Schur Complement Matrix And Its (Elementwise) Approximation: A Spectral Analysis Based On GLT Sequences

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Abstract. Using the notion of the so-called *spectral symbol* in the Generalized Locally Toeplitz (GLT) setting, we derive the GLT symbol of the sequence of matrices $\{A_n\}$ approximating the elasticity equations. Further, as the GLT class defines an algebra of matrix sequences and Schur complements are obtained via elementary algebraic operation on the blocks of A_n, we derive the symbol $f^{\mathcal{S}}$ of the associated sequences of Schur complements $\{S_n\}$ and that of its element-wise approximation.

1 Introduction and preliminaries

In this paper, the notions of (block)-Toeplitz matrices and related notations are used in their broadly accepted conventional meaning. We refer to [6] for details and include only some definitions for clarity and self-consistency of this paper.

Definition 1. [Generating function of Toeplitz sequences] *Denote by $f(\theta_1, \cdots, \theta_d)$ a d-variate complex-valued integrable function, defined over the domain $Q^d = [-\pi, \pi]^d, d \geq 1$. Denote by f_k the Fourier coefficients of f,*

$$f_k = \frac{1}{m\{Q^d\}} \int_{Q^d} f(\theta) e^{-i\,(k,\theta)}\, d\theta,\ k = (k_1, \cdots, k_d) \in \mathbb{Z}^d,\ i^2 = -1,$$

where $(k, \theta) = \sum_{j=1}^{d} k_j \theta_j$, $n = (n_1, \cdots, n_d)$, and $N(n) = n_1 \cdots n_d$. Following the multi-index notation in [18], with each f we can associate a sequence of Toeplitz matrices $\{T_n\}$, $T_n = \{f_{k-\ell}\}_{k,\ell=\mathbf{e}^T}^{n} \in \mathbb{C}^{N(n)\times N(n)}$, $\mathbf{e} = [1, 1, \cdots, 1] \in \mathbb{N}^d$.

The function f is referred to as the generating function (or the symbol of) T_n. Using a more compact notation, we say that the function f is the generating function of the whole sequence $\{T_n\}$ and we write $T_n = T_n(f)$.

Definition 2. [Spectral symbol of a matrix] *Given a sequence $\{A_n\}$, A_n of size n, we say that g is the (spectral) symbol of $\{A_n\}$ if all the eigenvalues of A_n are given, up to a small error and for large n, by an evaluation of g over a equispaced grid in the definition set of g.*

A notewhorty example is the Toepliz case where the spectral symbol of $\{T_n(f)\}$ is exactly the generating function f: in that case, for $d = 1$, the possible grid is given by $\{x_j^{(n)}\}$, $x_j^{(n)} = -\pi + \frac{2\pi j}{n}$, $j = 1, \ldots, n$.

1.1 Toeplitz matrices in the context of discrete PDEs

Consider a differential boundary value problem of the general form $\mathcal{L}u = f$ on Ω, complemented with proper boundary conditions, where $\mathcal{L}$ is a given differential operator and $\Omega \subset \mathbb{R}^d$, $d \geq 1$ is some open, bounded, connected domain.

The techniques to approximate partial differential equations (PDEs) by local methods such as the Finite Element method (FEM) ([4]) lead to sequences of matrices that admit a Toeplitz structure. When discretizing this problem for a sequence of discretization parameters h_n we obtain a corresponding sequence of matrices $\{A_n\}$ of size n that grows to infinity as the approximation error tends to zero. The study of the spectrum of A_n for fixed dimension and its behavior in an asymptotic sense is often a prerequisite for designing efficient solvers and preconditioners.

Most often, the theoretical results in the related literature (as well as all derivations in this paper) are done for PDEs with constant coefficients, square domains and uniform grids. At first glance, the limitations on square domains, uniform meshes, and constant coefficients seem quite strong. However, substantial steps for overcoming these limiting factors to variable coefficients, domains of arbitrary shape, nonequidistand discretization meshes, and preconditioning, have been done by Tilli [17] and by the third author [15,16]. There, the definition of Generalized Locally Toeplitz (GLT) sequences is introduced and characterized as follows.

