A preconditioner for optimal control problems, constrained by Stokes equation with a time-harmonic control

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Abstract

In an earlier paper preconditioning of stationary Stokes-constrained optimal control problems has been considered.

In this article we construct an efficient preconditioner for solving the algebraic systems arising from discretized optimal control problems with time-periodic Stokes equations. A simplified analysis of the derivation of the preconditioner and its properties is presented. The preconditioner is fully parameter independent and the condition number of the corresponding preconditioned matrix is bounded by 2. The so-constructed preconditioner is favourably compared with another preconditioner for the same problem.

Keywords: optimal control, time-harmonic Stokes problem, preconditioning

1 Introduction

Optimal control problems, constrained by a state equation in the form of a partial differential equation (PDE) arise in many applications, where one wants to steer the modelled process in order to have a solution close to some given target (desired) function, see for example [21]. The process is steered by a control function, normally the source function to the differential equation.

In an earlier publication ([4]) we have considered an efficient preconditioner, applied for control problems with a stationary state equation. In some applications, a more general problem with a time-harmonic desired state, arise. Such problems have been considered in e.g. [12, 13], see also the references therein. In the present paper we deal mainly with a Stokes time-harmonic problem as state equation. We present an efficient and fully robust preconditioner that involves just two solutions of a regularized Stokes problem and that leads to a spectrum of the preconditioned matrix clustered around the unit value, with a condition number bounded by 2.

This holds uniformly with respect to all parameters involved, namely, the meshsize (h) in the discretization mesh, the frequency (ω) of the harmonic wave and the control parameter (β). This result generalizes previous results for optimal control problems, constrained by stationary PDEs, see e.g. [17, 16, 15, 3, 4], to problems involving time-harmonic control functions and improves the results in [12].

To allow for the use of inner iteration procedures with variable tolerances when solving the Stokes equation, the preconditioner should preferably be coupled with a variable (flexible) version of a generalized conjugate gradient method, such as GCG ([5]) or FGMRES ([19]).

The general form of the preconditioner we aim at and its efficient implementation are recalled in Section 2 with a simplified derivation. In Section 3 we present the type of optimal control problems dealt with in this paper and their discretized versions. The generalization of the preconditioner from Section 2 to optimal control problems with Stokes control is done in Section 4. We illustrate the behaviour of the Stokes control preconditioner in Section 6 and give some concluding remarks.

2 A preconditioner for matrices with square blocks of the form and its efficient implementation

Certain problems, such as optimal control problems, lead to matrices in a special form,

$$\mathcal{A} = \begin{bmatrix} A & -b\,B_2 \\ a\,B_1 & A \end{bmatrix},$$

with square blocks. Here a and b are scalars with the same sign. It has been shown ([1, 2, 3, 4]) that for $\mathcal{A}$ an efficient preconditioner can be constructed, namely,

$$\mathcal{P}_F = \begin{bmatrix} A & -b\,B_2 \\ a\,B_1 & A + \sqrt{ab}(B_1 + B_2) \end{bmatrix}.$$

In the earlier papers, mentioned above, the quality of $\mathcal{P}_F$ as well as how to apply $\mathcal{P}_F$ in a computationally easy way has been analysed via the explicit form of its inverse,

$$\mathcal{P}_F^{-1} = \begin{bmatrix} H_1^{-1} + H_2^{-1} - H_2^{-1}AH_1^{-1} & -\sqrt{\frac{b}{a}}(I - H_2^{-1}A)H_1^{-1} \\ \sqrt{\frac{a}{b}}H_2^{-1}(I - AH_1^{-1}) & H_2^{-1}AH_1^{-1} \end{bmatrix}.$$

The matrices $H_i = A + \sqrt{ab}B_i, i = 1, 2$ are assumed to be nonsingular.

It can be seen that, besides one matrix-vector multiplication and four vector updates, the action of $\mathcal{P}_F^{-1}$ requires only a single solution of a system with H_1 and H_2, cf. Algorithm 2. We show this here in a simplified way, without use of the explicit form of the inverse of $\mathcal{P}_F^{-1}$.

An easy computation shows namely that $\mathcal{P}_F$ can be factorized as

$$\mathcal{P}_F = \begin{bmatrix} I & -\sqrt{\frac{b}{a}}I \\ 0 & I \end{bmatrix} \begin{bmatrix} A + \sqrt{ab}B_1 & 0 \\ aB_1 & A + \sqrt{ab}B_2 \end{bmatrix} \begin{bmatrix} I & \sqrt{\frac{b}{a}}I \\ 0 & I \end{bmatrix}. \tag{1}$$

This means that a system

$$\mathcal{P}_F \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \end{bmatrix}$$

can be solved as

$$\begin{bmatrix} A + \sqrt{ab}B_1 & 0 \\ aB_1 & A + \sqrt{ab}B_2 \end{bmatrix} \begin{bmatrix} \mathbf{y}_1 \\ \mathbf{x}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{g}_1 \\ \mathbf{f}_2 \end{bmatrix},$$

where $\mathbf{y}_1 = \mathbf{x}_1 + \sqrt{\frac{b}{a}}\mathbf{x}_2$ and $\mathbf{g}_1 = \mathbf{f}_1 + \sqrt{\frac{b}{a}}\mathbf{f}_2$.

The algorithm to compute $\mathbf{x}_1$ and $\mathbf{x}_2$ becomes as follows.

Algorithm 1 Solving the transformed $\mathcal{P}_F$

1: Compute $\mathbf{g}_1 = \mathbf{f}_1 + \sqrt{\frac{b}{a}}\mathbf{f}_2$.
2: Solve $H_1\mathbf{y}_1 = \mathbf{g}_1$.
3: Compute $\mathbf{h} = \mathbf{f}_2 - aB_1\mathbf{y}_1$.
4: Solve $H_2\mathbf{x_2} = \mathbf{h}$.
5: Compute $\mathbf{x}_1 = \mathbf{y}_1 - \sqrt{\frac{b}{a}}\mathbf{x}_2$.

For a comparison, we include below the algorithm, based on the inverse of $\mathcal{P}_F$, used in [3, 4] and in some earlier papers by the present authors.

