpha} + \|c_0\|_{L^\infty(\Omega)}\mu_0^{-1}$, appear in Lemma 8,
- $C_c = (1 + \|H\mathbf{b}\|_{L^\infty(\Omega)}\alpha^{-1})$, appears in Lemma 9 which shows continuity of the bilinear form on $\mathcal{V}^{\mathrm{f}} \times \mathcal{V}_h$,
- $C_\zeta = \left(C^2_{\beta/\alpha} + \|h\mathbf{b}\|_{L^\infty(\Omega)}\alpha^{-1}\right)^{1/2}$, appears in Lemma 11 using the stability property of the cut off function from Definition 10.
- $C_\phi = (1+\|H\mathbf{b}\|_{L^\infty(\Omega)}\alpha^{-1/2}+\|Hc\|_{L^\infty(\Omega)}\mu_0^{-1})$, appears in Lemma 13 using stability of the corrected basis functions.

3.2.1 Ideal discontinuous Galerkin multiscale method

An ideal multiscale method seeks $u_H^{ms} \in \mathcal{V}_H^{ms}$ such that

$$a_h(u_H^{ms}, v) = F(v) \quad \text{for all } v \in \mathcal{V}_H^{ms}. \tag{3.6}$$

Note that, to seek a solution in the space $\mathcal{V}_H^{ms}$, a variational problem has to be solved on the whole domain, Ω, for each local basis function, which is not feasible for real computations. The following theorem shows the convergence of the ideal multiscale method.

Theorem 1. *Let $u_h \in \mathcal{V}_h$ be the to solution to* (3.1)*, and $u_H^{ms} \in \mathcal{V}_H^{ms}$ be the to solution* (3.6)*, then*

$$|||u_h - u_H^{ms}||| \lesssim C_1 ||H(f - \Pi_H f)||_{L^2(\Omega)} \tag{3.7}$$

holds, where $C_1 = CC_c\alpha^{-1/2}$ and C is generic constant which do not depend on the mesh size or the problem data.

Proof. The proof is found in Section 4. □

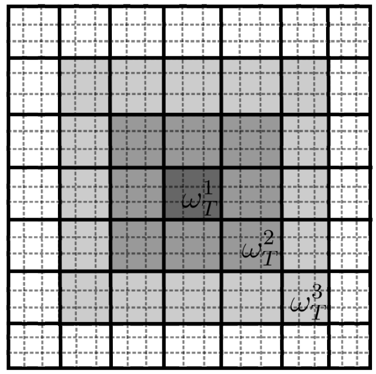


Figure 1: Example of a patch with size 1, ω_T^1, size 2 ω_T^2, and size ω_T^3, centered around element T.

3.2.2 Discontinuous Galerkin multiscale method

The fast decay of the corrected basis functions (Lemma 3), motivates us to solve the corrector functions on localized patches. This introduces a localization error, but choosing the patch size as $\mathcal{O}(H\log(H^{-1}))$ (as seen in Theorem 4) the localization error has the same convergence rate as the ideal multiscale method in Theorem 1. The corrector functions are solved on element patches, defined as follows.

Definition 2. For all $T \in \mathcal{T}_H$, let ω_T^L be a patch centered around element T and of size L, defined as,

$$\begin{aligned} &\omega_T^1 := \operatorname{int}(T), \\ &\omega_T^L := \operatorname{int}(\cup\{T' \in T_H \mid T \cap \bar{\omega}_T^{L-1} \neq 0\}), \quad L = 1, 2, \ldots. \end{aligned} \tag{3.8}$$

See Figure 1 for an illustration.

The localized corrector functions are calculated as follows: for all $\{T \in \mathcal{T}_H, j = 1, \ldots, r\}$ find $\phi_{T,j}^L \in \mathcal{V}^f(\omega_T^L) = \{v \in \mathcal{V}^{\mathrm{f}} \mid v|_{\Omega\setminus\omega_T^L} = 0\}$ such that

$$a_h(\phi_{T,j}^L, v) = a_h(\lambda_{T,j}, v), \quad \text{for all } v \in \mathcal{V}^f(\omega_T^L). \tag{3.9}$$

The decay of the corrected basis function is given in the following lemma.

Lemma 3. *For all $T \in \mathcal{T}_H$, $j = 1, \ldots, r$ where $\phi_{T,j}$ is the solution to* (3.5) *and $\phi_{T,j}^L$ is the solution to* (3.9)*, the following estimate*

$$\left|\left|\left|\phi_{T,j} - \phi_{T,j}^L\right|\right|\right|_h \leq C_2\gamma^L |||\lambda_{T,j} - \phi_{T,j}^L|||_{\mathrm{h}} \tag{3.10}$$

holds, where $L = \ell k$ is the size of the patch, $0 < \gamma = (\ell^{-1}C_3)^{\frac{k-1}{2\ell k}} < 1$, $C_2 = CC_cC_\zeta(1 + C_bC_s)$, $C_3 = C'(C_{\beta/\alpha}^2 + \|H\mathbf{b}\|_{L^\infty(\Omega)}\alpha^{-1} + C_cC_b\|\mathbf{b}\|_{L^\infty(\Omega)}\mu_0^{-1})$, and C, C' are generic constants neither depending on the mesh size, the size of the patches, or the problem data.

Proof. The proof is found in Section 4. □

The space of localized corrected basis function is defined by $\mathcal{V}_H^{ms,L} := \{\phi_{T,j}^L - \lambda_{T,j} \mid T \in \mathcal{T}_H, r = 1, \dots, r\}$. The dG multiscale method now reads: find $u_H^{ms,L} \in \mathcal{V}_H^{ms,L}$ such that

$$a_h(u_H^{ms,L}, v) = F(v) \quad \text{for all } v \in \mathcal{V}_H^{ms,L}. \tag{3.11}$$

An error bound for the dG multiscale method using a localized corrected basis is given in Theorem 4. Also, note that it is only the first term $|||u - u_h|||_{\mathrm{h}}$ in Theorem 4 that depends on the regularity of u.

Theorem 4. *Let $u_h \in \mathcal{V}_h$ be to solution to* (3.1)*, and $u_H^{ms,L} \in \mathcal{V}_H^{ms,L}$ be to solution* (3.11)*, then*

$$\begin{aligned} |||u - u_H^{ms,L}|||_{\mathrm{h}} \leq & |||u - u_h|||_{\mathrm{h}} + C_1\|H(f - \Pi_H f)\|_{L^2(\Omega)} \\ & + C_5\|H^{-1}\|_{L^\infty(\Omega)} L^{d/2}\gamma^L \|f\|_{L^2(\Omega)} \end{aligned} \tag{3.12}$$

holds, where L is the size of the patches, C_1 is a constant defined in Theorem 1, $0 < \gamma < 1$ and $C_5 = C_4^{1/2} C_2 C_\phi \alpha^{-1/2}$, where C_4 is defined in Lemma 12, and C_2 and γ are defined in Lemma 3.