GLT1 Each GLT sequence has a symbol (f).

GLT2 The set of GLT sequences form a $*$-algebra that is close under linear combinations, conjugation, products, inversion. Hence, the sequence obtained via algebraic operations on a finite set of GLT sequences is still a GLT sequence and its symbol is obtained by the same algebraic manipulations on the corresponding symbols of the input GLT sequences.

GLT3 Every Toeplitz sequence generated by a L^1 function f is a GLT sequence and its symbol is f, possessing the properties from **GLT1**.

GLT4 The approximation of PDEs with non-constant coefficients, general domains, nonuniform gridding by local methods (FDM, FEM, etc), under very mild assumptions leads also to GLT sequences (see [17,15,16,3,8].

The paper is organized as follows. Section 2 introduces the target problem, its discrete formulation and a preconditioner of interest. In Section 3, we use the GLT machinery to derive the corresponding symbols of the arising matrices, the exact Schur complement and its approximation.

2 Target problem, preconditioning

We simulate the so-called Glacial Isostatic Adjustment (GIA) model. It describes the response of the solid Earth to redistribution of mass due to alternating glaciation and deglaciation periods and is characterized by the coupled system

$$-\nabla \cdot (2\mu\varepsilon(\mathbf{u})) - \nabla(\mathbf{u}\cdot\mathbf{b}) + (\nabla\cdot\mathbf{u})\mathbf{c} - \mu\nabla p = \mathbf{f} \quad \text{in } \Omega, \tag{1a}$$

$$\mu\nabla\cdot\mathbf{u} - \frac{\mu^2}{\lambda}p = 0 \quad \text{in } \Omega, \tag{1b}$$

with $\mathbf{u}$ - the displacement vector, $\varepsilon(\mathbf{u}) = \frac{1}{2}\left(\nabla\mathbf{u} + \nabla\mathbf{u}^T\right)$, λ and μ - the Lamé (material) coefficients. It is assumed that $\mathbf{b} = \{b_i\},\ \mathbf{c} = \{c_i\},\ i = 1,2$ are some given vectors, for simplicity with constant coefficients. We note that the Lamé coefficients μ and λ depend on the material properties and can vary through the domain. Remarkably, the GLT machinery works also in presence of variable coefficients as already mentioned in **GLT4** and all the results, derived here, hold for variable problem parameters. System (1) is first formulated in variational terms and discretized with a stable pair of finite element spaces that satisfy the Ladyzhenskaya-Babuška-Brezzi (LBB) stability condition. We consider below the so-called Modified Taylor-Hood elements (Q1isoQ1, cf. [4]). The target geometry of the problem is rectangular, therefore a discretization with a square or a rectangular mesh is the natural choice. We use square grid and a lexicographical ordering of the node points. The variational setting and the discretization of (1) lead to the algebraic system of equations to be solved,

$$\mathcal{A}\begin{bmatrix}\mathbf{u}_h\\ \mathbf{p}_h\end{bmatrix} = \begin{bmatrix}\mathbf{f}\\ \mathbf{g}\end{bmatrix} \text{ where } \mathcal{A} = \begin{bmatrix}K & B^T\\ B & -\rho M\end{bmatrix} = \begin{bmatrix}K_{11} & K_{12} & B_1^T\\ K_{21} & K_{22} & B_2^T\\ B_1 & B_2 & -\rho M\end{bmatrix} \begin{matrix}\}\text{displ. in } x\\ \}\text{displ. in } y\,.\\ \}\text{pressure}\end{matrix} \tag{2}$$

Here M is the pressure mass matrix; $\rho = \frac{\mu^2}{\lambda} \neq 0$ for compressible materials, $\rho = 0$ for purely incompressible materials and $\rho \to 0$ in the nearly incompressible case. The block K is symmetric and positive definite when $\mathbf{b} = \mathbf{c} = \mathbf{0}$, otherwise it is nonsymmetric. The blocks B and B^T correspond to discrete divergence and gradient operators, correspondingly. Imposing separate displacement ordering (SDO) for $\mathbf{u}$, i.e., ordering first the displacements in x-direction and then the displacements in y-direction, we induce a two-by-two block structure of the block K and on B as $B = \begin{bmatrix}B_1 & B_2\end{bmatrix}$. The system matrix is depicted in (2), right.