Algorithm 2 Solving the factorized $\mathcal{P}_F$

1: Compute $\mathbf{g}_1 = \mathbf{f}_1 + \sqrt{\frac{b}{a}}\mathbf{f}_2$
2: Solve $\quad H_1\mathbf{g} = \mathbf{g}_1$.
3: Compute $\mathbf{v} = \mathbf{f}_1 - A\mathbf{g}$.
4: Solve $\quad H_2\mathbf{w} = \mathbf{v}$.
5: Compute $\mathbf{x}_1 = \mathbf{g} + \mathbf{w}$ and $\mathbf{x}_2 = -\sqrt{\frac{a}{b}}\mathbf{w}$.

Note that the major difference between the algorithms is that in Algorithm 1 we perform a matrix-vector multiplication with B_i, while in Algorithm 2 we multiply by A.

3 Optimal control problems with a time-harmonic desired state

Following [12], consider first the optimal control problem of finding the state $u(x,t)$ and the control $f(x,t)$ that minimize the functional

$$\mathcal{J}(u,f) = \frac{1}{2}\int_0^T\int_\Omega |u(x,t) - u_d(x,t)|^2\,dx\,dt + \frac{1}{2}\beta\int_0^T\int_\Omega |f(x,t)|^2\,dx\,dt$$

subject to the time-dependent parabolic problem

$$\begin{array}{rcll} \frac{\partial}{\partial t}u(x,t) - \Delta u(x,t) &=& f(x,t) & \text{in}\,\Omega\times(0,T), \\ u(x,t) &=& 0, & x\in\Gamma\times(0,T), \\ u(x,0) &=& u(x,T) & \text{in}\,\Omega, \\ f(x,0) &=& f(x,T) & \text{in}\,\Omega. \end{array}$$

Here Γ is the boundary of Ω and u_d is the desired state.

The target function is time-harmonic, $u_d(x,t) = u_d(x)e^{i\omega t}$ with period $\omega = 2\pi\,k/T$ for some positive integer $k\in\mathbb{Z}$. Then the solution and the control are also time-harmonic, $u(x,t) = u(x)e^{i\omega t}$ and $f(x,t) = f(x)e^{i\omega t}$. Hence, $u(x), f(x)$ are time-independent solutions of the following optimal control problem,

$$\text{minimize}\quad \frac{1}{2}\int_\Omega |u(x) - u_d(x)|^2\,dx + \frac{1}{2}\beta\int_\Omega |f(x)|^2\,dx$$

$$\text{subject to}\quad \left\{\begin{array}{rcll} i\omega u(x) &-& \Delta u(x) = f(x) & \text{in}\,\Omega, \\ u(x) &=& 0, & x\in\Gamma. \end{array}\right.$$

Here β is a positive regularization parameter. In the sequel we assume that $u(x)$ and $u_d(x)$ are real-valued but the control $f(x)$ must be complex-valued, $f(x) = f_0(x) + if_1(x)$.

Remark 1. *If also the function and its target have complex-valued coefficients then, as shown in [9], one can separate the real and imaginary parts of the equation which leads to two systems of the form (2), one for the real and one for the complex part of the solution.*

Using an appropriate finite element subspace V_h for both u and f and introducing the corresponding, complex-valued, Lagrange multiplier λ, the Lagrangian functional (L) for the discretized constrained optimization problem is given by

$$L(u,f,\lambda) = \frac{1}{2}(u-u_d)^T M(u-u_d) + \frac{1}{2}\beta f^* M f + \lambda^*(i\omega Mu + Ku - Mf).$$

Here M is the mass matrix, corresponding to the L_2-inner product in V_h, K is the negative discrete Laplacian and $*$ denotes conjugate transpose.

The first order necessary conditions, $\nabla L(u,f,\lambda) = 0$, which are also sufficient for the existence of the solution, lead to the system

$$\begin{bmatrix} M & 0 & K - i\omega M \\ 0 & \beta M & -M \\ K + i\omega M & -M & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{f} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} M\mathbf{u}_d \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}. \tag{2}$$

Using the relation $\boldsymbol{\lambda} = \beta\mathbf{f}$ we obtain the reduced system

$$\begin{bmatrix} M & \beta(K - i\omega M) \\ K + i\omega M & -M \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{f} \end{bmatrix} = \begin{bmatrix} M\mathbf{u}_d \\ \mathbf{0} \end{bmatrix} \tag{3}$$

Here $\mathbf{u}$ is real-valued whereas $\mathbf{f}$ contains imaginary part, i.e., $\mathbf{f} = \mathbf{f_0} + \mathbf{if_1}$. Thus, from the imaginary part, we have:

$$\begin{aligned} \beta K\mathbf{f_1} - \omega\beta M\mathbf{f_0} = \mathbf{0} \quad &\Rightarrow \quad K\mathbf{f_1} = \omega M\mathbf{f_0} \\ -M\mathbf{f_1} + \omega M\mathbf{u} = \mathbf{0} \quad &\Rightarrow \quad \mathbf{f_1} = \omega\mathbf{u}. \end{aligned} \tag{4}$$

By (4), $\beta\omega M\mathbf{f_1} - \beta\omega^2 M\mathbf{u} = \mathbf{0}$, that is, $\beta\omega M\mathbf{f_1} = \beta\omega^2 M\mathbf{u}$ and clearly $M\mathbf{f}_0 = K\mathbf{u}$. Finally, the real part of the reduced system (3) takes the form

$$\begin{bmatrix} (1+\beta\omega^2)M & \beta K \\ K & -M \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{f}_0 \end{bmatrix} = \begin{bmatrix} M\mathbf{u}_d \\ \mathbf{0} \end{bmatrix} \tag{5}$$

or

$$\mathcal{A}\begin{bmatrix} \mathbf{u} \\ -\mathbf{f}_0 \end{bmatrix} \equiv \begin{bmatrix} M & -\widetilde{\beta}K \\ K & M \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ -\mathbf{f}_0 \end{bmatrix} = \frac{1}{1+\beta\omega^2} \begin{bmatrix} M\mathbf{u}_d \\ \mathbf{0} \end{bmatrix}, \tag{6}$$

where $\widetilde{\beta} = \frac{\beta}{1+\beta\omega^2}$.

The matrix in system (6) has the form, discussed in, e.g., [3] and in Section 2, and can therefore be efficiently preconditioned by

$$\mathcal{P}_F = \begin{bmatrix} M & -\widetilde{\beta}K \\ K & M + 2\sqrt{\widetilde{\beta}}K \end{bmatrix}.$$

The inverse of this matrix involves only two matrix inverses, $(M + \sqrt{\widetilde{\beta}}K)^{-1}$.