Proof. The proof is found in Section 4. □

Remark 5. Theorem 4 is simplified to,

$$|||u - u_H^{ms,L}|||_{\mathrm{h}} \leq |||u - u_h|||_{\mathrm{h}} + C_1\|H\|_{L^\infty(\Omega)}. \tag{3.13}$$

given that the patch size is chosen as $L = \lceil C \log(H^{-1}) \rceil$ with an appropriate C and $\|f\|_{L^2} = 1$.

Remark 6. For $|||u_h - u_H^{ms,L}|||_{\mathrm{h}}$ to decay as $\mathcal{O}(H)$, it is sufficient that size of A and $\mathbf{b}$ fulfill the following relation, $\mathcal{O}(\beta) = \mathcal{O}(\|H\mathbf{b}\|_{L^\infty(\Omega)})$. If the convective part was omitted in the calculation of the corrected basis functions, using the same relation between the size of A and $\mathbf{b}$, the decay of $|||u_h - u_H^{ms,L}|||_{\mathrm{h}}$ would be $\mathcal{O}(1)$, see [12].

4 Proofs from Section 3

Before proving the the main results, Theorem 1, Lemma 3, and Theorem 4, we state a some definitions and technical lemmas which will be necessary in the proofs.

4.1 Some technical lemmas

The following inequalities will frequently be used in the error analysis.

Lemma 7. *For any $v \in \mathcal{V}_h$ and $T \in \mathcal{T}_H$, the approximation property*

$$H|_T^{-1}\|v - \Pi_H v\|_{L^2(T)} \lesssim \alpha^{-1/2}|||v|||_{h,T}, \tag{4.1}$$

and stability estimate

$$|||\Pi_H v|||_H \lesssim C_s|||v|||_{\mathrm{h}}, \tag{4.2}$$

is satisfied, with

$$C_s = \left(C_{\beta/\alpha}^2 + \frac{\|c_0\|_{L^\infty(\Omega)}^2}{\mu_0^2} + \frac{\|H\mathbf{b}\|_{L^\infty(\Omega)}}{\alpha}\right)^{1/2}. \tag{4.3}$$

Proof. Using the same procedure as Lemma 4 in [11], the lemma follows. □

Lemma 8. *For each $v_H \in \mathcal{V}_H$, there exist a $v \in \mathcal{V}_h \cap H^1$ such that $\Pi_H v = v_H$, $\|v\|_{L^2(\Omega)} \lesssim \|v_H\|_{L^2(\Omega)}$, $|||v|||_{\mathrm{h}} \lesssim C_b|||v_H|||_H$, $\mathrm{supp}(v) \subset \mathrm{supp}(v_H)$, and $C_b := C_{\beta/\alpha} + \|c_0\|_{L^\infty(\Omega)}\mu_0^{-1}$.*

Proof. Using the same procedure as Lemma 5 in [11], the lemma follows. □

Continuity of the dG bilinear form for convection-reaction problems are usually done on a orthogonal subset of $\mathcal{V}_h$. Since the space $\mathcal{V}^{\mathrm{f}}$ is an orthogonal subset of $\mathcal{V}_h$ we derive the following lemma.

Lemma 9 (Continuity in $(\mathcal{V}^{\mathrm{f}} \times \mathcal{V}_h)$ and $(\mathcal{V}_h \times \mathcal{V}^{\mathrm{f}})$)**.** *For all, $(u, v) \in \mathcal{V}^{\mathrm{f}} \times \mathcal{V}_h$, it holds*

$$a(v, w) \lesssim C_c|||v|||_{\mathrm{h}}|||w|||_{\mathrm{h}} \tag{4.4}$$

where

$$C_c = 1 + \|H\mathbf{b}\|_{L^\infty(\Omega)}\alpha^{-1/2}. \tag{4.5}$$

Proof. Since a_h^d is continuous in $(\mathcal{V}_h \times \mathcal{V}_h)$ continuity in $(\mathcal{V}^{\mathrm{f}} \times \mathcal{V}_h)$ follows from $\mathcal{V}^{\mathrm{f}} \subset \mathcal{V}_h$. For the convective part $a_h^{\mathrm{c\text{-}r}}$, we have

$$\begin{aligned}
a^{\mathrm{c\text{-}r}}(v, w) = &\sum_{T\in\mathcal{T}_h} \big(((c - \nabla\cdot\mathbf{b})v, w)_{L^2(T)} - (v, \mathbf{b}\cdot\nabla w)_{L^2(T)}\big) \\
&+ \sum_{e\in\mathcal{E}_k} (b_e[v], [w])_{L^2(e)} + \sum_{e\in\mathcal{E}_k(\Omega)} (\nu_e\cdot\{\mathbf{b}v\}, [w])_{L^2(e)} \\
&+ \sum_{e\in\mathcal{E}_k(\Gamma)} ((\nu_e\cdot\mathbf{b})v, w)_{L^2(e)} \\
\lesssim &\sum_{T\in\mathcal{T}_h} \big(\|c_0 v\|_{L^2(T)}\|c_0 w\|_{L^2(T)} + \|\|\mathbf{b}\|_{L^\infty(T)} v\|_{L^2(T)}\|\nabla w\|_{L^2(T)}\big) \\
&+ \sum_{e\in\mathcal{E}_k} \Big(\|\|\mathbf{b}\|_{L^\infty(e)} v\|_{L^2(T^+\cap T^-)} h^{-1/2}\|[w]\|_{L^2(e)}\Big).
\end{aligned} \tag{4.6}$$

Using a discrete Cauchy-Schwartz inequality and summing over the coarse elements, we have

$$
\begin{aligned}
a^{\text{c-r}}(v,w) &\lesssim |||v|||_{\text{h}}|||w|||_{\text{h}} + \alpha^{-1/2}\|H\mathbf{b}\|_{L^\infty(\Omega)}\|H^{-1}(v-\Pi_H v)\|_{L^2(\Omega)}|||w|||_{\text{h}}, \\
&\lesssim (1+\|H\mathbf{b}\|_{L^\infty(\Omega)}\alpha^{-1/2})|||v|||_{\text{h}}|||w|||_{\text{h}},
\end{aligned}
\tag{4.7}
$$

which concludes the proof for $(\mathcal{V}^{\text{f}} \times \mathcal{V}_h)$. A similar argument gives the proof of $(\mathcal{V}_h \times \mathcal{V}^{\text{f}})$. □

In the proof Lemma 3, which proves the decay of the corrected basis function, the following cut off function will be used.

Definition 10. The function $\zeta^{d,D} \in P_o(\mathcal{T}_h)$, for $D > d$, is a cut off function fulfilling the following condition

$$
\begin{aligned}
\zeta_T^{d,D}|_{\omega_T^d} &= 1, \\
\zeta_T^{d,D}|_{\Omega\setminus\omega_T^D} &= 0, \\
\|[\zeta_T^{d,D}]\|_{L^\infty(\mathcal{E}_h(T))} &\lesssim \frac{\|h\|_{L^\infty(T)}}{(D-d)H|_T},
\end{aligned}
\tag{4.8}
$$

and $|||[\zeta^{d,D}]|||_{L^\infty(\partial(\omega_T^D\setminus\omega_T^d))} = 0$, for all $T \in \mathcal{T}_H$.