To solve systems with $\mathcal{A}$ we consider preconditioned Krylov subspace iterative solution methods for general matrices, that are suitable for variable preconditioning schemes. We consider a preconditioner $\mathcal{B} = \begin{bmatrix}A_{11} & 0\\ A_{21} & S\end{bmatrix}$, known to be very efficient, provided that S is a high quality approximation of the exact Schur complement $S_{\mathcal{A}}$ of $\mathcal{A}$, $S_{\mathcal{A}} = A_{22} - A_{21}A_{11}^{-1}A_{12}$, cf. [2] and systems with A_{11} are solved accurately enough.

Various studies have shown (cf. [11,13,1,12,10]) that one particular approximation of $S_{\mathcal{A}}$, obtainable in the finite element context, is very efficient for the target problem, namely, the so-called element-wise Schur complement. To briefly describe it, we assume that the spatial discretization is done by the FEM method on some mesh with characteristic mesh-size h, denoted by $\mathcal{T}_h = \{\tau_\ell^e\}, \ell = 1, \cdots, L$, where τ_ℓ^e denote the individual elements (triangles, quadrilaterals, bricks etc.) and L is the number of the finite (macro-)elements in the discretization mesh.

It has been observed that the matrix $\mathcal{A}$ can be assembled based on local matrices that have the same structure as $\mathcal{A}$, namely, $\mathcal{A} = \sum\limits_{\ell=1}^{L} {R^{(\ell)}}^T A^{(\ell)} R^{(\ell)}$, $\mathcal{A} \in \mathbb{R}^{N\times N}$, $A^{(\ell)} \in \mathbb{R}^{n\times n}$, where

$$\mathcal{A}^{(\ell)} = \begin{bmatrix} A_{11}^{(\ell)} & A_{12}^{(\ell)} \\ A_{21}^{(\ell)} & A_{22}^{(\ell)} \end{bmatrix} \begin{matrix} \}n_1 \\ \}n_2 \end{matrix}, \qquad S = \sum_{\ell=1}^{L} {R_2^{(\ell)}}^T S^{(\ell)} R_2^{(\ell)}. \tag{3}$$

Here $n = n_1 + n_2$, $\ell = 1, \cdots, L$. The matrices $R^{(\ell)} \in \mathbb{R}^{n\times N}$ are the standard Boolean matrices which provide the local-to-global correspondence of the numbering of the degrees of freedom.

Based on (3) (left) we can compute the local Schur complements exactly and assemble those into a global matrix that is then used as an approximation of S ((3) (right)), where $S^{(\ell)} = A_{22}^{(\ell)} - A_{21}^{(\ell)} {A_{11}^{(\ell)}}^{-1} A_{12}^{(\ell)}$ and $R_2^{(\ell)}$ are the parts of $R^{(\ell)}$ corresponding to the degrees of freedom in A_{22}. The matrix S in (3) (right) is referred to as the element-wise Schur complement approximation. Without loss of generality we assume that all $A_{11}^{(\ell)}$ are invertible. Otherwise we add a diagonal perturbation of order h^2, where h is the characteristic discretization parameter.