The theory, derived in [1, 2, 3, 4] shows that all eigenvalues of the preconditioned matrix $\mathcal{P}_F^{-1}\mathcal{A}$ are real and positive, and also belong to the interval $[0.5, 1]$. In particular, to see the influence of β on the spectrum, we briefly repeat the analysis. Consider the generalized eigenvalue problem

$$\mu \begin{bmatrix} M & -\widetilde{\beta}K \\ K & M + 2\sqrt{\widetilde{\beta}}K \end{bmatrix} \begin{bmatrix} \mathbf{v} \\ \mathbf{w} \end{bmatrix} = \begin{bmatrix} M & -\widetilde{\beta}K \\ K & M \end{bmatrix} \begin{bmatrix} \mathbf{v} \\ \mathbf{w} \end{bmatrix}, \quad \|\mathbf{v}\| + \|\mathbf{w}\| \neq 0.$$

Since the matrix $\mathcal{A}$ is nonsingular, here $\mu \neq 0$. It shows that $\mu = 1$ for $\mathbf{v} \neq \mathbf{0}$ and $\mathbf{w} = \mathbf{0}$ and

$$\left(\frac{1}{\mu} - 1\right) \begin{bmatrix} M & -\widetilde{\beta}K \\ K & M \end{bmatrix} \begin{bmatrix} \mathbf{v} \\ \mathbf{w} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ 2\sqrt{\widetilde{\beta}}K\mathbf{w} \end{bmatrix}.$$

Thus, for $\mu \neq 1$ it holds $M\mathbf{v} = \widetilde{\beta}K\mathbf{w}$, $\mathbf{w} \neq \mathbf{0}$ and

$$\left(\frac{1}{\mu} - 1\right) \mathbf{w}^T \left[\widetilde{\beta}KM^{-1}K + M\right] \mathbf{w} = 2\sqrt{\widetilde{\beta}}\mathbf{w}^T K\mathbf{w} = 2\sqrt{\widetilde{\beta}}\mathbf{w}^T KM^{-1/2}M^{1/2}\mathbf{w}.$$

That is, $\frac{1}{\mu} - 1 > 0$, or $\mu < 1$. Further, by the Cauchy-Schwarz inequality it follows that $\frac{1}{\mu} - 1 \leq 1$, i.e.,

$$\frac{1}{2} \leq \mu \leq 1.$$

It can be seen that the eigenvalues cluster at $\mu = 1$ for small values of β and even further for large values of ω^2. Therefore, the convergence of the preconditioned iteration method is expected to be rapid. Preferably, a method that incorporates variable preconditioner should be used (GCG or FGMRES) as systems with the matrices $M + \sqrt{\widetilde{\beta}}K$ are to be solved by inner iterations with variable accuracy.

Consider now the corresponding optimal control problem with a state equation of Stokes type,

$$\begin{array}{rcll}\mathcal{L}(u,p) \equiv \dfrac{\partial}{\partial t}u(x,t) - \Delta u(x,t) + \nabla p(x,t) & = & f(x,t) & \text{in } \Omega \times [0,T] \\ \nabla \cdot u(x,t) & = & 0 & \text{in } \Omega \times [0,T] \\ u(x,t) & = & 0 & \text{on } \Gamma \times [0,T] \\ u(x,0) & = & u(x,T) & \text{in } \Omega \\ p(x,0) & = & p(x,T) & \text{in } \Omega \\ f(x,0) & = & f(x,T) & \text{in } \Omega \end{array} \quad (7)$$

where f is the vectorial control function (for the velocity and the pressure component) and let $g(x,t)$ and $q(x,t)$ be Lagrangian multipliers.

The Lagrangian functional becomes

$$\begin{aligned} L(u,p,f,g,q) &= \tfrac{1}{2}\int_0^T\int_\Omega |u(x,t)-u_d(x,t|^2\,dx\,dt + \int_0^T\int_\Omega g(x,t)(\mathcal{L}(u,p)-f)\,dx\,dt + \\ &\int_0^T\int_\Omega \nabla\cdot u(x,t)\,q(x,t)\,dx\,dt + \tfrac{1}{2}\beta\int_0^T\int_\Omega |f(x,t)|^2\,dx\,dt \end{aligned}$$

As above, u_d and u are assumed to be time-harmonic, i.e., $u(x,t) = u(x)e^{i\omega t}$, $\omega = 2\pi\, k/T$, etc.

From $\nabla\, L(u,p,f,g,q) = 0$, after a suitable discretization, there arise five necessary conditions, written in the matrix form (8), see [12],

$$\begin{bmatrix} M & 0 & 0 & K-i\omega M & -D^T \\ 0 & 0 & 0 & D & 0 \\ 0 & 0 & \beta M & -M & 0 \\ K+i\omega M & -D^T & -M & 0 & 0 \\ -D & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{p} \\ \mathbf{f} \\ \mathbf{g} \\ \mathbf{q} \end{bmatrix} = \begin{bmatrix} M\mathbf{u}_d \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad (8)$$

where M, K and D are the mass matrix, the discrete negative Laplacian and the discrete divergence matrix, correspondingly.

By use of the relation $M\mathbf{g} = \beta M\mathbf{f}$ we eliminate $\mathbf{g}$ and reduce to four equations which, together with some sign changes, take the form

$$\begin{bmatrix} M & 0 & \beta(K-i\omega M) & -\beta D^T \\ 0 & 0 & -\beta D & 0 \\ K+i\omega M & -D^T & -M & 0 \\ -D & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{p} \\ \mathbf{f} \\ \widetilde{\mathbf{q}} \end{bmatrix} = \begin{bmatrix} M\mathbf{u}_d \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad (9)$$

where $\widetilde{\mathbf{q}} = \mathbf{q}/\beta$.

