On a Limitation in Networked Flow Control

Torbjörn Wigren
Division of Systems and Control, Department of Information Technology,
Uppsala University, P.O Box 337, SE-75105 Uppsala, Sweden e-mail: torbjorn.wigren@it.uu.se.

Abstract

The paper analyzes a continuous time flow control system, with flow of a general quantity from a source node to a sink node. The flow is one-directional, meaning that there is a saturation between the nodes that limits the flow to be positive and below a maximum. The controlled plant is located in the sink node and the controller is located in the source node. The plant and the controller are modeled by linear filters parameterized with poles and zeros. Feed forward control from measured disturbances is included. Delays affect both the downlink control signal and the uplink measurement signals. The paper proves that for large delays, $\mathcal{L}_2$-stability does not follow from the Popov criterion unless the quotient of the products of all zeros and the product of all poles is less than $1/kG_p$, where k is the slope of the saturation and G_p is the gain constant of the loop gain. In case the plant models a leaky reservoir, the conclusion is that the amount of low frequency gain of the controller cannot be arbitrarily high at the same time as the amount of leakage of the reservoir is arbitrarily low. In communications this means that an increased requirement to regulate static errors of the reservoir needs to be accompanied by a reduced flow capacity.

Index Terms

I. Introduction

Networked control systems (NCS) very often study situations where a plant is remotely controlled from a distinct site where the controller is located. This setup requires a joint study of communications and control to fully assess the properties of the closed loop system [4]. The present paper is focused on a one-directional networked flow control system, for which a restriction on the low frequency loop gain is derived. In wireless internet flow control this limitation boils down to a tradeoff between static regulation performance and data flow capacity, see Fig. 4 and Fig. 6 for an immediate illustration. NCS has also found other widespread applications, e.g. in power control and scheduling for cellular communication systems [9].

The interplay between communications and control for NCS is particularly visible in the data rate theorem that states the bit rate needed to stabilize a system with a pre-specified number of unstable poles [2], [3], [5], [11]. Apart from communication constraints, delays are a fundamental consequence of the NCS setup. In flow control over the internet, these delays may dominate the dynamics [21]. The data flow is also one-directional meaning that there is a flow rate saturation between the controller and the plant. This has further implications on closed loop stability. The main purpose of the paper is therefore to study the fundamental limitations of such a general continuous time flow control problem in more detail. Although the NCS here is restricted to have a specific non-linear structure, there are a number of obvious applications including internet packet flow control [21], general fluid flow control [19], mixing dynamics [8], [22], as well as traffic flow control [10]. The paper is also relevant for general control systems with actuator saturation, see e.g. [1] [12].

The closed loop system is one where a static nonlinearity is combined with linear dynamics. Since the static nonlinearity has a zero minimal gain, the Popov criterion is a suitable starting point for the analysis [14], [20]. Since the delays make the linear dynamics infinite dimensional, the results based on passivity and/or input-output stability theory are the ones underpinning the analysis, see e.g. [6], [15], [16], [23], [24].

The main contribution of the paper is a result that limits the total low frequency gain of the linear loop gain. The analysis studies the Popov criterion [20] on the negative real axis of the complex plane, in the case where the delays tend to infinity. As shown in the paper the Popov inequality then collapses to a condition on the quotient between the product of the zeros and the product of the poles of the linear loop gain, where the quotient needs to be small enough for the Popov inequality to hold. The limiting value is the inverse of the product of the gain constant of the linear loop gain and the slope of the static saturation. A consequence of the result is that $\mathcal{L}_2$-stability does not follow when the plant and the controller simultaneously have poles very close to zero. This translates into the statement that it is not possible to control arbitrarily well statically, without increasing the distance of the stable plant poles from the imaginary axis. This effect was experimentally observed in [21], where the tradeoff turns out to be one between regulation of static disturbances and flow capacity. To the best of the authors knowledge these insights are new.

The analysis presented here is not yet complete since the Popov criterion may not be necessary for $\mathcal{L}_2$-stability. It is therefore a very interesting topic to investigate the obtained result further and to seek more stringent ones. Papers do e.g. exist on the limiting situation where there is a pure integrator in the loop [7]. Another restriction of the result is that it is limited to linear control combined with a static saturation. Therefore it cannot be ruled out that other strategies for control could reveal workarounds. Despite these limitations, the result should serve as one starting point for further research on limitations in NCS when nonlinear effects come into play.

The paper is organized as follows. It starts by defining the notation for input-output stability analysis in section II. Section III defines the closed loop system, and the imposed conditions. The result is then derived in section IV, and numerically illustrated in section V. The paper ends with conclusions in section VI.

II. Notation

This analysis is based on the input-output stability theory presented in [20], chapter 6, which builds on [15]-[18], [23]-[25], or on passivity [6]. The reason is that the delays make the NCS infinite dimensional.

Definition 1: For all $p \in [0, \infty)$, $\mathcal{L}_p[0, \infty)$ denotes the set of all measurable functions $f(\cdot) : [0, \infty) \rightarrow R$, such that

$$\|f(\cdot)\|_p^p = \int_0^\infty |f(t)|^p dt < \infty.$$

Definition 2: The set of all measurable functions $f(\cdot) : [0, \infty) \rightarrow R$, such that their truncations

$$f_T(t) = \begin{cases} f(t), & 0 \leq t \leq T \\ 0, & t > T \end{cases} \in \mathcal{L}_p[0, \infty),$$

$\forall T$, is denoted the extension $\mathcal{L}_{pe}[0, \infty)$ of $\mathcal{L}_p[0, \infty)$.