For the cut off function defined in Definition 10, we have the following stability condition.

Lemma 11. *For any $v \in \mathcal{V}_h$ and $\zeta_T^{d,D}$ from Definition 10, the estimate,*

$$
|||\zeta_T^{d,D} v|||_{\text{h}} \lesssim C_\zeta |||v|||_{h,\omega_T^D}, \tag{4.9}
$$

holds, where $C_\zeta = (C_{\beta/\alpha}^2 + \|h\mathbf{b}\|_{L^\infty(\Omega)})^{1/2}$.

Proof. Let us use the following for the diffusion part in [11],

$$
|||(1-\zeta_T^{d,D})v|||_{\text{h,d}} \lesssim C_{\beta/\alpha}|||v|||_{h,\Omega\setminus\omega_T^{L-1}} \tag{4.10}
$$

and focus on the convection-reaction part, where $e = S^+ \cap S^- \in \mathcal{E}_h$

$$
\begin{aligned}
&|||(1-\zeta_T^{d,D})v|||_{\mathrm{h,c\text{-}r}}^2 \\
&\leq \|c_0 v\|_{L^2(\Omega\setminus\omega_T^{L-1})}^2 + \sum_{e\in\mathcal{E}_h} \|b_e^{1/2}[(1-\zeta_T^{d,D})v]\|_{L^2(e)}^2 \\
&\leq \|c_0 v\|_{L^2(\Omega\setminus\omega_T^{L-1})}^2 \\
&\qquad + \sum_{\substack{e\in\mathcal{E}_h:\\ e\cap\omega_T^{L-1}\neq 0}} \left(\|b_e^{1/2}[v]\|_{L^2(e)}^2 + \frac{\|h\|_{L^\infty(T)}^2}{\|H\|_{L^\infty(S^+\cup S^-)}^2} \|b_e^{1/2}\{v\}\|_{L^2(e)}^2 \right) \qquad (4.11) \\
&\lesssim |||v||||_{\mathrm{h},\Omega\setminus\omega^{L-1}}^2 + \sum_{\substack{T\in\mathcal{T}_H:\\ e\cap\omega_T^{L-1}\neq 0}} \frac{\|h\mathbf{b}\|_{L^\infty(T)}}{\|H\|_{L^\infty(S^+\cup S^-)}^2} \|v-\Pi_H v\|_{L^2(T)}^2 \\
&\leq |||v||||_{\mathrm{h},\Omega\setminus\omega^{L-1}}^2 + \frac{\|h\mathbf{b}\|_{L^\infty(\Omega)}}{\alpha} |||v|||_{h,\Omega\setminus\omega^{L-1}}^2,
\end{aligned}
$$

using $[vw] = \{v\}[w] + \{w\}[v]$, the triangle inequality, and a trace inequality. The proof is concluded using (4.10) and (4.11). □

The following lemmas will be necessary in order to prove Theorem 4.

Lemma 12. *The following estimate,*

$$
|||\sum_{T\in\mathcal{T}_H,\, j=1,\dots,r} v_j(\phi_{T,j}-\phi_{T,j}^L)|||_{\mathrm{h}}^2 \leq C_4 L^d \sum_{T\in\mathcal{T}_H,\, j=1,\dots,r} |v_j|^2 |||\phi_{T,j}-\phi_{T,j}^L|||_{\mathrm{h}}^2, \tag{4.12}
$$

holds, where $C_4 = CC_c^2C_\zeta^2(1+C_bC_s)^2$ and C is a generic constant neither depending on the mesh size, the size of the patches, or the problem data.

Proof. Defining $\eta_T := \zeta_T^{L,L+1}$ and let $w \in \mathcal{V}^{\mathrm{f}}$. From Lemma 8 there exist a b_T such that $\Pi_H b_T = \Pi_H(\eta_T w)$ such that $|||b_T|||_{\mathrm{h}} \lesssim C_b |||\Pi_H(\eta_T w)|||_H$. We have the following relation

$$
a_h(\phi_{T,j}-\phi_{T,j}^L, w-\eta_T w + b_T) = 0, \tag{4.13}
$$

since $w - \eta_T w + b_T \in \mathcal{V}^{\mathrm{f}}$ with no support in ω_T^L. Let
$w = \sum_{T\in\mathcal{T}_H,\, j=1,\dots,r} v_j(\phi_{T,j}-\phi_{T,j}^L)$, we obtain

$$
\begin{aligned}
|||w|||_{\mathrm{h}}^2 &\lesssim a_h(w,w) \\
&= \sum_{T\in\mathcal{T}_H,\, j=1,\dots,r} v_j a_h(\phi_{T,j}-\phi_{T,j}^L, w) \\
&= \sum_{T\in\mathcal{T}_H,\, j=1,\dots,r} v_j a_h(\phi_{T,j}-\phi_{T,j}^L, \zeta_T w - b_T) \qquad (4.14) \\
&= C_c \sum_{T\in\mathcal{T}_H,\, j=1,\dots,r} |v_j| |||\phi_{T,j}-\phi_{T,j}^L|||_{\mathrm{h}} \left(|||\zeta_T w|||_{\mathrm{h}} + |||b_T|||_{\mathrm{h}}\right).
\end{aligned}
$$

Furthermore, using Lemma 8, Lemma 7, and Lemma 11, we have

$$|||b|||_{\mathrm{h}} \lesssim C_b|||\Pi_H\zeta_T w|||_H \lesssim C_bC_s|||\zeta_T\phi_h|||_{\mathrm{h}} \lesssim C_bC_sC_\zeta|||w|||_{h,\omega_T^{L+1}} \tag{4.15}$$

and obtain,

$$|||w|||_{\mathrm{h}}^2 \lesssim C_cC_\zeta(1+C_bC_s)\sum_{T\in\mathcal{T}_H,\, j=1,\dots,r}|v_j||||\phi_{T,j}-\phi_{T,j}^L|||_{\mathrm{h}}|||w|||_{h,\omega_T^{L+1}} \tag{4.16}$$

using (4.14), (4.15), and (4.17). Also, note that

$$\sum_{T\in\mathcal{T}_H,\, j=1,\dots,r}|||w|||_{h,\omega_T^{L+1}}^2 \lesssim L^d|||w|||_{\mathrm{h}}^2. \tag{4.17}$$

and using a Cauchy-Schwartz inequality for the sum, concludes the proof with $C_4 = CC_c^2C_\zeta^2(1+C_bC_s)^2$. Where C is a generic constant hidden in '$\lesssim$'. □

Lemma 13 (Stability of the corrected basis function)**.** *For all $T \in \mathcal{T}_H$, $j = 1,\dots,r$, the following estimate*

$$|||\phi_{T,h}-\lambda_{T,j}|||_{\mathrm{h}} \le C_\phi\|H^{-1}\lambda_{T,j}\|_{L^2(\Omega)} \tag{4.18}$$

holds, where $C_\phi = C(1+\|H\mathbf{b}\|_{L^\infty(\Omega)}\alpha^{-1/2}+\|Hc\|_{L^\infty(\Omega)}\mu_0^{-1})$ and C is generic constant neither depending on the mesh size or the problem data.