Here $\mathbf{u}, \mathbf{p}$ and $\mathbf{q}$ are real valued but $\mathbf{f}$ is complex-valued. With $\mathbf{f} = \mathbf{f}_0 + i\mathbf{f}_1$, by putting together the imaginary parts in the first and the third equations in (9), we see that $\beta K \mathbf{f}_1 = \beta\omega M \mathbf{f}_0$ and $\omega M \mathbf{u} = M \mathbf{f}_1$. Thus, $\mathbf{f}_1 = \omega \mathbf{u}$ and $K\mathbf{u} = M\mathbf{f}_0$. Hence, the matrix equation (9) takes the form

$$\begin{bmatrix} (1+\beta\omega^2)M & 0 & \beta K & -\beta D^T \\ 0 & 0 & -\beta D & 0 \\ K & -D^T & -M & 0 \\ -D & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{p} \\ \mathbf{f}_0 \\ \widetilde{\mathbf{q}} \end{bmatrix} = \begin{bmatrix} M\mathbf{u}_d \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

or

$$\begin{bmatrix} M & 0 & -\widetilde{\beta} K & \widetilde{\beta} D^T \\ 0 & 0 & \widetilde{\beta} D & 0 \\ K & -D^T & M & 0 \\ -D & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{p} \\ -\mathbf{f}_0 \\ -\widetilde{\mathbf{q}} \end{bmatrix} = \frac{1}{1+\beta\omega^2} \begin{bmatrix} M\mathbf{u}_d \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{10}$$

with $\widetilde{\beta} = \beta/(1+\beta\omega^2)$.

In this way we obtain a block system of the type $\begin{bmatrix} A & -cB \\ B & A \end{bmatrix}$ and, following the idea of the method in Section 2, choose a preconditioner of the form

$$\mathcal{P}_F = \begin{bmatrix} M & 0 & -\widetilde{\beta} K & \widetilde{\beta} D^T \\ 0 & 0 & \widetilde{\beta} D & 0 \\ K & -D^T & M + 2\sqrt{\widetilde{\beta}} K & -2\sqrt{\widetilde{\beta}} D^T \\ -D & 0 & -2\sqrt{\widetilde{\beta}} D & 0 \end{bmatrix}. \tag{11}$$

Let denote the matrix in (10) by $\mathcal{A}$. An observation reveals that we can transform $\mathcal{A}$ into a spectrally equivalent matrix $\tilde{\mathcal{A}}$ with the help of a diagonal matrix $\mathcal{D}$, namely, we let $\widetilde{\mathcal{A}} = \mathcal{D}\mathcal{A}\mathcal{D}^{-1}$, where $\mathcal{D} = blockdiag(I, I, \sqrt{\widetilde{\beta}}, \sqrt{\widetilde{\beta}}\, I)$. After scaling, the transformed system reads

$$\begin{bmatrix} M & 0 & -\sqrt{\widetilde{\beta}} K & \sqrt{\widetilde{\beta}} D^T \\ 0 & 0 & \sqrt{\widetilde{\beta}} D & 0 \\ \sqrt{\widetilde{\beta}} K & -\sqrt{\widetilde{\beta}} D^T & M & 0 \\ -\sqrt{\widetilde{\beta}} D & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{p} \\ -\sqrt{\widetilde{\beta}} \mathbf{f}_0 \\ -\sqrt{\widetilde{\beta}} \widehat{\mathbf{q}} \end{bmatrix} = \frac{1}{1+\beta\omega^2} \begin{bmatrix} M\mathbf{u}_d \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{12}$$

Numerical experiments show that solutions with the matrix $\widetilde{\mathcal{A}}$ require less iterations than that with the matrix $\mathcal{A}$.

4 Spectral properties of the preconditioned matrix for Stokes-type optimal control problems

We analyse now the preconditioner in Section 3 in the framework of Stokes control problems (cf. [4]) and amend it for the time-harmonic case. We compare it with a preconditioner as presented in [12], for which we simplify the analysis.

4.1 The block matrix preconditioner applied to a Stokes optimal control problem

The arising matrix has the form (10). We analyse it below in a slightly more general form, replacing K by F, scaling and incorporating the coefficient β in the matrix blocks, as

$$\begin{bmatrix} M & 0 & -F & -D^T \\ 0 & 0 & -D & 0 \\ F & D^T & M & 0 \\ D & 0 & 0 & 0 \end{bmatrix} \tag{13}$$

indicating that the results hold also when $F \neq F^T$. It is preconditioned by a preconditioner, $\mathcal{P}_{FS}$, formed from the matrix in (13), to which we add

$$\begin{bmatrix} F & D^T \\ D & 0 \end{bmatrix} + \begin{bmatrix} F^T & D^T \\ D & 0 \end{bmatrix}$$

to the lower $(2,2)$ diagonal block, namely

$$\mathcal{P}_{FS} = \begin{bmatrix} M & 0 & -F & -D^T \\ 0 & 0 & -D & 0 \\ F & D^T & M+F+F^T & 2D^T \\ D & 0 & 2D & 0 \end{bmatrix}. \tag{14}$$

As we have shown in [4] and also follows from Section 2, an application of the preconditioner $\mathcal{P}_{FS}$ involves the solution of one system with the matrix $\begin{bmatrix} M+F & D^T \\ D & 0 \end{bmatrix}$ and one system with the matrix $\begin{bmatrix} M+F^T & D^T \\ D & 0 \end{bmatrix}$, both of Stokes type.

For completeness, we include a brief account for the analysis of the spectrum of the preconditioned matrix $\mathcal{P}_{FS}^{-1}\mathcal{A}$, where $\mathcal{A}$ is as in (13). The following result holds true.

Theorem 1. *Consider the generalized eigenvalue problem* $\lambda \mathcal{P}_{FS}\mathbf{v} = \mathcal{A}\mathbf{v}$*, where* $\mathcal{A}$ *and* $\mathcal{P}_{FS}$ *are defined in (13) and (14), correspondingly, Assume that* M *and* $F+F^T$ *are symmetric and positive definite and* D *has full rank. Then all eigenvalues* λ *are real and positive and*

$$1/3 \leq \lambda \leq 1$$

If F *is symmetric, then*

$$1/2 \leq \lambda \leq 1$$

A detailed proof can be found in [4].