Definition 3: The mapping $A : \mathcal{L}_{pe} \rightarrow \mathcal{L}_{pe}$ is $\mathcal{L}_p$-stable if i) $Af \in \mathcal{L}_p$ whenever $f \in \mathcal{L}_p$, and ii) there exist finite constants k and c, such that

$$\|Af\|_p \leq k\|f\|_p + c, \quad \forall f \in \mathcal{L}_p.$$

Definition 4: $\mathcal{A}$ denotes the set of generalized functions of the form

$$f(t) = \begin{cases} 0, & t < 0 \\ \sum_{i=0}^\infty f_i \delta(t - t_i) + f_a(t), & t \geq 0, \end{cases}$$

where $\delta(\cdot)$ is the unit delta distribution, t_i are non-negative constant delays, $f_a(t)$ is measurable and

$$\textstyle\sum_{i=0}^\infty |f_i| < \infty, \qquad \int_0^\infty |f_a(t)| dt < \infty.$$

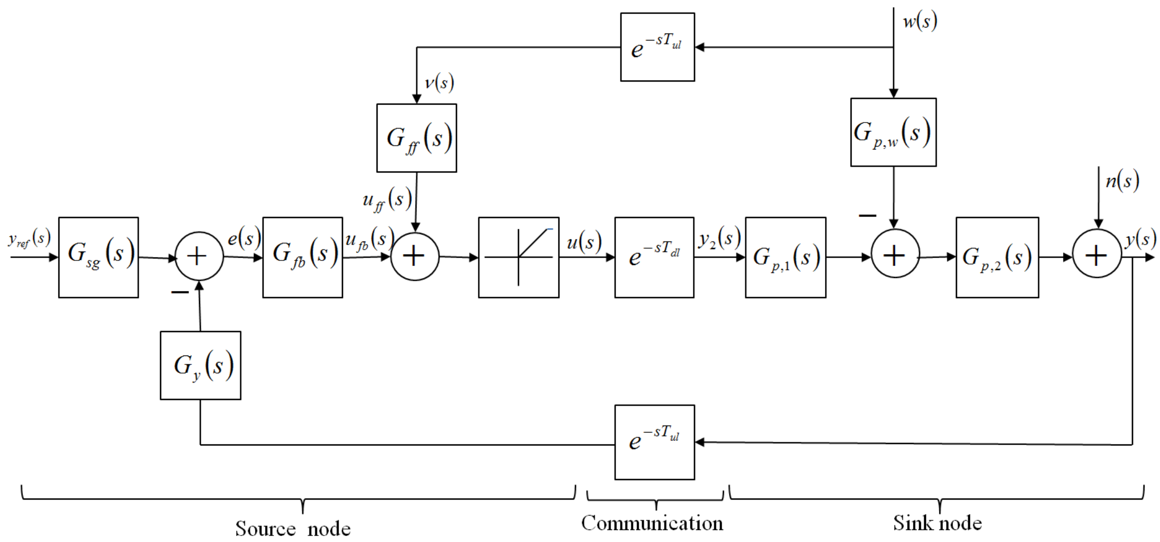

Fig. 1. Block diagram of the system model. The notation $y_2(s)$ for the signal before $G_{p,1}(s)$ is intentional and used in section IV.

Definition 5: $\hat{\mathcal{A}}$ denotes the set of all function $\hat{f} : C_+ \to C$ that are Laplace transforms of elements of $\mathcal{A}$.

Note that in case transient servo control problems are considered, the input signals appearing in the application will easily meet the integrability condition defining $\mathcal{L}_p[0, \infty)$.

III. System Model

The system model used in the paper can be summarized by Fig. 1. The controller resides in the source node and processes delayed measurements of the output $y(t - T_{ul})$, together with delayed measurements of a disturbance useful for feed forward, $w(t - T_{ul})$. The feedback part of the controller acts on a control error formed by filtered versions of the reference signal $y_{ref}(t)$ and the delayed output signal $y(t - T_{ul})$. The measurements are performed in the sink node. The control signal, typically the flow rate, is limited to be positive and below a maximum rate to model the one-directional flow. In the receiving node the dynamics is divided into one part before the disturbance $w(t)$ enters, and one part after that. Finally the measurement disturbance $n(t)$ is added to produce the output signal $y(t)$.

The analysis is based on the following structural assumptions on the blocks in the feedback loop of Fig. 1,

$$G_{p,1}(s) = G_p \frac{(s + b_{1,1})...(s + b_{1,nb_1})}{(s + a_{1,1})...(s + a_{1,na_1})} \tag{1}$$

$$G_{p,2}(s) = \frac{(s+b_{2,2})...(s+b_{2,nb_2})}{(s+a_{2,2})...(s+a_{2,na_2})} \tag{2}$$

$$G_y(s) = \frac{(s+d_1)...(s+d_{nd})}{(s+c_1)...(s+c_{nc})} \tag{3}$$

$$G_{fb}(s) = \frac{(s+f_1)...(s+f_{nf})}{(s+e_1)...(s+e_{ne})} \tag{4}$$

Any complex zeros and poles are assumed to appear as complex conjugate pairs. The assumptions A1-A16 are then introduced for the components and signals of the block diagram of Fig. 1.Here the subscript $^{(i)}$ denotes differentiation i times. The left half of the complex plane is denoted the LHP.

A1) $Re[a_{1,i}] > 0, \quad i = 1, ..., na_1, \quad Re[b_{1,i}] > 0, \quad i = 1, ..., nb_1.$

A2) $Re[a_{2,i}] > 0, \quad i = 1, ..., na_2, \quad Re[b_{2,i}] > 0, \quad i = 1, ..., nb_2.$

A3) $Re[c_i] > 0, \quad i = 1, ..., nc, \quad Re[d_i] > 0, \quad i = 1, ..., nd.$

A4) $Re[e_i] > 0, \quad i = 1, ..., ne, \quad Re[f_i] > 0, \quad i = 1, ..., nf.$

A5) $G_{ff}(s)$ is strictly proper with all poles strictly in the LHP.