For coupled systems of equations of the form (2) that are discretized with mixed finite elements, the macroelement is tightly related to the choice of the stable finite element pair of spaces we use. For the Q1isoQ1 case we have two meshes, based on one consecutive regular refinement, characterized by a mesh size H and $h = H/2$. Using Linear Algebra tools it is possible to explain the experimentally observed high qualities of the element-wise Schur complement for the case when $\mathcal{A}$ is symmetric and positive definite as well as when it is symmetric indefinite and A_{11} is positive semi-definite. Those tools and the available results for Schur complements are not applicable for both definite or indefinite nonsymmetric matrices. Therefore, to get a better insight in the above, we apply the GLT framework.

3 The symbols of $\mathcal{A}$, $\mathcal{S}_{\mathcal{A}}$ and $\mathcal{S}$

The matrix $\mathcal{A}$ in (2) can be seen as a generalized block Toeplitz matrix. Note that here we deal with matrices which are Toeplitz up to low rank corrections E_n, i.e., these can be written as $T_n(f) + E_n$ for some function f, where E_n is a low rank perturbation matrix. If the matrices are unilevel then $\text{rank}(E_n)$ is bounded by a constant independent of n. Therefore by **GLT2**, the whole sequence $\{T_n(f) + E_n\}$ is a GLT sequence with the same symbol as $\{T_n(f)\}$. Hence, again by **GLT2**, we deduce that the symbol of $\{T_n(f) + E_n\}$ is the generating function of Toeplitz part i.e. the function f.

We next show the related symbols for the blocks and for the whole matrix in (2). Under the lexicographical ordering, all matrix blocks can be seen as stencil-based. All stencils are and the detailed derivation of the symbols can be found in [6].

The mass matrix M is block-tridiagonal and each block has a tridiagonal structure. The block-symbol of M, $f^M(\theta_1,\theta_2)$ is $f^M(\theta_1,\theta_2) = 4(2+\cos(\theta_1))(2+\cos(\theta_2))$, where θ_1 and θ_2 are generic angles between 0 and π.

Symbols of $\boldsymbol{K}$, $\boldsymbol{B}$ and the Schur complement for Q1isoQ1 The symbols for K_{11}, K_{22} and K_{12} read as follows:

$$\begin{aligned} f^{K_{11}}(\theta_1,\theta_2) &= 4 - 2\cos(\theta_1)(1+\cos(\theta_2)),\\ f^{K_{22}}(\theta_1,\theta_2) &= 4 - 2(1+\cos(\theta_1))\cos(\theta_2),\\ f^{K_{12}}(\theta_1,\theta_2) &= \ 4\sin(\theta_1)\sin(\theta_2). \end{aligned} \tag{4}$$

Correspondingly, the symbol of the block K has the matrix form

$$f^K = \mu \begin{bmatrix} 4-2\cos(\theta_1)(1+\cos(\theta_2)) & \sin(\theta_1)\sin(\theta_2) \\ \sin(\theta_1)\sin(\theta_2) & 4-2(1+\cos(\theta_1))\cos(\theta_2) \end{bmatrix}. \tag{5}$$

The derivation of the symbols of the blocks B_1 and B_2 deserves a special attention as these blocks are rectangular. For the case of Q1isoQ1, the blocks B_ℓ^T, $\ell = 1,2$ are of size $n^2 \times m^2$, where m and n are the number of mesh points in one direction, on two consecutive meshes, i.e., $n = 2(m-1)+1$. As the symbol can be related only to square matrices, in order to use the technique, we represent B_ℓ as a result of *downsampling* of larger square matrices $\widetilde{B}_\ell$ of size $n \times n$, namely, $B_\ell(n,m) = \widetilde{B}_\ell(n,n)H(n,m)$, where H has a particular structure used in various contexts, including multigrid methods, cf., e.g., [7], where it referred to as the *cutting* matrix. For the considered discretization and ordering, $\widetilde{B}_\ell$ are five-diagonal block matrices, where each block is itself five-diagonal of size (n,n). The term *downsampling* describes a particular size reduction of a square matrix (of odd size), obtained by deleting each second column, deleting every second block column, or both. More details on how the sampling matrices work can be found in [6]. The corresponding symbols of $\widetilde{B}$ are found to be $f^{\widetilde{B}_1}(\theta_1,\theta_2) = -4i\phi(\theta_1)\psi(\theta_2)$, $f^{\widetilde{B}_2}(\theta_1,\theta_2) = -4i\psi(\theta_1)\phi(\theta_2)$, where $\phi(\theta) = 2\sin(\theta)+\sin(2\theta)$ and $\psi(\theta) = 5+6\cos(\theta)+\cos(2\theta)$.