4.2 An alternative preconditioner for the time-harmonic optimal control problem for Stokes equation

In [12], as a part of a more general analysis of saddle point matrices and related preconditioning techniques, a block-diagonal preconditioner is presented for the complex-valued matrix arising from the time-harmonic parabolic problem. As we compare our preconditioner for that, derived in [12], for completeness we briefly describe their approach. Consider the system, arising from the first order necessary conditions in a compressed form and, compared to the system in (9), scaled and permuted as

$$\begin{bmatrix} M & \sqrt{\beta}(K - i\omega M) & 0 & -\sqrt{\beta}D^T \\ \sqrt{\beta}(K - i\omega M) & -M & -\sqrt{\beta}D & 0 \\ 0 & -\sqrt{\beta}D^T & 0 & 0 \\ -\sqrt{\beta}D & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{p} \\ \frac{1}{\sqrt{\beta}}\mathbf{f} \\ \frac{1}{\sqrt{\beta}}\mathbf{q} \end{bmatrix} = \begin{bmatrix} M\mathbf{u}_d \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}$$

and denote the matrix by $\mathcal{M}$, where $\mathcal{M} = \begin{bmatrix} \mathbf{A} & \mathbf{B}^* \\ \mathbf{B} & -\mathbf{C} \end{bmatrix}$, $\mathbf{B} = -\sqrt{\beta}\begin{bmatrix} 0 & D \\ 0 & D \end{bmatrix}$, $\mathbf{A} = \begin{bmatrix} M & \sqrt{\beta}(K - i\omega M) \\ \sqrt{\beta}(K - i\omega M) & -M \end{bmatrix}$, and $\mathbf{C} = \mathbf{0}$. Here, $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are generic names for the matrix blocks of $\mathcal{M}$ and $\mathbf{A}$ and $\mathbf{B}$ are complex. In [12], the following block-diagonal preconditioner for $\mathcal{M}$ is analysed, based on non-standard norm techniques,

$$\mathcal{P}_{nsn} = \begin{bmatrix} \mathbf{P} & 0 \\ 0 & \mathbf{S} \end{bmatrix},$$

where $\mathbf{P} = \begin{bmatrix} P & 0 \\ 0 & P \end{bmatrix}$, $\mathbf{S} = \beta \begin{bmatrix} S & 0 \\ 0 & S \end{bmatrix}$, $P = M + \sqrt{\beta}(F + \omega M)$ and $S = D(M + \sqrt{\beta}(F + \omega M))^{-1}D^T$. Clearly, $\mathbf{P}$ and $\mathbf{S}$ are real and symmetric positive definite. The analysis in [12] shows that the spectrum of $\mathcal{P}_{nsn}^{-1}\mathcal{M}$ is real and symmetric around zero, contained in the intervals (approximately) $[-1.618, -0.306] \cup [0.306, 1.618]$, correspondingly.

We present here a simplified analysis of the preconditioner $\mathbf{P}$, used to approximate the block $\mathbf{A}$ in $\mathcal{M}$. We recall that both F and M are symmetric positive definite. Given

$$\mathbf{A_o} = \begin{bmatrix} M & F - i\omega M \\ F + i\omega M & -\frac{1}{\beta} M \end{bmatrix}.$$

We scale first $\mathbf{A_o}$ by multiplication from both sides by $\begin{bmatrix} I & 0 \\ 0 & \sqrt{\beta} I \end{bmatrix}$ to obtain

$$\mathbf{A} \equiv \begin{bmatrix} M & \sqrt{\beta}(F - i\omega M) \\ \sqrt{\beta}(F + i\omega M) & -M \end{bmatrix}.$$

For $\mathbf{A}$, a block-diagonal preconditioner is applied,

$$\mathbf{P} = \begin{bmatrix} (1+\sqrt{\beta}\omega)M + \sqrt{\beta}F & 0 \\ 0 & (1+\sqrt{\beta}\omega)M + \sqrt{\beta}F \end{bmatrix},$$

resulting in the preconditioned matrix

$$\mathbf{G} = \mathbf{P^{-1}A} = \begin{bmatrix} \frac{1}{1+\sqrt{\beta}\omega}\widetilde{M} & I - \widetilde{M} - i\frac{\sqrt{\beta}\omega}{1+\sqrt{\beta}\omega}\widetilde{M} \\ I - \widetilde{M} + i\frac{\sqrt{\beta}\omega}{1+\sqrt{\beta}\omega}\widetilde{M} & -\frac{1}{1+\sqrt{\beta}\omega}\widetilde{M} \end{bmatrix}, \tag{15}$$

where $\widetilde{M} = [(1+\sqrt{\beta}\omega)M + \sqrt{\beta}F]^{-1}(1+\sqrt{\beta}\omega)M$ and we have used that $\sqrt{\beta}F = \big((1+\sqrt{\beta}\omega)M + \sqrt{\beta}F\big) - (1+\sqrt{\beta}\omega)M$.

A calculation shows that $\mathbf{G^2} = \begin{bmatrix} G & 0 \\ 0 & G \end{bmatrix}$ where $G = \left(\frac{1}{1+\sqrt{\beta}\omega}\right)^2 \widetilde{M}^2 + (I - \widetilde{M})^2 + \frac{\beta\omega^2}{(1+\sqrt{\beta}\omega)^2}\widetilde{M}^2$.

Theorem 2. *Let* $\mathbf{G}$ *be defined by (15). Then the eigenvalues of* $\mathbf{G^2}$ *satisfy*

$$\frac{\alpha}{1+\alpha} \le \lambda(\mathbf{G^2}) \le \mathbf{1}, \tag{16}$$

where $\alpha = (1+\beta\omega^2)/(1+\beta\omega^2+2\sqrt{\beta}\omega)$, *i.e.,* $\frac{1}{2} \le \alpha \le 1$. *Hence, for the condition number of* $\mathbf{G^2}$ *there holds*

$$\varkappa(\mathbf{G^2}) \le \frac{\mathbf{1}+\alpha}{\alpha} \le \mathbf{3}.$$

Proof. Since $0 < \widetilde{M} \leq I$, it follows that the eigenvalues of $\mathbf{G^2}$ satisfy

$$\lambda = \alpha\mu^2 + (1-\mu)^2,$$

where μ is an eigenvalue of $\widetilde{M}$. The minimal value of λ is taken for $\mu = \frac{1}{1+\alpha}$, namely,

$$\lambda = \frac{\alpha}{(1+\alpha)^2} + \left(1 - \frac{1}{1+\alpha}\right)^2 = \frac{\alpha}{1+\alpha}.$$

This shows that

$$\frac{\alpha}{1+\alpha} \leq \lambda(\mathbf{G^2}) \leq \mathbf{1}.$$

Since $\frac{1}{2} \leq \alpha \leq 1$ it follows that $\frac{1}{3} \leq \lambda(\mathbf{G^2}) \leq \mathbf{1}$ and the condition number of $\mathbf{G^2}$ is bounded by $\frac{1+\alpha}{\alpha} \leq 3$. ∎

A minimal residual method to solve a system with $\mathbf{A}$, preconditioned by $\mathbf{P}$ will, hence, require twice the number of iterations for a CG-like method to solve a system with the matrix $\mathbf{G^2}$. The above is a simplification and clarification of the corresponding analysis in [12].