A6) $G_{p,w}(s)$ is proper with all poles strictly in the LHP.

A7) $G_{sg}(s)$ is proper with all poles strictly in the LHP.

A8) $nb_1 \leq na_1$.

A9) $nb_2 \leq na_2$.

A10) $nd \leq nc$.

A11) $nf \leq ne$.

A12) At least one of $G_{p,1}(s)$, $G_{p,2}(s)$, $G_y(s)$ and $G_{fb}(s)$ is strictly proper.

A13) $y_{ref}(t)$ is constant.

A14) $w^{(na_1-nb_1)} \in \mathcal{L}_2$, $w^{(na_1-nb_1+1)} \in \mathcal{L}_2$.

A15) $n^{(na_1-nb_1+na_2-nb_2)} \in \mathcal{L}_2$, $n^{(na_1-nb_1+na_2-nb_2+1)} \in \mathcal{L}_2$.

A16) The loop gain intersects the negative real axis at least once.

The conditions A1-A4 ensure that the loop gain is asymptotically stable, as required by the Popov criterion used below. The need for this follows from the fact that the static nonlinearity is given by

$$u(t) = \begin{cases} 0, & u_{ff}(t) + u_{fb}(t) \leq 0 \\ k(u_{ff}(t) + u_{fb}(t)), & 0 < u_{ff}(t) + u_{fb}(t) < u_{max} \\ u_{max}, & u_{ff}(t) + u_{fb}(t) \geq u_{max}, \end{cases} \tag{5}$$

where u_{max} is the maximal flow rate from the source to the sink node. This means that the sector condition of the non-linearity $\Phi(\cdot)$ of (5) reads (cf. [20])

$$0 \leq \sigma\Phi(\sigma) \leq k\sigma^2. \tag{6}$$

It should be noted that (5) actually need not be linear in the its middle interval. In case it is nonlinear all results hold as long as (6) holds. The conditions A1-A4 also ensure that the inverse transfer functions are stable, i.e. minimum phase. This requirement is needed to ensure that the signals of the redrawn and equivalent block diagram of Fig. 2 meet the conditions of the corresponding Popov criterion. The remaining conditions A5-A15 also serve this purpose, the detailed reasons will become clear below. The condition A12 ensures that the loop gain tends to zero when $\omega \to \infty$, a fact that is used in the derivation of the main result of the paper. The conditions A16 is needed to make the analysis in section IV.C relevant.

IV. The Low Frequency Gain Limitation

A. The Popov Criterion

The forthcoming analysis is based on the $\mathcal{L}_2$ version of the Popov criterion valid for the block diagram of Fig. 2. The result is

Lemma 1: (Popov Criterion, [20] Theorem 6.7.63). Consider the system of Fig. 2. Assume that the inverse Laplace transform of the transfer function $\hat{g}(s)$ fulfils

$$g(\cdot) \in \mathcal{A}, \qquad \dot{g}(\cdot) \in \mathcal{A},$$

that the time invariant static nonlinearity $\Phi(\cdot)$ fulfils

$$0 \leq \sigma\Phi(\sigma) \leq k\sigma^2,$$

and that $\dot{u}_2 \in \mathcal{L}_2$. Under these conditions the system is $\mathcal{L}_2$-stable if there exist constants q, δ, such that the Popov plot

$$\omega \in [0, \infty) \to Re[\hat{g}(j\omega)] + j\omega Im[\hat{g}(j\omega)] \in C$$

lies entirely to the right of a line through $-1/k + \delta + j0$ with slope $1/q$, for some $q \geq 0$ and some $\delta > 0$.

Proof: See [20], section 6.7.

Remark 1: The geometrical condition on the Popov plot is equivalent to the following inequality

$$Re[(1 + j\omega q)\hat{g}(j\omega)] + \frac{1}{k} \geq \delta > 0, \ \forall \omega \geq 0, \text{some } q > 0. \tag{7}$$

B. Block Diagram Transformation

Lemma 2: Consider the NCS of Fig. 1 and assume that the conditions A1)-A15) hold. Then the NCS is $\mathcal{L}_2$-stable if there exist constants q, $\delta > 0$, such that the Popov plot of

$$\hat{g}(s) = e^{-s(T_{ul}+T_{dl})} G_y(s) G_{fb}(s) G_{p,1}(s) G_{p,2}(s),$$

given by

$$\omega \in [0, \infty) \to Re[\hat{g}(j\omega)] + j\omega Im[\hat{g}(j\omega)] \in C$$

lies entirely to the right of a line through $-1 + \delta + j0$ with slope $1/q$, for some $q \geq 0$ and some $\delta > 0$.

Proof: Lemma 1 is valid for the block diagram of Fig. 2, therefore the block diagram of Fig. 1 is redrawn to resemble that of Fig. 2. The downlink delay block $e^{-sT_{dl}}$ is first moved to before the static nonlinearity, noting that for the *time invariant* static nonlinearity (5) it holds that if $\alpha(t) = \Phi(\beta(t)) \Rightarrow q^{-1}\alpha(t) = \alpha(t - T) = \Phi(\beta(t - T)) = \Phi(q^{-1}\beta(t))$. Linearity and time invariance are then used to move the delay block $e^{-sT_{dl}}$ into the feedback and feed forward branches of Fig. 1. The quantities connected to the summation points after the nonlinearity, i.e. the dynamics $G_{p,w}(s)$, $G_{p,1}(s)$, $G_{p,2}(s)$, the measurement error $n(s)$, the uplink delay block $e^{-sT_{ul}}$ of the feedback path, and $G_y(s)$ are all moved clockwise to after the summation point generating the control error $e(s)$ in Fig. 1. At the same time the signals $w(s)$, $n(s)$ and $y_{ref}(s)$ are merged. Linearity motivates these transformations. It follows that

$$\begin{aligned} e(s) &= e^{-sT_{ul}} G_y(s) G_{p,1}(s) G_{p,2}(s) \\ &\cdot \Big(e^{sT_{ul}} G_{sg}(s) \left(G_y(s) G_{p,1}(s) G_{p,2}(s) \right)^{-1} y_{ref}(s) \\ &+ G_{p,1}^{-1}(s) \Big(G_{p,w}(s) w(s) - G_{p,2}^{-1}(s) n(s) \Big) - y_2(s) \Big). \end{aligned} \tag{8}$$