Proof. Let $v = \lambda_{T,j} - b_T \in \mathcal{V}^{\mathrm{f}}$, where $b_T \in H_0^1(T)$ and $\Pi_H b_{T,j} = \lambda_{T,j}$ from Lemma 8. We have

$$\begin{aligned}
&|||\phi_{T,h}-\lambda_{T,j}|||_{\mathrm{h}}^2 \lesssim a_h(\phi_{T,h}-\lambda_{T,j},\phi_{T,h}-\lambda_{T,j})\\
&= a_h(\phi_{T,h}-\lambda_{T,j}, v-\lambda_{T,j}) = a_h(\phi_{T,h}-\lambda_{T,j}, b_{T,j})\\
&= a_h^d(\phi_{T,h}-\lambda_{T,j}, b) + a_h^{\text{c-r}}(\phi_{T,h}-\lambda_{T,j}, b_{T,j})\\
&= a_h^d(\phi_{T,h}-\lambda_{T,j}, b) + (\mathbf{b}\cdot\nabla_h(\phi_{T,h}-\lambda_{T,j}) + c(\phi_{T,h}-\lambda_{T,j}), b_{T,j})_{L^2(\Omega)}
\end{aligned} \tag{4.19}$$

Using that the diffusion part in (4.19) of the bilinear form is continuous in $(\mathcal{V}_h\times\mathcal{V}_h)$, Lemma 8, and a inverse inequality, we have

$$\begin{aligned}
a_h^d(\phi_{T,h}-\lambda_{T,j}, b_{T,j}) &\lesssim |||\phi_{T,h}-\lambda_{T,j}|||_{\mathrm{h}}|||b_{T,j}|||_{\mathrm{h}}\\
&\lesssim C_b|||\phi_{T,h}-\lambda_{T,j}|||_{\mathrm{h}}|||\lambda_{T,j}|||_H\\
&\lesssim C_b\beta^{1/2}|||\phi_{T,h}-\lambda_{T,j}|||_{\mathrm{h}}\|H^{-1}\lambda_{T,j}\|_{L^2(T)}.
\end{aligned} \tag{4.20}$$

For the convection-reaction part in (4.19), we have

$$\begin{aligned}
&(\mathbf{b}\cdot\nabla_h(\phi_{T,h}-\lambda_{T,j}) + c(\phi_{T,h}-\lambda_{T,j}), b_{T,j})_{L^2(\Omega)}\\
&\lesssim \left(\|\mathbf{b}\cdot\nabla_h(\phi_{T,h}-\lambda_{T,j})\|_{L^2(\Omega)} + \|c(\phi_{T,h}-\lambda_{T,j})\|_{L^2(\Omega)}\right)\|b_{T,j}\|_{L^2(\Omega)}\\
&\lesssim \big(\|H\mathbf{b}\|_{L^\infty(\Omega)}\|\nabla_h(\phi_{T,h}-\lambda_{T,j})\|_{L^2(\Omega)}\\
&\qquad + \|Hc\|_{L^\infty(\Omega)}\mu_0^{-1}\|c_0(\phi_{T,h}-\lambda_{T,j})\|_{L^2(\Omega)}\big)\|H^{-1}\lambda_{T,j}\|_{L^2(\Omega)}.
\end{aligned} \tag{4.21}$$

We obtain

$$|||\phi_{T,h}-\lambda_{T,j}|||_{\mathrm{h}} \le C_\phi\|H^{-1}\lambda_{T,j}\|_{L^2(\Omega)}. \tag{4.22}$$

where $C_\phi = C(1+\|H\mathbf{b}\|_{L^\infty(\Omega)}\alpha^{-1/2}+\|Hc\|_{L^\infty(\Omega)}\mu_0^{-1})$ and C is generic constant hidden in '$\lesssim$'. □

4.2 Proof of main results

We are now ready prove, Theorem 1, Lemma 3, and Theorem 4.

Proof of Theorem 1. Let us decompose u_h into a coarse contribution, $v_H^{ms} \in \mathcal{V}_H^{ms}$, and a fine scale remainder, $v^f \in \mathcal{V}^{\mathrm{f}}$, i.e., $u_h = v_H^{ms} + v^f$. For v^f we have

$$\begin{aligned} |||v^f|||_{\mathrm{h}}^2 &\lesssim a_h(v^f, v^f) = a_h(u_h, v^f) = (f, v^f)_{L^2(\Omega)} \\ &= (f - \Pi_H f, v^f - \Pi_H v^f)_{L^2(\Omega)} \\ &\leq \|H(f - \Pi_H f)\|_{L^2(\Omega)} \|H^{-1}(v^f - \Pi_H v^f)||_{L^2(\Omega)} \\ &\leq \alpha^{-1/2} \|H(f - \Pi_H f)\|_{L^2(\Omega)} |||v^f|||_{\mathrm{h}}. \end{aligned} \tag{4.23}$$

Using continuity, we have

$$\begin{aligned} |||u_h - u_H^{ms}|||_{\mathrm{h}}^2 &\lesssim a_h(u_h - u_H^{ms}, u_h - u_H^{ms}) = a_h(u_h - u_H^{ms}, u_h - v_H^{ms}) \\ &\lesssim C_c |||u_h - u_H^{ms}|||_{\mathrm{h}} |||u_h - v_H^{ms}|||_{\mathrm{h}} \end{aligned} \tag{4.24}$$

which concludes the proof together with (4.23). □

Proof of Lemma 3. Define $e := \phi_{T,j} - \phi_{T,j}^L$ where $\phi_{T,j} \in \mathcal{V}^{\mathrm{f}}$ and $\phi_{T,j}^L \in \mathcal{V}^f(\omega_T^L)$. We have

$$\begin{aligned} |||e|||_h^2 &\lesssim a_h(e, \phi_{T,j} - \phi_{T,j}^L) = a_h(e, \phi_{T,j} - v) \\ &\lesssim C_c |||e|||_{\mathrm{h}} |||\phi_{T,j} - v|||_{\mathrm{h}}. \end{aligned} \tag{4.25}$$

Furthermore from Lemma 8, there exist a $v = \zeta_T^{L-1,L} \phi_{T,j} - b_T \in \mathcal{V}^f(\omega_T^L)$ such that $\Pi_H b_T = \Pi_H(\zeta_T^{L-1,L} \phi_{T,j})$ and $|||b_T|||_{\mathrm{h}} \lesssim C_b |||\Pi_H(\zeta_T^{L-1,L} \phi_{T,j})|||_H$, we have

$$|||e|||_h \lesssim C_c \left(|||(1 - \zeta_T^{L-1,L}) \phi_{T,j}|||_{\mathrm{h}} + |||b_T|||_{\mathrm{h}} \right), \tag{4.26}$$

where

$$\begin{aligned} |||b_T|||_{\mathrm{h}} &\lesssim C_b |||\Pi_H \zeta_T^{L-1,L} \phi_{T,j}|||_H = C_b |||\Pi_H(1 - \zeta_T^{L-1,L}) \phi_{T,j}|||_H \\ &\lesssim C_b C_s |||(1 - \zeta_T^{L-1,L}) \phi_{T,j}|||_{\mathrm{h}} \lesssim C_b C_s C_\zeta |||\phi_{T,j}|||_{h, \Omega \setminus \omega_T^{L-1}}. \end{aligned} \tag{4.27}$$

using Lemma 8, Lemma 7, and Lemma 11. We obtain,

$$|||e|||_h \lesssim C_2 |||\phi_{T,j}|||_{h, \Omega \setminus \omega_T^{L-1}}, \tag{4.28}$$

where $C_2 = C C_c C_\zeta (1 + C_b C_s)$ from (4.26) and (4.27). Where C is the generic constant hidden in '$\lesssim$'.