5 Computational complexity of the preconditioners $\mathcal{P}_F$ and $\mathcal{P}_{nsn}$

5.1 The preconditioner $\mathcal{P}_F$

As follows from Section 2 and Section 4, the solution of linear systems with $\mathcal{P}_F$ involves the solution of two saddle point systems with the matrix

$$\mathcal{H} = \begin{bmatrix} M + \sqrt{\widetilde{\beta}}K & -\sqrt{\widetilde{\beta}}D^T \\ -\sqrt{\widetilde{\beta}}D & 0 \end{bmatrix}. \tag{17}$$

In order to efficiently solve systems with $\mathcal{H}$ we introduce

$$\mathcal{P}_{\mathcal{H}} = \begin{bmatrix} M + \sqrt{\widetilde{\beta}}K & 0 \\ -\sqrt{\widetilde{\beta}}D & -\widetilde{S} \end{bmatrix} \tag{18}$$

where $\widetilde{S}$ is an approximation of the Schur complement S of H, $S = \widetilde{\beta}D(M + \sqrt{\widetilde{\beta}}K)^{-1}D^T$. Since the stiffness matrix K can be written in the form $D^T M_p^{-1} D$, where M_p is the pressure (i.e., scalar) mass matrix, we can use the relation

$$\begin{aligned} S^{-1} &= \left(D(M + \sqrt{\widetilde{\beta}}D^T M_p^{-1} D)^{-1} D^T\right)^{-1} = \sqrt{\widetilde{\beta}}M_p^{-1} + (DM^{-1}D^T)^{-1} \\ &\approx \sqrt{\widetilde{\beta}}M_p^{-1} + K_p^{-1} = \widetilde{S}^{-1} \end{aligned},$$

where K_p is the scalar Laplace matrix. For details see [8]. Actually, the relation follows from

$$\left(\widetilde{D}(I+\widetilde{D}^T\widetilde{M}_p\widetilde{D}^T)^{-1}\widetilde{D}\right)^{-1}=\widetilde{M}_p^{-1}+(\widetilde{D}\widetilde{D}^T)^{-1},$$

where $\widetilde{D}=DM^{1/2}$, $\widetilde{M}_p=\frac{1}{\beta}M_p$, which simply follows from

$$\begin{aligned}&\widetilde{D}(I+\widetilde{D}^T\widetilde{M}_p^{-1}\widetilde{D})^{-1}\widetilde{D}^T(\widetilde{M}_p^{-1}+(\widetilde{D}\widetilde{D}^T)^{-1})=\\&\widetilde{D}(I+\widetilde{D}^T\widetilde{M}_p^{-1}\widetilde{D})^{-1}(\widetilde{D}^T\widetilde{M}_p^{-1}\widetilde{D}+I)\widetilde{D}^T(\widetilde{D}\widetilde{D}^T)^{-1}=I.\end{aligned}$$

Hence,

$$\widetilde{S}^{-1}=\sqrt{\widetilde{\beta}}M_p^{-1}+K_p^{-1}.$$

Moreover, the preconditioner $P_{\mathcal{H}}$ in (18) has been shown to possess a mesh-independent convergence, cf. [23, 24].

Solving systems with the two-by-two block matrix $\mathcal{H}$ block

Systems with $\mathcal{H}$ can be solved in various ways. We test here two such methods.

(1) Solve $\mathcal{H}$ by (inner) FGMRES, preconditioned by as an preconditioning it with $\mathcal{P}_H$.

(2) Solve $\mathcal{H}$ using the *inexact Uzawa* method, cf. i.e., [10, 7], in which case $\mathcal{P}_H$ then acts as a splitting matrix for the iteration,

$$\boldsymbol{\rho_{n+1}}=\boldsymbol{\rho_n}+\mathcal{P}_H^{-1}\mathcal{H}\mathbf{r_n} \tag{19}$$

where $\boldsymbol{\rho_n}=(\mathbf{x},\mathbf{y})^{\mathbf{T}}$ and $\mathbf{r_n}$ are the solution and residual vectors at the n-th iteration. However, a parameter τ is introduced with S_p in $\mathcal{P}_H$ to improve performance. This approach has been previously used in [18] in a similar context with $\tau=3/5$.

5.2 The Preconditioner $\mathcal{P}_{nsn}$

Following the approach described in [13], the complex valued system is transformed into

$$\mathcal{A}=\mathbf{T}^*\mathcal{A}_T\mathbf{T} \tag{20}$$

where

$$\mathcal{A}_T=\begin{bmatrix}(1+\beta\omega^2)^{1/2}M & K & 0 & -D^T\\ K & \beta^{-1}(1+\beta\omega^2)^{1/2}M & -D & 0\\ 0 & -D^T & 0 & 0\\ -D & 0 & 0 & 0\end{bmatrix} \tag{21}$$

and $\mathbf{T} = \begin{bmatrix} T \otimes I_n & 0 \\ 0 & T \otimes I_m \end{bmatrix}$ with $T = (1+\beta\omega^2)^{1/4} \begin{bmatrix} (1+\beta\omega^2)^{1/2} & -i \\ 0 & 1 \end{bmatrix}$. The operator $\otimes$ is a Kronecker product and $I_k \in R^k$ is the identity matrix. This transformation splits the the original system into

$$\begin{aligned} \mathcal{A}_T \mathbf{y_1} &= \mathbf{c_1} \\ \mathcal{A}_T \mathbf{y_2} &= \mathbf{c_2}. \end{aligned} \tag{22}$$

where $c = c_1 + ic_2 = (\mathbf{T}^*)^{-1}\mathbf{b}$ and $y = y_1 + ic_2 = \mathbf{T}^{-1}\mathbf{x}$. Hence one can solve two real systems, which may be done in parallel, instead of solving one single complex valued linear system. As noted in [12], an effective preconditioner to solve (21) is

$$\mathcal{P}_{nsn} = \begin{bmatrix} \mathbf{P} & 0 \\ 0 & \mathbf{S} \end{bmatrix}$$

where $\mathbf{P} = \begin{bmatrix} P & 0 \\ 0 & \beta P \end{bmatrix}$, $\mathbf{S} = \beta \begin{bmatrix} S & 0 \\ 0 & \frac{1}{\beta} S \end{bmatrix}$, $P = M + \sqrt{\beta}(F + \omega M)$ and $S = D(M + \sqrt{\beta}(F + \omega M))^{-1} D^T$.