This means that the plant, delay dynamics and $G_y(s)$ are located *after* a redefined summation point, with $y_2(s)$ being subtracted from the re-defined reference

$$\begin{aligned} u_1(s) \equiv{} & G_{sg}(s) \left(G_y(s) G_{p,1}(s) G_{p,2}(s) \right)^{-1} e^{sT_{ul}} y_{ref}(s). \\ & + G_{p,1}^{-1}(s) \Big(G_{p,w}(s) w(s) - G_{p,2}^{-1}(s) n(s) \Big) \end{aligned} \tag{9}$$

After a multiplication with $G_{fb}(s)$ the block diagram of Fig. 1 can therefore be represented by the block diagram of Fig. 2, using (9) and

$$u_2(s) \equiv e^{-s(T_{ul}+T_{dl})} G_{ff}(s) w(s), \tag{10}$$

and with $\Phi(\cdot)$ given by the saturation of (5).

It remains to prove that the technical conditions of Lemma 1 hold. Because of A1-A4 all poles of the transfer function $\hat{g}(s)$ of Lemma 2 are in the left half plane. Using A8-A12 it follows that $\hat{g}(s)$ of Lemma 2 is strictly proper and hence $s\hat{g}(s)$ is at least proper. It then follows that $g(\cdot) \in \mathcal{A}$ and $\dot{g}(\cdot) \in \mathcal{A}$, see Definition 4.

Next it needs to be checked that $u_1 \in \mathcal{L}_2$ since this is necessary for stating $\mathcal{L}_2$ stability in Lemma 2, cf. definition 3. To see that $u_1 \in \mathcal{L}_2$, (9) is multiplied out, resulting in

$$u_1(s) = G_{sg}(s) \left(G_y(s)G_{p,1}(s)G_{p,2}(s)\right)^{-1} e^{sT_{ul}} y_{ref}(s)$$
$$+G_{p,1}^{-1}(s)G_{p,w}(s)w(s) - G_{p,1}^{-1}(s)G_{p,2}^{-1}(s)n(s). \quad (11)$$

The first term of (11) is then considered. The condition A13 implies that the factor $e^{sT_{ul}}$ has negligible effect and can be removed. The combined transfer function of the first term is asymptotically stable by A1-A3 and A7. By A13 $s^i y_{ref}(s)$ vanishes in the time domain for $i > 0$. This means that any differentiators that may affect $y_{ref}(t)$ due to the fact that $\left(G_y(s)G_{p,1}(s)G_{p,2}(s)\right)^{-1}$ may not be proper can be disregarded. Condition A7 then shows that the first term of (11) $\in \mathcal{L}_2$. The second term is asymptotically stable because of A1 and A6. By incorporation of differentiators that may be due to $G_{p,1}^{-1}(s)$ not being proper into direct products with $w(s)$, and using A14, shows that also the second term of (11), $G_{p,1}^{-1}(s)G_{p,2}^{-1}(s)n(s) \in \mathcal{L}_2$. The third term can be treated in the same manner. Hence asymptotic stability follows from A1 and A2, while A15 shows that also the third term of (11) $\in \mathcal{L}_2$. Since $\mathcal{L}_2$ is a linear space, it follows that the sum of the three terms of (11) $\in \mathcal{L}_2$. Hence $u_1 \in \mathcal{L}_2$.

The input signal $u_2(t)$ of (10) is obtained as the convolution $u_2 = h \star w$, where h according to (10) and A5 is the inverse Laplace transform of the strictly proper transfer function

$$H(s) = e^{-s(T_{ul}+T_{dl})} G_{ff}(s). \quad (12)$$

By A5, $H(s)$ has all poles in the LHP, which shows that $h(.) \in \mathcal{A}$. By A14 and [20], Theorem 6.5.37, it follows that $u_2 \in \mathcal{L}_2$. Since $\dot{u}_2$ is obtained by convolution of w with the inverse Laplace transform of the at least proper transfer function $sH(s) \in \hat{\mathcal{A}}$, also $\dot{u}_2 \in \mathcal{L}_2$. Observing that the static nonlinearity $\Phi(.)$ obeys (6) completes the proof of Lemma 2.

C. Main Result

With Lemma 2 proved, the main result can now be derived from the Popov criterion. Referring to Fig. 3, the Popov criterion states that the closed loop system is $\mathcal{L}_2$ stable when the Popov plot is to the right of any line passing through the point $-1/k + \delta + j0$ with any positive slope $1/q$, where k is the

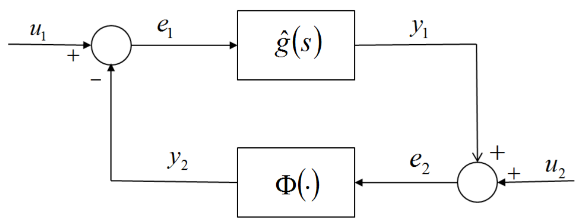

Fig. 2. Block diagram for which the Popov criterion is proved.