The next step in the proof is to construct a recursive relation which will be used to prove the decay in the correctors. To this end, let $\ell k = L - 1$, and define another the cut off function, $\eta_T^m := (1 - \zeta^{\ell(k-m), \ell(k-m+1)})$ and the patch $\tilde{\omega}_T^m := \omega_T^{\ell(k-m+1)}$, for $m = 1, \ldots, k-1$. Note that $\tilde{\omega}_T^{m+1} \subset \tilde{\omega}_T^m$. We have

$$|||\phi_{T,j}|||_{h, \Omega \setminus \tilde{\omega}_T^m} \leq |||\eta_T^m \phi_{T,j}|||_{\mathrm{h}} \lesssim a_h(\eta_T^m \phi_{T,j}, \eta_T^m \phi_{T,j}) \tag{4.29}$$

So shorten the proof we refer to the following inequality

$$a^{\mathrm{d}}(\eta_T^m \phi_{T,j}, \eta_T^m \phi_{T,j}) \lesssim a^{\mathrm{d}}(\phi_{T,j}, (\eta_T^m)^2 \phi_{T,j} - b_T) + \frac{C_{\beta/\alpha}^2}{\ell} |||\phi_{T,j}|||_{h,\omega_T^m \setminus \bar{\omega}_T^{m+1}}^2. \tag{4.30}$$

where $(\eta_T^m)^2 \phi_{T,j} - b_T \in \mathcal{V}^{\mathrm{f}}$, in the proof of Lemma 10 in [11]. We focus on the term convection-reaction term, i.e.,

$$\begin{aligned} &a^{\text{c-r}}(\eta_T^m \phi_{T,j}, \eta_T^m \phi_{T,j}) \\ &= \sum_{\substack{S \in \mathcal{T}_h: \\ S \cap (\Omega \setminus \tilde{\omega}_T^{m+1}) \neq 0}} ((\gamma - \nabla \cdot \mathbf{b}) \eta_T^m \phi_{T,j}, \eta_T^m \phi_{T,j})_{L^2(S)} - (\eta_T^m \phi_{T,j}, \mathbf{b} \cdot \nabla \eta_T^m \phi_{T,j})_{L^2(S)} \\ &\quad + \sum_{\substack{e \in \mathcal{E}_h: \\ e \cap (\Omega \setminus \tilde{\omega}_T^{m+1}) \neq 0}} (b_e [\eta_T^m \phi_{T,j}], [\eta_T^m \phi_{T,j}])_{L^2(e)} \\ &\quad + \sum_{\substack{e \in \mathcal{E}_h(\Omega): \\ e \cap (\Omega \setminus \tilde{\omega}_T^{m+1}) \neq 0}} ((\nu_e \cdot \{\mathbf{b} \eta_T^m \phi_{T,j}\}, [\eta_T^m \phi_{T,j}])_{L^2(e)} \\ &\quad + \sum_{\substack{e \in \mathcal{E}_h(\Gamma): \\ e \cap (\Omega \setminus \tilde{\omega}_T^{m+1}) \neq 0}} \frac{1}{2} ((\nu_e \cdot \mathbf{b}) \eta_T^m \phi_{T,j}, \eta_T^m \phi_{T,j})_{L^2(e)} \end{aligned} \tag{4.31}$$

Since the cut of function is piecewise constant it follows that

$$\begin{aligned} &((\gamma - \nabla \cdot \mathbf{b}) \eta_T^m \phi_{T,j}, \eta_T^m \phi_{T,j})_{L^2(S)} - (\eta_T^m \phi_{T,j}, \mathbf{b} \cdot \nabla \eta_T^m \phi_{T,j})_{L^2(S)} \\ &= (\gamma - \nabla \cdot \mathbf{b}) \phi_{T,j}, (\eta_T^m)^2 \phi_{T,j})_{L^2(S)} - (\phi_{T,j}, \mathbf{b} \cdot \nabla (\eta_T^m)^2 \phi_{T,j})_{L^2(S)} \end{aligned} \tag{4.32}$$

for all $S \in \mathcal{T}_h$. Using the following equalities from (Appendix A in [11])

$$\begin{aligned} \{vw\}[vw] &= \{w\}[v^2 w] - [v]\{w\}\{v\}\{w\} + 1/4[v]\{v\}[w][w] \\ [vw][vw] &= [w][v^2 w] - 1/4[v]^2[w]^2 + [v]^2\{w\}^2 \end{aligned} \tag{4.33}$$

and (4.32), we obtain

$$\begin{aligned} &a^{\text{c-r}}(\eta_T^m \phi_{T,j}, \eta_T^m \phi_{T,j}) = a^{\text{c-r}}(\phi_{T,j}, (\eta_T^m)^2 \phi_{T,j}) \\ &+ \sum_{\substack{e \in \mathcal{E}_h(\Omega): \\ e \cap (\tilde{\omega}_T^m \setminus \tilde{\omega}_T^m) \neq 0}} \Big(- (\nu_e \cdot \mathbf{b}[\eta_T^m]\{\phi_{T,j}\}, \{\eta_T^m\}\{\phi\})_{L^2(e)} \\ &\qquad + 1/4(\nu_e \cdot \mathbf{b}[\eta_T^m]\{\phi_{T,j}\}, \{\eta_T^m\}[\phi_{T,j}])_{L^2(e)} \\ &\qquad - 1/4(b_e[\eta_T^m]^2, [\phi_{T,j}]^2)_{L^2(e)} + (b_e[\eta_T^m]^2, \{\phi_{T,j}\}^2)_{L^2(e)} \Big) \end{aligned} \tag{4.34}$$