In [13], the action of the block S in $\mathbf{S}$ is replaced using two steps. In the first step each block S is replaced by

$$S = \widehat{S}_{CH} = (\sqrt{\beta} M_p^{-1} + (1 + \sqrt{\beta}\omega) K_p^{-1})^{-1} \tag{23}$$

using [8]. In the second step the action of S is replaced by preconditioned Richardson iteration given by

$$\widehat{S} = S \left(I - \prod_{i=1}^{r} (I - \tau_i \widehat{S}_{CH} S)^i \right)^{-1} \tag{24}$$

for $i = 1, 2, 3, ...$ Here I is an identity matrix of proper order. Moreover, it is noted that the condition

$$1 - \prod_{i=1}^{r} (1 - \tau_i \lambda)^i > 0 \text{ for any } \lambda \in (0, 1] \tag{25}$$

is sufficient to guarantee that $\widehat{S}$ is positive definite. Furthermore, if $\tau_1 > 0$ and $\tau_i > 1$ then $\widehat{S}$ is symmetric positive definite and $\widehat{S} \sim S$. While replacing S with $\widehat{S}$ yields a good approximation, incorporating preconditioned Richard iteration yields further improvements. This can be confirmed by the numerical experiments in [13].

5.3 Spectrally equivalent approximations

In the numerical tests in Section 6, the following approximations are used.

(i) Based on the analysis in [22], a system with a mass matrix, whether in the velocity space or the pressure space, i.e., $M_{\vec{y}}$ and M_p, is solved by the Chebyshev semi-iteration method, terminated after 20 iterations.

(ii) All blocks of the form K_p, $M_{\vec{y}} + \sqrt{\widetilde{\beta}} K_{\vec{y}}$, $M_{\vec{y}} + \sqrt{\beta}(K_{\vec{y}} + \omega M_{\vec{y}})$, $M_p + \sqrt{\widetilde{\beta}} K_p$ are replaced by one V-cycle of an Algebraic Multigrid (AMG) solver.

Table 1 summarizes the computational costs for each preconditioner per iteration.

Preconditioner	Operations
$\mathcal{P}_{nsn}$	Four AMG solves and two solves with Chebyshev semi-iterations.
$\mathcal{P}_F(Uzawa)$	Four Uzawa iterations, each requiring one solution with $P_{\mathcal{H}}$ (eight AMG solves and four solves with Chebyshev semi-iterations).
$\mathcal{P}_F(FGMRES)$	Four FGMRES iterations, preconditioned by $\mathcal{P}_{\mathcal{H}}$ (eight AMG solves and four solves with Chebyshev semi-iterations).

Table 1: Computational complexity of the preconditioners, summary

6 Numerical tests

We illustrate the performance of the preconditioners using the test problem from [12], see also [11].

Problem 1. *Consider the Stokes optimal control problem on the unit square. The target velocity is $\vec{y}_d(x_1, x_2) = (y_{d,1}(x_1, x_2), y_{d,2}(x_1, x_2))$ is chosen as*

$$y_{d,1}(x_1, x_2) = 10\frac{\partial}{\partial x_2}(\varphi(x_1)\varphi(x_2)) \quad \text{and} \quad y_{d,2}(x_1, x_2) = -10\frac{\partial}{\partial x_1}(\varphi(x_1)\varphi(x_2)),$$

where $\varphi(z) = (1 - cos(0.8\pi z))(1 - z)^2$.

As in [3, 4] we discretize the problem using the stable Taylor-Hood Q2-Q1 pair of finite element spaces, namely, the state **u**, the control **f** and the adjoint **g** are discretized using piece-wise quadratic (Q2) basis functions, while the

Size	β								
	10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-6}	10^{-7}	10^{-8}	10^{-9}	10^{-10}
$\omega = \mathbf{10^{-4}}$									
4934	8	10	12	12	11	11	9	7	7
	0.098	0.118	0.140	0.140	0.129	0.128	0.107	0.086	0.086
19078	8	10	12	12	12	11	10	8	6
	0.370	0.454	0.537	0.537	0.538	0.498	0.456	0.372	0.291
75014	8	10	12	12	12	11	10	8	6
	1.549	1.901	2.241	2.243	2.244	2.072	1.897	1.553	1.211
297478	8	10	12	12	12	11	10	8	6
	6.870	8.381	9.899	9.898	9.896	9.145	8.380	6.849	5.335
$\omega = \mathbf{1}$									
4934	8	10	12	12	11	11	9	7	7
	0.098	0.118	0.14	0.14	0.129	0.129	0.107	0.087	0.087
19078	8	10	12	12	12	11	10	8	6
	0.371	0.455	0.539	0.538	0.539	0.497	0.455	0.373	0.291
75014	8	10	12	12	12	11	10	8	6
	1.553	1.897	2.24	2.242	2.243	2.074	1.895	1.552	1.209
297478	8	10	12	12	12	11	10	8	6
	6.846	8.363	9.875	9.897	9.888	9.129	8.363	6.846	5.332
$\omega = \mathbf{10^{4}}$									
4934	9	9	9	9	9	9	9	7	7
	0.108	0.107	0.107	0.108	0.108	0.107	0.108	0.104	0.087
19078	10	10	10	10	10	10	9	8	6
	0.454	0.454	0.455	0.454	0.455	0.456	0.414	0.374	0.292
75014	10	10	10	10	10	10	9	8	6
	1.893	1.897	1.912	1.897	1.901	1.898	1.725	1.558	1.214
297478	10	10	10	10	10	10	9	8	6
	8.361	8.346	8.362	8.368	8.374	8.368	7.599	6.850	5.334

Table 2: Preconditioner $\mathcal{P}_F$, outer solver FGMRES; each $\mathcal{H}$ block is approximated by 4 Uzawa iterations

pressure $\mathbf{p}$ and its corresponding adjoint $\mathbf{q}$ are discretized using piece-wise linear (Q1) basis functions.