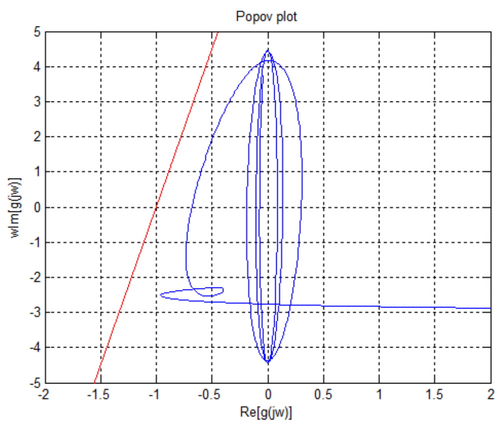

Fig. 3. Illustration of the Popov criterion. The Popov plot (blue) is to be right of the Popov line (red), in order for the closed loop system to be $\mathcal{L}_2$ stable. The Popov plot represents the data flow control system of [21].

maximum sector slope of (5). Since the interest here is to derive low frequency limitations of $\hat{g}(s)$ of Lemma 2, it is desirable to study situations where the Popov criterion does not hold. Therefore the value of the Popov inequality is addressed on the negative real axis of the complex plane.

As a first step, the condition for the Popov plot of Lemma 2 to intersect the negative real axis is quantified to be

$$\arg\left(Re[\hat{g}(j\omega)] + j\omega Im[\hat{g}(j\omega)]\right) = -\pi. \tag{13}$$

Since $\omega = 0$ represents the static gain, this solution can be excluded and it follows that (13) is equivalent to

$$Re[\hat{g}(j\omega)] < 0, \quad Im[\hat{g}(j\omega)] = 0. \tag{14}$$

Defining

$$\hat{g}(s) = e^{-sT}\hat{g}_o(s), \tag{15}$$

where $T = T_{ul} + T_{dl}$, the condition (14) implies that

$$|\hat{g}_o(j\omega)| \sin\left(\arg(\hat{g}_o(j\omega) - \omega T)\right) = 0, \tag{16}$$

$$|\hat{g}_o(j\omega)| \cos\left(\arg(\hat{g}_o(j\omega) - \omega T)\right) < 0. \tag{17}$$

Since A1-A4 hold, $|\hat{g}_o(j\omega)| \neq 0$, and it follows from (16) and (17) that

$$\arg(\hat{g}_o(j\omega_l) - \omega_l T = \pm l\pi, \quad l = 1, ..., n_0, ..., \tag{18}$$

where l is odd and n_0 is an odd number that is guaranteed to exist due to A16. Note that the evaluation of transfer function values on the negative real axis are hence tied to ω_l.

The conditions A1-A4 imply that $\arg(\hat{g}_0(j\omega)) < \varphi_0 < \infty$. Furthermore, by A12 it follows that $|\hat{g}_o(j\omega)| \to 0$ as $\omega \to \infty$. Therefore there is a *largest* n_0 of (18) and a maximal $\omega_m(n_0)$, for which $Re[\hat{g}(j\omega_m(n_0))] \leq -1/k$ i.e for which the Popov criterion may not hold, i.e.

$$\omega_l = \frac{1}{T}\left(\arg(\hat{g}_o(j\omega_l)) \pm l\pi\right), \quad l = 1, ...n_0. \tag{19}$$

Equation (19) applied to $\omega_m(n_0)$ then gives

$$\begin{aligned} \lim_{T\to\infty} \omega_m(n_0) &= \lim_{T\to\infty} \omega_{n_0} \\ &= \lim_{T\to\infty} \frac{1}{T}\left(\varphi_0 \pm n_0\pi\right) = 0. \end{aligned} \tag{20}$$

To proceed the following Lemma is useful

Lemma 3: Assume that

$$h(s) = K\frac{(s+z_1)...(s+z_m)}{(s+p_1)...(s+p_n)},$$

where complex zeros and poles appear as complex conjugate pairs. Then

$$\begin{aligned} \lim_{\omega\to 0} Re[h(j\omega)] &= K\frac{z_1...z_m}{p_1...p_n} \\ \lim_{\omega\to 0} Im[h(j\omega)] &= 0. \end{aligned}$$

Proof: A straightforward computation gives

$$Re[h(j\omega)] = K\frac{z_1...z_m\bar{p_1}...\bar{p_n} + \omega^2(...}{(\omega^2+|p_1|^2)...(\omega^2+|p_n|^2)} \tag{21}$$

$$Im[h(j\omega)] = K\frac{\omega(...}{(\omega^2+|p_1|^2)...(\omega^2+|p_n|^2)}, \tag{22}$$

where $\bar{p_i}$ denotes the complex conjugate of p_i. Letting $\omega \to 0$ gives the result of Lemma 3.

Next, using the conclusion (20) and applying Lemma 3 to $\hat{g}_o(j\omega_l)$ where $\omega_l \leq \omega_m(n_0)$, results in

$$\lim_{T\to\infty} \arg(\hat{g}_o(j\omega_l)) = 0, \quad l = 1, ..., n_0. \tag{23}$$

Therefore it follows from (18) and (19) that

$$\lim_{T\to\infty} \omega_l T = \pm l\pi, \quad l = 1, ..., n_0. \tag{24}$$

Exploiting (15) and Lemma 3 implies that

$$\lim_{T\to\infty} Re[\hat{g}(j\omega_l)] = \lim_{T\to\infty} Re[e^{-j\omega_l T}\hat{g}_o(j\omega_l)]$$

$$= \lim_{T\to\infty} (\cos(\omega_l T) Re[\hat{g}_o(j\omega_l)] - \sin(\omega_l T) Im[\hat{g}_o(j\omega_l)])$$

$$= -\lim_{T\to\infty} Re[\hat{g}_o(j\omega_l)], \quad l = 1, ...n_0. \tag{25}$$

By (19) and (20) the last equation can be written

$$\lim_{T\to\infty} Re[\hat{g}(j\omega_l)] = -\lim_{\omega_l\to 0} Re[\hat{g}_o(j\omega_l)], \quad l = 1, ..., n_0. \tag{26}$$