Next we bound the terms in (4.34) in two steps, first the sum over the edges, and then the bilinear form. The sum over the edges terms can be bounded by using that $\|[\eta_T^m]\|_{L^\infty(T)} \lesssim \|h\|_{L^\infty(T)}/H|_T$, $\|\{\eta_T^m\}\|_{L^\infty(\Omega)} \lesssim 1$, $\|h\|_{L^\infty(T)}/H|_T \ell < 1$, and a trace inequality. Let $e = S^+ \cap S^- \in \mathcal{E}_h$, we obtain

$$
\begin{aligned}
&\sum_{\substack{e\in\mathcal{E}_h(\Omega):\\ e\cap(\tilde{\omega}_T^m\setminus\tilde{\omega}_T^m)\neq 0}} \frac{\|\mathbf{b}\|_{L^\infty(S^+\cup S^-)}}{\|H\|_{L^\infty(S^+\cap S^-)}\ell}\Big(\|h^{1/2}\{\phi_{T,j}\}\|_{L^2(e)}\|h^{1/2}\{\phi\}\|_{L^2(e)}+\\
&\qquad \|h^{1/2}\{\phi_{T,j}\}\|_{L^2(e)}\|h^{1/2}[\phi_{T,j}]\|_{L^2(e)} + \|h^{1/2}[\phi_{T,j}]\|^2_{L^2(e)}\\
&\qquad + \|h^{1/2}\{\phi_{T,j}\}\|^2_{L^2(e)}\Big)\\
&\lesssim \sum_{\substack{e\in\mathcal{E}_H(\Omega):\\ e\cap(\tilde{\omega}_T^m\setminus\tilde{\omega}_T^m)\neq 0}} \frac{\|\mathbf{b}\|_{L^\infty(T^+\cup T^-)}}{\ell}\|H^{-1}\phi_{T,j}\|^2_{L^2(T^+\cup T^-)}\\
&\lesssim \sum_{\substack{T\in\mathcal{T}_H:\\ T\cap(\tilde{\omega}_T^m\setminus\tilde{\omega}_T^m)\neq 0}} \frac{\|\mathbf{b}\|_{L^\infty(T)}}{\ell}\|H^{-1}(\phi_{T,j}-\Pi_H\phi_{T,j})\|^2_{L^2(T)}\\
&\lesssim \frac{\|H\mathbf{b}\|_{L^\infty(\Omega)}}{\ell\alpha}|||\phi_{T,j}|||^2_{h,(\tilde{\omega}_T^m\setminus\tilde{\omega}_T^{m+1})}
\end{aligned}
\tag{4.35}
$$

For the bilinear form, using Lemma 8 there exist a b_T with support in $\tilde{\omega}_T^m\setminus\tilde{\omega}_T^{m+1}$, such that $(\eta_T^m)^2\phi_{T,j} - b_T \in \mathcal{V}^{\mathrm{f}}$ and $|||b_T|||_{\mathrm{h}} \lesssim C_b|||\Pi_H((\eta_T^m)^2\phi_{T,j})|||_H$. We have

$$
\begin{aligned}
&a^{\text{c-r}}(\phi_{T,j},(\eta_T^m)^2\phi_{T,j}) = a^{\text{c-r}}(\phi_{T,j},(\eta_T^m)^2\phi_{T,j}-b_T) + a^{\text{c-r}}(\phi_{T,j},b_T)\\
&\lesssim a^{\text{c-r}}(\phi_{T,j},(\eta_T^m)^2\phi_{T,j}-b_T)\\
&\qquad + C_c|||\phi_{T,j}|||_{\mathrm{a},h,\tilde{\omega}_T^m\setminus\tilde{\omega}_T^{m+1}}\|c_0 b_T\|_{L^2(\tilde{\omega}_T^m\setminus\tilde{\omega}_T^{m+1})}\\
&\qquad + C_c|||\phi_{T,j}|||_{\mathrm{a},h,\tilde{\omega}_T^m\setminus\tilde{\omega}_T^{m+1}}\|c_0 b_T\|_{L^2(\tilde{\omega}_T^m\setminus\tilde{\omega}_T^{m+1})}
\end{aligned}
\tag{4.36}
$$

which can be further estimated, using Lemma 7. For all $T \in \mathcal{T}_H$ the operator Π_H is stable in the $L^2(T)$-norm, we have

$$
\begin{aligned}
&\|b_T\|_{L^2(\tilde{\omega}_T^m\setminus\tilde{\omega}_T^{m+1})}\\
&\lesssim C_b\|Hc_0\|_{L^\infty(\tilde{\omega}_T^m\setminus\tilde{\omega}_T^{m+1})}\|H^{-1}\Pi_H((\eta_T^m)^2\phi_{T,j})\|_{L^2(\tilde{\omega}_T^m\setminus\tilde{\omega}_T^{m+1})}\\
&= C_b\|\Pi_H(((\eta_T^m)^2-\Pi_0(\eta_T^m)^2)H^{-1}\phi_{T,j})\|_{L^2(\tilde{\omega}_T^m\setminus\tilde{\omega}_T^{m+1})}\\
&\leq C_b\|(\eta_T^m)^2-\Pi_0(\eta_T^m)^2\|_{L^\infty(\tilde{\omega}_T^m\setminus\tilde{\omega}_T^{m+1})}\|H^{-1}\phi_{T,j}\|_{L^2(\tilde{\omega}_T^m\setminus\tilde{\omega}_T^{m+1})}\\
&\lesssim HC_b\ell^{-1}\mu_0^{-1}|||\phi_{T,j}|||_{h,\tilde{\omega}_T^m\setminus\tilde{\omega}_T^{m+1}}.
\end{aligned}
\tag{4.37}
$$

using Lemma 8 and that $\Pi_H\phi = 0$. We obtain

$$
\begin{aligned}
&|||\phi_{T,j}|||^2_{h,\Omega\setminus\omega_T^m}\\
&\lesssim \Big(a^{\mathrm{d}}(\phi_{T,j},(\eta_T^m)^2\phi_{T,j}-\tilde{b}) + a^{\text{c-r}}(\phi_{T,j},(\eta_T^m)^2\phi_{T,j}-\tilde{b})\Big)\\
&+ (C^2_{\beta/\alpha} + \|H\mathbf{b}\|_{L^\infty(\Omega)}\alpha^{-1} + C_cC_b\|c_0\|\mu_0^{-1})_{L^\infty(\Omega)}\ell^{-1}|||\phi_{T,j}|||^2_{h,\omega_T^m\setminus\omega_T^{m+1}}\\
&= C_3\ell^{-1}|||\phi_{T,j}|||^2_{h,\omega_T^m\setminus\omega_T^m} \leq C_3\ell^{-1}|||\phi_{T,j}|||^2_{h,\Omega\setminus\omega_T^{m+1}},
\end{aligned}
\tag{4.38}
$$

where $C_3 = C'(C^2_{\beta/\alpha} + \|H\mathbf{b}\|_{L^\infty(\Omega)}\alpha^{-1} + C_c C_b\|c_0\|_{L^\infty(\Omega)}\mu_0^{-1})$, using (4.30), (4.36), and (4.37). Where C' is the generic constant hidden in '$\lesssim$'. Implying that

$$|||\phi_{T,j}|||_{h,\Omega\setminus\omega_T^m} \lesssim C_3\ell^{-1}|||\phi_{T,j}|||_{h,\Omega\setminus\omega_T^{m+1}}, \quad \text{for any } m = 1,2,\ldots,k-1. \tag{4.39}$$