The preconditioners are implemented in C++ and executed using the open source finite element library deal.ii ([6]) and the Algebraic Multigrid (AMG) solver from the Trilinos library ([20]). All experiments are performed on an Intel(R) Core(TM) i5 CPU 750 @ 2.67GHz-2.80GHz computer with 4GB RAM.

We compare the performance of $\mathcal{P}_F$ with that of $\mathcal{P}_{nsn}$, to match the numerical experiments from [12]. We solve the system, preconditioned by $\mathcal{P}_F$ with FGMRES and the system, preconditioned by $\mathcal{P}_{nsn}$ with MINRES. The results are presented in Tables 2 and 3 for $\mathcal{P}_F$, and in Table 4 for $\mathcal{P}_{nsn}$.

Size	β								
	10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-6}	10^{-7}	10^{-8}	10^{-9}	10^{-10}
$\omega = \mathbf{10^{-4}}$									
4934	8	9	9	8	7	6	5	4	2
	0.141	0.180	0.156	0.140	0.125	0.110	0.094	0.079	0.061
19078	8	9	9	8	7	6	5	4	3
	0.512	0.571	0.570	0.515	0.459	0.401	0.345	0.288	0.236
75014	8	9	9	8	7	6	5	4	3
	2.057	2.285	2.288	2.060	1.839	1.607	1.380	1.153	0.924
297478	8	9	9	8	7	6	5	4	3
	8.909	9.880	9.891	8.908	7.926	6.920	5.933	4.971	3.984
$\omega = \mathbf{1}$									
4934	8	9	9	8	7	6	5	4	2
	0.141	0.161	0.155	0.141	0.125	0.109	0.094	0.092	0.057
19078	8	9	9	8	7	6	5	4	3
	0.513	0.569	0.570	0.516	0.455	0.399	0.343	0.288	0.230
75014	8	9	9	8	7	6	5	4	3
	2.058	2.291	2.291	2.058	1.831	1.603	1.376	1.148	0.922
297478	8	9	9	8	7	6	5	4	3
	8.906	9.894	9.889	8.896	7.913	6.921	5.936	4.952	3.969
$\omega = \mathbf{10^{4}}$									
4934	5	5	5	5	5	5	5	4	2
	0.140	0.134	0.094	0.094	0.093	0.094	0.095	0.079	0.048
19078	5	5	5	5	5	5	5	4	3
	0.342	0.342	0.343	0.344	0.343	0.343	0.342	0.285	0.251
75014	5	5	5	5	5	5	5	4	3
	1.373	1.374	1.378	1.405	1.378	1.377	1.377	1.149	0.924
297478	5	5	5	5	5	5	5	4	3
	5.941	5.945	5.965	5.970	5.950	5.952	5.937	4.961	3.969

Table 3: Preconditioner $\mathcal{P}_F$, outer solver FGMRES; each $\mathcal{H}$ block is approximated by four FGMRES iterations preconditioned by the block-lower triangular preconditioner $\mathcal{P}_{\mathcal{H}}$

Size	β								
	10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-6}	10^{-7}	10^{-8}	10^{-9}	10^{-10}
$\omega = \mathbf{10^{-4}}$									
4934	36	37	37	35	33	29	27	25	23
	0.188	0.206	0.191	0.182	0.186	0.151	0.141	0.131	0.121
19078	38	40	41	39	37	33	29	27	25
	0.75	0.788	0.828	0.767	0.731	0.652	0.573	0.533	0.498
75014	40	44	45	43	39	37	33	29	25
	3.327	3.65	3.73	3.573	3.248	3.073	2.747	2.413	2.083
297478	44	45	47	45	41	39	35	31	27)
	16.289	16.603	17.37	16.625	15.143	14.425	12.949	11.472	10.002
$\omega = \mathbf{1}$									
4934	36	37	37	35	32	29	27	25	23
	0.188	0.207	0.191	0.182	0.166	0.151	0.141	0.131	0.12
19078	38	40	41	39	37	33	29	27	25
	0.75	0.786	0.81	0.774	0.733	0.658	0.576	0.533	0.496
75014	40	44	45	43	39	37	33	29	25
	3.327	3.656	3.742	3.577	3.238	3.069	2.745	2.418	2.083
297478	42	45	47	45	41	39	35	31	27
	15.548	16.633	17.388	16.641	15.173	14.416	12.932	11.476	10.012
$\omega = \mathbf{10^{4}}$									
4934	19	21	23	23	23	25	27	23	21
	0.175	0.127	0.158	0.145	0.161	0.131	0.179	0.159	0.125
19078	21	23	25	25	25	27	29	25	23
	0.415	0.457	0.495	0.497	0.493	0.535	0.575	0.495	0.455
75014	25	25	27	27	27	31	33	27	23)
	2.083	2.08	2.248	2.247	2.236	2.586	2.748	2.249	1.919
297478	27	28	29	31	31	33	37	29	25
	9.995	10.373	10.731	11.47	11.473	12.221	13.695	10.738	9.265

Table 4: Preconditioner $\mathcal{P}_{nsn}$, outer solver MINRES; the Schur complement approximation S is approximated by two Richardson's iterations

7 Conclusions

The block matrix preconditioner, previously applied to Stokes optimal control problems has been shown to have the same robust behaviour when applied for time-harmonic state equations. It outperforms previously published methods, such as the method proposed in [12].

The results show that our preconditioner converges with a number of iterations less than twice times the difference in number of iterations for a matrix with condition number 3 instead of condition number 2. In addition, the method in [12] is more complicated. It involves a block-diagonal preconditioner for the block 4×4 block matrix in (8). The preconditioning does not show a fully parameter-independent behaviour and needs many iterations, see [12, 13]. The method is not competitive with our method that involves solving two Stokes problems per outer iteration. Since for our method the spectrum of the outer preconditioned matrix is clustered and corresponds to a condition number, bounded by 2, it needs a few outer iterations.

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