Having constructed all the symbols, using symbolic computations, we compute the symbol of $\tilde{B}K^{-1}\tilde{B}^T$ as $G = f^{\tilde{B}K^{-1}\tilde{B}^T} = v^*(f^K)^{-1}v$ with the vector v such that $v_1 = f^{\widetilde{B}_1}$ and $v_2 = f^{\widetilde{B}_2}$.

Finally we consider the effect of H and H^T on the underlying symbol. The symbol of the exact Schur $f^{\mathcal{S}}$ is computed by the formula below

$$f^{\mathcal{S}}(\theta_1,\theta_2) = f^M(\theta_1,\theta_2) + \frac{1}{4}\left(\sum_{l=0}^{1}\sum_{m=0}^{1} G\left(\frac{\theta_1}{2}+l\pi, \frac{\theta_2}{2}+m\pi\right)\right). \tag{6}$$

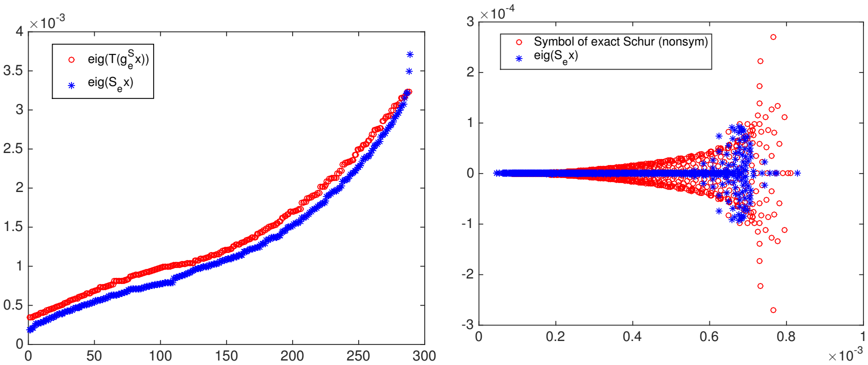
As already mentioned, the detailed derivation is shown in [6].

Next we deal with the advection term in the 11-block of the matrix $\mathcal{A}$. We consider only a term of the form $\nabla(\mathbf{b}\cdot\mathbf{u})$, with an advection vector $\mathbf{b} = [b_1, b_2]$. We denote the matrix, arising from the discretization of $\nabla(\mathbf{b}\cdot\mathbf{u})$ by A. Similarly to K, SDO induces a two-by-two structure on A, where the blocks $A_{k,\ell},\ k,\ell =$

1, 2 are block-tridiagonal and each block is also block tridiagonal. The symbol of the block A is found to be

$$f^A = -4\,i \begin{bmatrix} b_1\ \sin(\theta_1)(2+\cos(\theta_2))\ b_2\ \sin(\theta_1)(2+\cos(\theta_2)) \\ b_1\ \sin(\theta_2)(2+\cos(\theta_1))\ b_2\ \sin(\theta_2)(2+\cos(\theta_1)) \end{bmatrix}. \tag{7}$$

In an analogous way we can derive the symbol of the matrix, arising from the term $(\nabla \cdot \mathbf{u})\,\mathbf{c}$ with $\mathbf{c} = [c_1, c_2]$. The symbol of the nonsymmetric Schur complement is obtained in the same way as in the symmetric case. To illustrates how



(a) Symmetric exact Schur (b) Nonsymmetric exact Schur

Fig. 1. The spectrum of the symmetric and nonsymmetric $\mathcal{S}$ vs sampling of their symbols

well the symbols describe the spectral properties of the corresponding matrices, in Figure 1 we show the spectrum of the exact symmetric and the nonsymmetric ($\mathbf{b} = [0, 1]$) Schur complements (in blue) and the sampled symbols (in red).