Using (4.39) recursively, we have

$$|||\phi_{T,j}|||^2_{h,\Omega\setminus\tilde{\omega}_T^1} \lesssim (C_3\ell^{-1})^{k-1}|||\phi_{T,j}|||^2_{h,\Omega\setminus\tilde{\omega}_T^k} = (C_3\ell^{-1})^{k-1}|||\phi_{T,j} - \lambda_{T,j}|||^2_{h,\Omega} \tag{4.40}$$

Equation (4.28) together with (4.39), gives

$$|||\phi_{T,j} - \phi_h^L|||_{\mathrm{h}} \le C_2(C_3\ell^{-1})^{\frac{k-1}{2}}|||\phi_{T,j} - \lambda_{T,j}|||_{\mathrm{h}}. \tag{4.41}$$

which concludes the proof is concluded. □

Proof of Theorem 4. Using the triangle inequality, we have

$$|||u - u_H^{ms,L}|||_{\mathrm{h}} \le |||u - u_h|||_{\mathrm{h}} + |||u_h - u_H^{ms,L}|||_{\mathrm{h}}. \tag{4.42}$$

Note that, $u_h \in \mathcal{V}_h$, can be decomposed into a coarse, $v_H^{ms} \in \mathcal{V}_H^{ms}$, and a fine, $v^f \in \mathcal{V}^{\mathrm{f}}$, scale contribution, i.e., $u_h = v_H^{ms} + u^f$. Also, let $v_H^{ms,L} \in \mathcal{V}_H^{ms,L}$ be chosen such that $\Pi_H v_H^{ms,L} = \Pi_H v_H^{ms}$. We have

$$\begin{aligned} |||u_h - u_H^{ms,L}|||_{\mathrm{h}} &\lesssim a_h(u_h - u_H^{ms,L}, u_h - u_H^{ms,L}) \\ &= a_h(u_h - u_H^{ms,L}, u_h - v_H^{ms,L}) \\ &\lesssim C_c|||u_h - u_H^{ms,L}|||_{\mathrm{h}}|||u_h - v_H^{ms,L}|||_{\mathrm{h}}, \end{aligned} \tag{4.43}$$

and obtain

$$\begin{aligned} |||u - u_H^{ms,L}|||_{\mathrm{h}} \le& |||u - u_h|||_{\mathrm{h}} \\ &+ C_c\left(|||u_h - v_H^{ms}|||_{\mathrm{h}} + |||v_H^{ms} - v_H^{ms,L}|||_{\mathrm{h}}\right). \end{aligned} \tag{4.44}$$

The first term in (4.44) implies that the reference mesh need to be sufficiently fine to get a sufficient approximation. The second term is approximated using using (4.23), i.e.

$$|||u_h - v_H^{ms}|||_{\mathrm{h}} \lesssim \alpha^{-1/2}\|H(1 - \Pi_H)f\|_{L^2(\Omega)}, \tag{4.45}$$

and for the last term in we have,

$$\begin{aligned} |||v_H^{ms} - v_H^{ms,L}|||^2_{\mathrm{h}} &= |||\sum_{T\in\mathcal{T}_H,\, j=1,\ldots,r} v^{ms}_{H,T}(x_j)(\phi_{T,h} - \phi^L_{T,j})|||^2_{\mathrm{h}} \\ &\lesssim C_4 L^d \sum_{T\in\mathcal{T}_H,\, j=1,\ldots,r} |v^{ms}_{H,T}(x_j)|^2|||\phi_{T,h} - \phi^L_{T,j}|||^2_{\mathrm{h}} \\ &\lesssim C_4 C_2^2 L^d\gamma^{2L} \sum_{T\in\mathcal{T}_H,\, j=1,\ldots,r} |v^{ms}_{H,T}(x_j)|^2|||\phi_{T,h} - \lambda_{T,j}|||^2_{\mathrm{h}} \end{aligned} \tag{4.46}$$

using Lemma 12 and Lemma 3.

We obtain, using Lemma 13, that

$$
\begin{aligned}
&\sum_{T\in\mathcal{T}_H,\, j=1,\dots,r} |v_{H,T}^{ms}(x_j)|^2 |||\phi_{T,h} - \lambda_{T,j}|||_{\mathrm{h}}^2 \\
&\le C_\phi^2 \sum_{T\in\mathcal{T}_H,\, j=1,\dots,r} |v_{H,T}^{ms}(x_j)|^2 \|H^{-1}\lambda_{T,j}\|_{L^2(T)}^2 \\
&= C_\phi^2 \sum_{T\in\mathcal{T}_H,\, j=1,\dots,r} \|H^{-1} v_{H,T}^{ms}(x_j)\lambda_{T,j}\|_{L^2(\Omega)}^2 \\
&\lesssim C_\phi^2 \| \sum_{T\in\mathcal{T}_H,\, j=1,\dots,r} H^{-1} v_{H,T}^{ms}(x_j)\lambda_{T,j}\|_{L^2(\Omega)}^2 \\
&= C_\phi^2 \| \sum_{T\in\mathcal{T}_H,\, j=1,\dots,r} H^{-1} v_{H,T}^{ms}(x_j)\Pi_H(\lambda_{T,j} - \phi_{T,j})\|_{L^2(\Omega)}^2 \\
&\le C_\phi^2 \| \sum_{T\in\mathcal{T}_H,\, j=1,\dots,r} H^{-1} v_{H,T}^{ms}(x_j)(\lambda_{T,j} - \phi_{T,j})\|_{L^2(\Omega)}^2 \\
&\le C_\phi^2 \|H^{-1} v_H^{ms}\|_{L^2(\Omega)}^2 \\
&\le C_\phi^2 \left(\|H^{-1}u_h\|_{L^2(\Omega)} + \|H^{-1}u^f\|_{L^2(\Omega)}\right)^2 \\
&\le C_\phi^2 \left(\|H^{-1}\|_{L^\infty(\Omega)} |||u_h|||_{\mathrm{h}} + \alpha^{-1/2}|||u^f|||_{\mathrm{h}}\right)^2 .
\end{aligned}
\tag{4.47}
$$