By Lemma 2, the Popov criterion hence cannot hold in the high delay limit if

$$\lim_{T\to\infty} Re[\hat{g}(j\omega_l)] = -\lim_{\omega_l\to 0} Re[\hat{g}_o(j\omega_l)] \leq -1/k, \quad l = 1, ..., n_0, \tag{27}$$

i.e if

$$\lim_{\omega_l\to 0} Re[\hat{g}_o(j\omega_l)] > 1/k, \quad l = 1, ..., n_0, \tag{28}$$

By Lemma 2, (1)-(4) and Lemma 3, the limiting value of (28) can be expressed as

$$\lim_{\omega_l\to 0} Re[\hat{g}_o(j\omega_l)] \tag{29}$$

$$= G_p \frac{b_{1,1}...b_{1,nb_1} b_{2,1}...b_{2,nb_2} d_1...d_{nd} f_1...f_{nf}}{a_{1,1}...a_{1,na_1} a_{2,1}...a_{2,na_2} c_1...c_{nc} e_1...e_{ne}}, \quad l = 1, ..., n_0. \tag{30}$$

This proves the following main result:

Theorem 1: Consider the flow control system given by Fig. 1 and (1)-(6). Assume that the conditions A1-A16 hold. Then the Popov condition does not imply $\mathcal{L}_2$-stability in case

$$\frac{b_{1,1}...b_{1,nb_1} b_{2,1}...b_{2,nb_2} d_1...d_{nd} f_1...f_{nf}}{a_{1,1}...a_{1,na_1} a_{2,1}...a_{2,na_2} c_1...c_{nc} e_1...e_{ne}} \geq \frac{1}{kG_p}.$$

Remark 2: It follows from the derivation and A1-A4 that the result hold also for pairs of complex conjugate zeros and poles.

V. Discussion

Given the plant transfer functions and the nominal delays, the feedback controller design needs to determine the transfer functions $G_y(s)$ and $G_{fb}(s)$ so that the controller meets the specification, cf. [21]. Here, one important specification point is that of $\mathcal{L}_2$-stability. Therefore, in case the delays are large, the constraint of Theorem 1 also comes into play. In order to keep the right hand side low enough so that the constraint of Theorem 1 does not affect $\mathcal{L}_2$-stability, it is evident that the controller poles cannot be placed arbitrarily close to zero. Therefore, the result of Theorem 1 represents a limitation of the low frequency gain of the loop gain, when subject to saturations in the feedback path. One should keep in mind though that the Popov criterion is not necessary for $\mathcal{L}_2$-stability, therefore further studies are needed to see wether the result of Theorem 1 could be made more stringent, cf. e.g. [7].

Theorem 1 is easily interpreted since (30) is related to the static loop gain. Theorem 1 could then be seen as a static interpretation of the Nyquist criterion [20]. However, this interpretation is misleading since the Nyquist criterion can hold even if there are intersections with the negative real axis to the left of $-1/k + 0j$. Furthermore, the nonlinear analysis performed in the paper needs to introduce conditions that are not needed for the Nyquist criterion to hold. The result of the paper therefore goes beyond the Nyquist criterion.

The constraint that controller poles cannot be placed arbitrarily close to 0 means that the suppression of a static disturbance may be compromised. A remedy to that situation would be to re-build the plant by moving the asymptotically poles away from the imaginary axis, or by reducing the static gain of the plant. Both these actions do hovever represent a flow capacity reduction. To see that this is actually the case, note that a modification of a pole $p \to p+\varepsilon$ can be seen as addition of a feedback loop, feeding back the general quantity that is flow controlled to an earlier stage in the chain. To see this mathematically note that for a first order unity gain system

$$\frac{dy}{dt} = -py + (1-p)u, \tag{31}$$

the pole transfer replaces (31) by

$$\frac{dy}{dt} = -py - \varepsilon y + (1-p-\varepsilon)u. \tag{32}$$

In the data flow control example of [21] this effect corresponds to a reduced data transfer capacity.

VI. Numerical Results

An example from [21] is first used to illustrate the obtained result.

Example 1: The data flow controller of [21] is a special case of the controller treated in the present paper. In [21] the plant is given by a leaky queue where

$$G_{p,1} = 1 \tag{33}$$

$$G_{p,2} = \frac{1}{s+\varepsilon} \tag{34}$$

The feedback controller is a lead-lag controller, hence

$$G_y = 1 \tag{35}$$

$$G_{fb} = K\frac{s+a}{s+aK\Delta_{we}}N\frac{s+b}{s+Nb}. \tag{36}$$

To tune the controller the algorithm proposed in [21] was used in order to precompute the $\mathcal{L}_2$ stability region of the non-linear system as a function of a number of parameters, including ε and the dominating "integration pole" $aK\Delta_{we}$. The latter pole is expressed in terms of a relative requirement Δ_{we} on the regulation of static disturbances. The important thing here is that the smaller the value of Δ_{we}, the closer the controller is to integral control and the better the regulation of static disturbances can be expected to become. The delay T used for tuning was $0.30\ s$ and the slope of the saturation was $k = 1$.

Fig. 4 illustrates one subset of the $\mathcal{L}_2$-stability region. The figure plots the actual tolerable delay of the system, as a function of the pole related quantities ε and Δ_{we}. Regions where the actual tolerable delays become zero represent parameter combinations where the Popov criterion is not met. As can be seen in the plot, it is not possible to make ε and Δ_{we} small at the same time, exactly as predicted by Theorem 1. It can also be noted that the result of Theorem 1 is conservative, since it considers the case where $T \to \infty$. This can be seen numerically as follows. The left hand side of Theorem 1 becomes $1/(NK\varepsilon\Delta_{we})$ while the right hand side becomes $1/(kKN)$. The condition of Theorem 1 therefore becomes

$$\varepsilon\Delta_{we} < 1. \tag{37}$$

Note that (37) defines the condition for the Popov condition not to hold. Equation (37) is clearly more restrictive than what is indicated by Fig. 4.