The symbol of the element-wise Schur complement approximation Using exactly the same machinery, we derive the symbols of the elementwise Schur complement approximation S, both in the symmetric and nonsymmetric case. In contrast to the exact Schur complement, the symbol of S depends on h as we add a diagonal perturbation of order h^2 to A_{11}^{ℓ} in order to invert them. When constructing the symbol, we use the matrices, corresponding to seven refinements ($h = 0.002$). The symbols read as follows

$$\begin{aligned} f^{S_{sym}} &= (0.7145 + 0.3766\cos(\theta_1)) + 2\cos(\theta_2)(0.1883 + 0.0996\cos(\theta_1)) \\ f^{S_{nonsym}} &= (0.7145 + 0.3765\cos(\theta_1) + 0.0001i\sin(\theta_1)) \\ &\quad +0.2878\cos(\theta_1))(\cos(\theta_2) + i\sin(\theta_2)) + (0.7145 + 0.0995\cos(\theta_1) \\ &\quad -i0.0001\sin(\theta_1))(\cos(\theta_2) - i\sin(\theta_2)). \end{aligned}$$

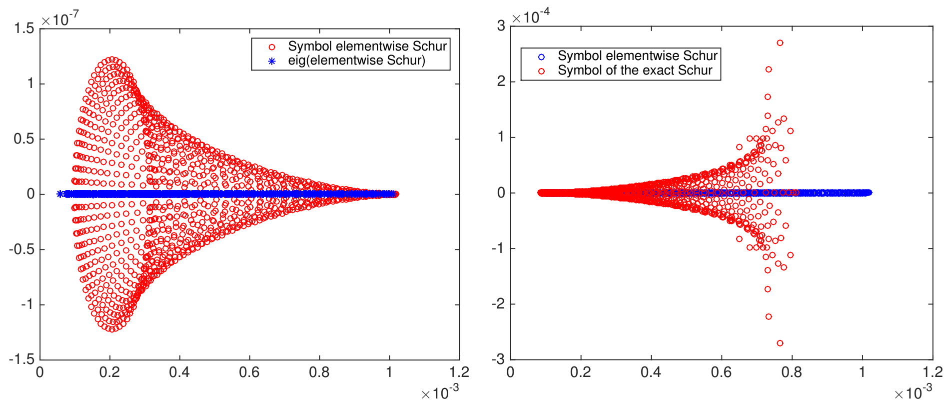
The nonsymmetric case is illustrated in Figure 2.

4 Conclusions and open problems

In this work, using the notion of the so-called *spectral symbol* in the Generalized Locally Toeplitz (GLT) setting, we identify the GLT symbol of the sequence of matrices $\{\mathcal{A}_n\}$ approximating the elasticity equations. Further, by exploiting the property that the GLT class defines an algebra of matrix sequences and the fact that Schur complements are obtained via elementary operation on the blocks of $\mathcal{A}_n$, we derived the symbols g_s of the associated sequences of Schur complements $\{\mathcal{S}_n\}$. As a consequence of the GLT theory, the eigenvalues of $\mathcal{S}_n$ for large n are described by a sampling of g_s on a uniform grid of its domain of definition.

We derive the symbols of $\mathcal{A}_n$ and $\mathcal{S}_n$ for the Q1isoQ1 stable FEM pair and the corresponding symbols for the case where the PDE problem includes an advection term and the corresponding system matrix, and respectively, the Schur complement matrix are nonsymmetric. Further, we derive the symbol of the elementwise Schur complement approximation and visually compare it with that of the exact Schur complement. One unexpected result of the study is that the elementwise Schur complement approximation for the considered problem converges to a symmetric matrix when $h \to 0$.