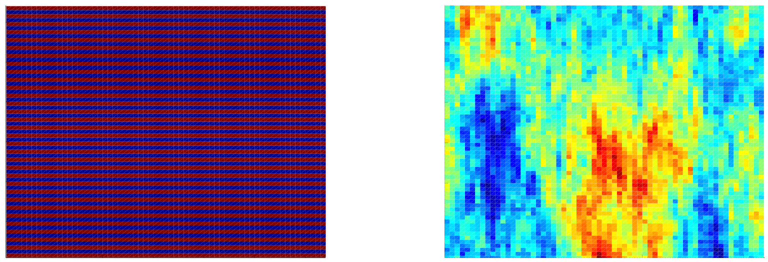
Using (4.23), we have

$$
\begin{aligned}
&\sum_{T\in\mathcal{T}_H,\, j=1,\dots,r} |v_{H,T}^{ms}(x_j)|^2 |||\phi_{T,h} - \lambda_{T,j}|||_{\mathrm{h}}^2 \\
&\lesssim C_\phi^2 \left(\|H^{-1}\|_{L^\infty(\Omega)}\|f\|_{L^2(\Omega)} + \alpha^{-1}\|H(1-\Pi_H)f\|_{L^2(\Omega)})\right)^2 \\
&\lesssim C_\phi^2 \left(\|H^{-1}\|_{L^\infty(\Omega)}^2\|f\|_{L^2(\Omega)}^2 + \alpha^{-1}\|Hf(1-\Pi_H f)\|^2\right) .
\end{aligned}
\tag{4.48}
$$

which concludes the proof. □

5 Numerical experiment

We consider the domain $\Omega = [0,1]\times[0,1]$ and the forcing function $f = 1 + \cos(2\pi x)\cos(2\pi y)$. The localization parameter, which determine the size of the patches, is chosen as $L = \lceil 2\log(H^{-1})\rceil$, i.e., the size of the patches are $2H\log(H^{-1})$. Consider a coarse quadrilateral mesh, $\mathcal{T}_H$, of size $H = 2^{-i}$, $i = 2,3,4,5$. The corrector functions are solved on sub-grids of the quadrilateral mesh, $\mathcal{T}_h$, where $h = 2^{-7}$. We consider three different permeabilities: $A_1 = 1$, $A_2 = A_2(y)$ which is piecewise constant with respect to a Cartesian grid of width 2^{-6} in y-direction taking the values 1 or 0.01, and $A_3 = A_3(x,y)$ which is piecewise constant with respect to a Cartesian grid of width 2^{-6} both in the x- and y-directions, bounded below by $\alpha = 0.05$ and has a maximum ratio $\beta/\alpha = 4\cdot 10^5$. The permeability A_3 is taken from the 31 layer in the SPE benchmark



(a) A_2 (b) A_3

Figure 2: The diffusion coefficients A_2 and A_3 in log scale.

problem, see `http:www.spe.org/web/csp/`. The diffusion coefficients A_2 and A_3 are illustrated in Figure 2. For the convection term we consider: $\mathbf{b} = [C, 0]$, for different values of C.

To investigate how the error in relative energy-norm, $|||u_h - u_H^{ms,L}|||/|||u_h|||$, depends on the magnitude of the convection we consider: A_1 and $\mathbf{b} = [C, 0]$ with $C = \{32, 64, 128\}$. Figure 3 shows the convergence in energy-norm as a function of the coarse mesh size H for the different values of C.

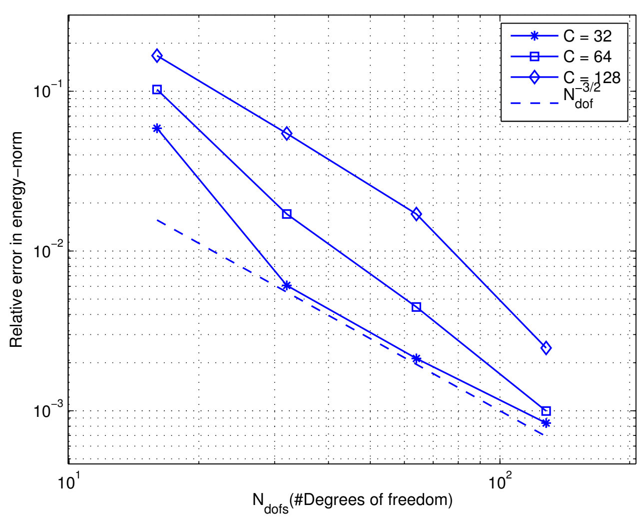


Figure 3: The number degrees of freedom (N_{dof}) vs. the relative error in energy-norm, for different sizes of the convection term, C.

Also, to see the effect of heterogeneous diffusion of the error in the relative energy-norm, $|||u_h - u_H^{ms,L}|||/|||u_h|||$, we consider: Figure 4 which shows the error in relative energy-norm using A_2 and $\mathbf{b} = [1, 0]$ and Figure 5 which shows the error in relative energy-norm using A_3 and $\mathbf{b} = [512, 0]$.

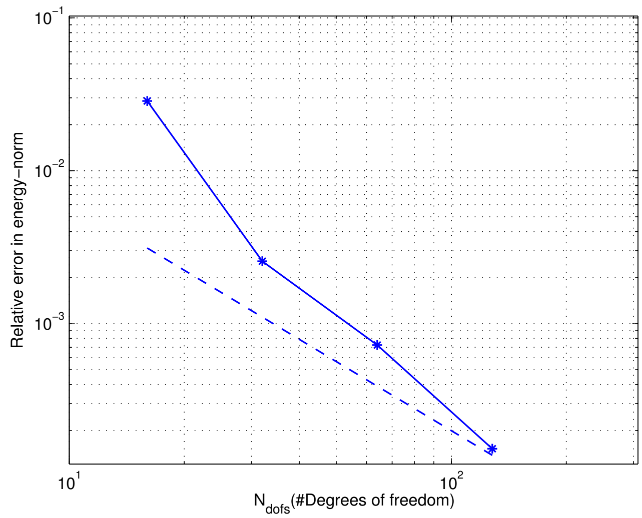


Figure 4: The number degrees of freedom (N_{dof}) vs. the relative error in energy-norm, using a high contrast diffusion coefficients A_2 and $\mathbf{b} = [1, 0]$. The dotted line corresponds to $N_{dof}^{-3/2}$.

We obtain H^3 convergence of the dG multiscale method to a reference solution in the relative energy-norm, $|||u_h - u_H^{ms,L}|||/|||u_h|||$, independent of the variation in the coefficients or regularity of the underlying solution.

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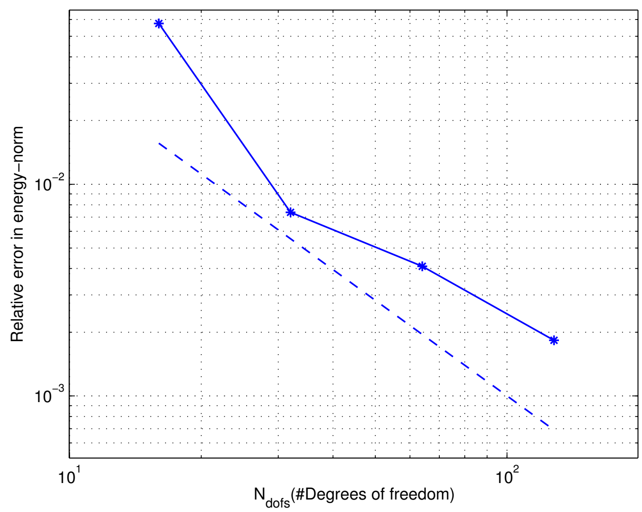

Figure 5: The number degrees of freedom (N_{dof}) vs. the relative error in energy-norm, using a high contrast diffusion coefficients A_3 and $\mathbf{b} = [512, 0]$. The dotted line corresponds to $N_{dof}^{-3/2}$.

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