Example 2: To further illustrate the limiting character of Theorem 1, a simple example is useful. A system consisting of the leaky reservoir

$$G_{p,2} = \frac{1}{s+\varepsilon}, \tag{38}$$

was therefore considered. The plant was assumed to be controlled by the proportional controller

$$G_{fb} = K. \tag{39}$$

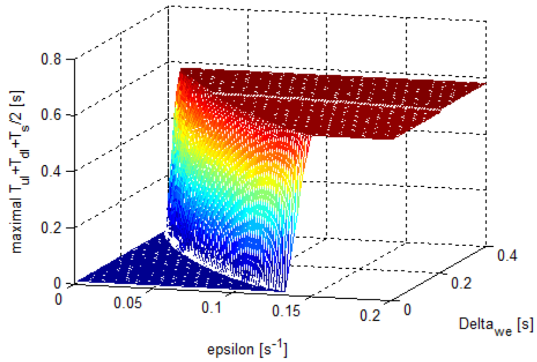

Fig. 4. A subspace of the pre-computed $\mathcal{L}_2$-stability region. The figure plots the actual tolerable delay of the system, as a function of the pole related quantities ε and Δ_{we}. The actual delay differs from the one of 0.30 s used as the nominal system assumption.

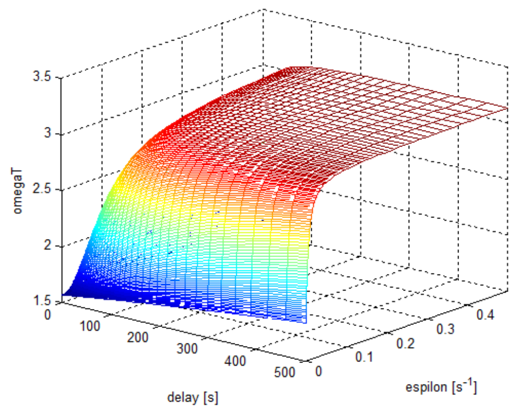

Fig. 5. An illustration of the convergence speed of $\omega_l T$.

The sector condition parameter was $k = 1$. In this example the pole $s = -\varepsilon$ and the proportinal gain K therefore controls the low frequency loop gain. To illustrate the speed of convergence as a function of the delay $T = T_{ul} + T_{dl}$, towards the region where Theorem 1 holds, (19) was first solved for ω_l, for $\varepsilon = 0.001, ..., 0.5$ s^{-1} and $T = 0.1, ..., 500$ s. After that $\omega_l T$ was computed. The result is depicted in Fig. 5, where it can be seen that $\lim_{T\to\infty} \omega_l T = \pi$ as predicted by (24). As can be seen T needs to become quite large before the asymptotic result applies. The convergence is particularly slow when the pole of the system is close to the imaginary axis. This provides an explanation of the conservativeness of Theorem 1 that was appearent in Example 1.

For each ε and T, the critical proportional gain K_{max} where the Popov criterion does no longer hold

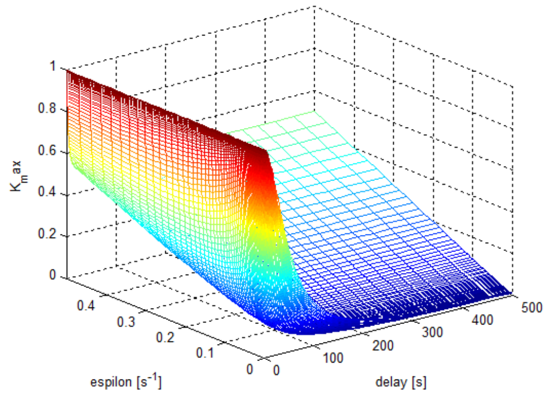

Fig. 6. The maximum proportional gain K_{max} of the controller as a function of the delay T and the pole parameter ε. The trade off between K_{max} and ε is obvious - both cannot generate high low frequency gains.

was then computed. This gain limit follows from

$$Re[K_{max}\frac{1}{j\omega_l+\varepsilon}e^{-j\omega_l T}] = -1, \tag{40}$$

which leads to

$$K_{max} = -\frac{\sqrt{\omega_l^2+\varepsilon^2}}{\varepsilon\cos(\omega_l T)-\omega_l\sin(\omega_l T)} \tag{41}$$

The result is depicted in Fig. 6. Again it can be seen that the delay needs to become quite large before the asymptotic region where Theorem 1 holds is reached. The trade off between the low frequeny gain contribution of the pole parameter ε and the controller gain K_{max} is also clear.

As a general conclusion, it could be stated that that the low frequency loop gain needs to be carefully considered in netqworked flow control systems with significant delays.

VII. Conclusions

The paper studied a general flow control problem, where large delays are a major effect. The problem is nonlinear since it is assumed that the flow is one-directional. The Popov criterion was applied to derive a limitation on the low frequency gain of the loop gain. The limiting quantity turns out to be related to the static feedback loop gain. More precicely the paper proves that for large delays, $\mathcal{L}_2$-stability does not follow from the Popov criterion when the quotient of the products of all zeros and the product of all poles is larger than $1/kG_p$, where k is the slope of the saturation and G_p is the gain constant of the loop gain. In case the plant models a leaky reservoir, the conclusion is that the amount of low frequency gain of the controller cannot be arbitrarily high at the same time as the amount of leakage of the reservoir is arbitrarily low. In communications this means that an increased requirement to regulate static errors of the reservoir needs to be accompanied by a reduced flow capacity.