All numerical experiments show that, for the studied discrete problems, the sampling of the symbol agrees very well with the computed spectrum even for a relatively small-sized matrices.



(a) Nonsymmetric element-wise Schur, symbol and eigenvalues

(b) Symbol of the nonsymmetric Schur, exact vs elementwise

Fig. 2. Four refinements

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References

1. O. Axelsson, R. Blaheta, M. Neytcheva. *Preconditioning of boundary value problems using elementwise schur complements*, SIAM J. Matrix Anal. Appl., **31** (2009), 767–789.
2. O. Axelsson, M- Neytcheva. Preconditioning methods for linear systems arising in constrained optimization problems, *Numer. Linear Algebra Appl.*, **10** (2003), 3-31.
3. B. Beckermann, S. Serra Capizzano, *On the asymptotic spectrum of Finite Elements matrices*, SIAM J. Num. Anal. **45-2** (2007), 746–769.
4. D. Braess. *Finite elements*, Cambridge University Press, Cambridge, third edition, 2007. Theory, fast solvers, and applications in elasticity theory.
5. M. Donatelli, C. Garoni, S. Serra Capizzano, D. Sesana, *Spectral behavior of preconditioned non-Hermitian block Toeplitz matrices with matrix-valued symbols*, Appl. Math. Comput., **245** (2014), 158-173.
6. A. Dorostkar, M. Neytcheva, S. Serra-Capizzano, Spectral analysis of coupled PDEs and of their Schur complements via the notion of Generalized Locally Toeplitz sequences, TR 2015-008, Dept. Information Technology, Uppsala U., February 2015.
7. G. Fiorentino, S. Serra-Capizzano, *Multigrid methods for Toeplitz matrices*, Calcolo, **28-3/4** (1991), 283–305.
8. C. Garoni, S. Serra Capizzano, D. Sesana, *Spectral analysis and symbol of d-variate* $\mathbf{Q}_r$ *Lagrangian FEM stiffness matrices*, TR 2014-19; Dept. Information Technology, U. Uppsala, november 2014.
9. C. Garoni, S. Serra Capizzano, D. Sesana, *Tools for the asymptotic spectrum of non-Hermitian perturbations of Hermitian matrix-sequences and applications*, Integral Eq. Oper. Theory, (2014) 10.1007/s00020-014-2157-6.
10. J. Kraus. *Additive Schur complement approximation and application to multilevel preconditioning*, SIAM J. Sci. Comput., **34** (2012), A2872–A2895.
11. Kraus J. *Algebraic multilevel preconditioning of finite element matrices using local Schur complements*, Numer.l Linear Algebra Appl., **13** (2006), 49-70.
12. M. Neytcheva. *On element-by-element Schur complement approximations*, Linear Algebra Appl., **434** (2011), 2308–2324.
13. M. Neytcheva, E. Bängtsson. *Preconditioning of nonsymmetric saddle point systems as arising in modelling of viscoelastic problems*, Electr. Trans. Numer.l Anal., **29**(2008), 193–211.
14. S. Serra Capizzano, *An ergodic theorem for classes of preconditioned matrices*, Linear Algebra and its Applications, **282** (1998), 161–183.
15. S. Serra Capizzano, *Generalized Locally Toeplitz sequences: spectral analysis and applications to discretized Partial Differential Equations*, Linear Algebra Appl., **366-1** (2003) 371–402.
16. S. Serra Capizzano, *GLT sequences as a Generalized Fourier Analysis and applications*, Linear Algebra Appl., **419-1** (2006), 180–233.
17. P. Tilli, *Locally Toeplitz sequences: spectral properties and applications*, Linear Algebra Appl., **278** (1998), 91–120.
18. E.E. Tyrtyshnikov, *A uniform approach to some old and new theorems on distribution and clustering*, Linear Algebra Appl., **232** (1996), 1–43.