The obtained result is by no means final, since the Popov criterion is not necessary for $\mathcal{L}_2$-stability. The obtained condition was also shown to be conservative, this being believed to be an effect of the study of the limiting case when the delays tend to infinity. Other controller structures than linear ones may also circumvent the limitation, at least partly. Therefore, further studies are suggested to possibly strengthen the result and the knowledge about fundamental limitations for the one-directional flow control problem. Despite of the above the derived result was able to qualitatively explain effects observed in a previous data flow control application.

Acknowledgment

The author would like to thank Professor Thomas Schön and Professor Bengt Carlsson for comments and suggestion on the paper.

References

[1] K. J. Åström, ”Reglerteknik - Olinjära system”, Lund Institute of Technology, Lund, 1968. (in Swedish)

[2] J. Baillieul, ”Feedback designs for controlling device arrays with communication channel bandwidth constraints”, in Proc. ARO Workshop Smart Structures, Pennsylvania State Univ., Aug. 16-18, 1999.

[3] J. Baillieul, ”Feedback coding for information based control: Operating near the data-rate limit”, *in Proc. CDC*, Las Vegas, NV, Dec. 2002, pp. 3229-3236.

[4] J. Baillieul and P. J. Antsaklis, ”Control and communication challenges in networked real-time systems”, *Proc. IEEE*, vol. 95, no. 1, pp. 9-28, 2007.

[5] R. W. Brocket and D. Liberzon, ”Quantized feedback stabilization of linear systems”, *IEEE Trans. Automat. Contr.*, vol. 45, no. 7, pp. 1279-1289, 2000.

[6] B. Brogliato, R. Lozano, B. Maschke and O. Egeland, *Dissipative Systems Analysis and Control - Theory and Applications, 2:nd ed.*. Springer: London, UK, 2007.

[7] R. F. Curtain, H. Logemann, and O. Staffans, ”Stability results of Popov-type for infinite dimensional systems with application to integral control”, *Proc. London Math. Society*, no. 3, vol. 86, pp. 779-816, 2003.

[8] D. d'Alessandro, M. Dahleh, and I. Mezic, ”Control of mixing in fluid flow: a maximum entropy approach”, *IEEE Trans. Automat. Contr.*, vol. 44, no. 10, pp. 1852-1863, 1999.

[9] S. V. Hanly and D.-N. Tse, ”Power control and capacity of spread spectrum wireless networks”, *Automatica*, vol. 35, pp. 1987-2012, 1999.

[10] F.-S. Ho and P. Ioannou, ”Traffic flow modeling and control using artificial neural networks”, *IEEE Control Systems*, vol. 16, no. 5, pp. 16-26, 1996.

[11] G. N. Nair and R. J. Evans, ”Stabilization with data-rate-limited feedback: tightest attainable bound”, *Syst. Contr. Lett.*”, vol. 41, no. 1, pp. 49-56, 2000.

[12] K. S. Narendra and J. H. Taylor, *Frequency Domain Criteria for Absolute Stability.* New York, NY: Academic Press, 1973.

[13] D. E. Quevedo and T. Wigren, ”Design of embeeded filters for inner-loop power control in wireless CDMA communication systems”, *Asian J. Contr.*, vol. 14, no. 4, pp. 891-900, 2012.

[14] V. M. Popov, ”Nouveaux criteriums de stabilité pour les systemés automatiques non-linèaries”, *Revue d´Electrotechnique et d´Energetique, Acad. de la Rep. Populaire Romaine*, vol. 5, no. 1, 1960.

[15] I. W. Sandberg, ”On the $\mathcal{L}_2$-boundedness of solutions of nonlinear functional equations”, *Bell Sys. Tech. J.*, vol. 43, pp. 1581-1599, 1964.

[16] I. W. Sandberg, ”A frequency-domain condition for the stability of feedback systems containing a single time-varying element”, *Bell Sys. Tech. J.*, vol. 43, pp. 1601-1608, 1964.

[17] I. W. Sandberg, ”Some results on the theory of physical systems goverened by nonlinear functional equations”, *Bell Sys. Tech. J.*, vol. 44, no. 5, 1965.

[18] I. W. Sandberg, ”On generalizations and extensions of the Popov criterion”, *IEEE Trans. Circuit Theory*, vol. CT-13, no. 1, 1966.

[19] N. Vandelli, D. Wroblewski, M. Velonis, T. Bifano, ”Development of a MEMS microvalve array for fluid flow control”, *J. of Microelectromechanical Systems*, vol. 7, no. 4, pp. 395-403, 1998.

[20] M. Vidyasagar, *Nonlinear Systems Analysis*. Englewood Cliffs, NJ: Prentice-Hall, 1978.

[21] T. Wigren, ”Robust $\mathcal{L}_2$ stable networked control of wireless packet queues over delayed internet connections”, *submitted*, March, 2014.

[22] T. Wigren and L. Brus, ”Reduction of amplitude dependent gain variations in control of non-linear Wiener type systems”, *7th IFAC Symposium on Nonlinear Control Systems, NOLCOS 2007*, Pretoria, South Africa, pp. 730-735, August 22-24, 2007.

[23] G. Zames, ”Functional analysis applied to nonlinear feedback systems”, *IEEE Trans. Circuit Theory*, vol. CT-10, no. 3, 1963.

[24] G. Zames, ”On the input-output stability of time-varying nonlinear feedback systems, Part1: Conditions derived using concepts of loop-gain, conicity and positivity”, *IEEE Trans. Automat. Contr.*, vol. AC-11, pp. 228-238, 1966.

[25] G. Zames, ”On the input-output stability of time-varying nonlinear feedback systems, Part2: Conditions involving circles in the frequency plane and sector nonlinearities”, *IEEE Trans. Automat. Contr.*, vol. AC-11, pp. 465-476, 1966.