Spectral analysis of finite-difference approximations of $1-d$ waves in non-uniform grids

Davide Bianchi and Stefano Serra-Capizzano

Preserving the finite positive velocity of propagation of continuous solutions of wave equations is one of the key issues, when building numerical approximation schemes for control and inverse problems. And this is hard to achieve uniformly on all possible ranges of numerical solutions. In particular, high frequencies often generate spurious numerical solutions, behaving in a pathological manner and making the propagation properties of continuous solutions fail. The latter may lead to the divergence of the "most natural" approximation procedures for numerical control or identification problems.

On the other hand, the velocity of propagation of high frequency numerical wave-packets, the so-called group velocity, is well known to be related to the spectral gap of the corresponding discrete spectra. Furthermore most numerical schemes in uniform meshes fail to preserve the uniform gap property and, consequently, do not share the propagation properties of continuous waves.

However, recently, S. Ervedoza, A. Marica and the E. Zuazua have shown that, in $1-d$, uniform propagation properties are ensured for finite-difference schemes in suitable non-uniform meshes behaving in a monotonic manner. The monotonicity of the mesh induces a preferred direction of propagation for the numerical waves. In this way, meshes that are suitably designed can ensure that all numerical waves reach the boundary in an uniform time, which is the key for the fulfillment of boundary controllability properties.

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Supported in part by INDAM-GNCS and in part by the grant FAR 2015/16 of the Università dell'Insubria.

In this paper we study the gap of discrete spectra of the Laplace operator in $1-d$ for non-uniform meshes, analysing the corresponding spectral symbol, which allows to show how to design the discretization grid for improving the gap behaviour. The main tool is the study of an univariate monotonic version of the spectral symbol, obtained by employing a proper rearrangement.

The analytical results are illustrated by a number of numerical experiments. We conclude discussing some open problems.

Key words: Wave equation, boundary control and observation, finite-differences, velocity of propagation, non-uniform grids, spectral symbol, spectral gap.

AMS Classification: 65F10, 65N22, 15A18, 15A12, 47B65

1 Motivation

Even if our analysis applies to wave equations with variable coefficients, for the sake of simplicity, we focus on the constant coefficient $1-D$ wave equation

$$\begin{cases} \partial_{tt}u - \partial_{xx}u = 0, & (t,x) \in (0,T)\times(0,1), \\ u(t,0) = u(t,1) = 0, & t \in (0,T), \\ u(0,x) = u^0(x),\ \partial_t u(0,x) = u^1(x), & x \in (0,1). \end{cases} \tag{1}$$

As a consequence of the Fourier or d'Alembert explicit representation formulas, it is easy to see that the solutions of this model fulfill the so-called *observability inequality*

$$||(u^0,u^1)||^2_{H^1_0(0,1)\times L^2(0,1)} \leq C_{obs}\int_0^T |\partial_x u(t,1)|^2 dt, \tag{2}$$

for a suitable constant $C_{obs} = C_{obs}(T) > 0$ provided $T \geq 2$.

Inequality (2) ensures that all waves propagating in space-time according to the wave equation reach the extreme $x=1$ in time $T=2$. This is in agreement with the well-known propagation properties of solutions of the wave equation. Solutions of the wave equation can be decomposed into waves traveling towards left and right in the x-direction at velocity $=1$ and bouncing back and forth when reaching the boundary points ($x=0$ and $x=1$) in an odd manner. This dynamical behaviour allows showing also that the observability inequality fails for $T<2$.

This inequality, by duality, ensures the exact controllability of the wave equation, when the control acts on the boundary $x=1$, see e.g. [19].

In this paper we analyze to which extent finite-difference approximations of (1) preserve this property, uniformly with respect to the mesh-size parameter. We focus on semi-discrete space-discretizations in which the time-variable is kept continuous.

In the pioneering works [11, 12] it was shown that this uniform (with respect to the mesh-size parameter) observability property may fail for classical numerical methods (such as finite-difference or finite-elements) on uniform meshes.

This phenomenon has been further analyzed and explained for finite-difference and finite-element methods on uniform meshes and it is by now rather well-understood. We refer to the survey articles [8, 31] for an account of the most recent developments in the subject and to [18] for the first results in this direction, based on the use of discrete multiplier techniques.

In those articles, in particular, it is shown that the uniform propagation of numerical waves is related to the preservation of an uniform gap for the corresponding discrete spectra. More precisely, it is shown that the lack of uniform observability is due to the existence of high-frequency numerical wave packets that propagate with a vanishing (as the mesh-size tends to zero) group velocity, which is an obstruction for uniform observability. It was also proved that this vanishing group velocity is due to the lack of uniform gap properties of the discrete spectra.

Thus, roughly, on uniform meshes, the discrete Laplacian looses the needed uniform gap condition, producing high frequency numerical wave packets that propagate very slowly and that, accordingly, are not uniformly observable from the boundary in a finite time horizon independent of the mesh-size.

More recently, in [20], this problem was addressed in the more general context of non-uniform meshes built through a diffeomorphic deformation of a uniform one. Using microlocal tools, the notions of numerical symbol and bicharacteristic rays were introduced in agreement with, for instance, [30]. It was also shown both analytically and through numerical simulations, that high-frequency numerical wave packets follow, in space-time, the path indicated by the numerical bicharacteristics rays.

Inspired on the analysis in [20], more recently, in [7], the problem of building non-uniform meshes that preserve the uniform observability inequality was addressed successfully. In that paper, using discrete multiplier techniques, it was proved that meshes that enjoy a suitable monotonicity condition ensure the uniform velocity of propagation of numerical waves and preserve the property of observability. In that paper it was also shown through numerical experiments that the spectra of the corresponding discretizations of the Laplacian seem to preserve the uniform gap condition.

In this article we focus on the spectral problem. Employing specific tools allowing to analyse the asymptotic properties of the spectra of the tri-diagonal matrices arising in the finite-difference discretization of the Laplacian by means of non-uniform meshes (see [25]), the so-called spectral symbols that we introduce in the next section, we show that, indeed, the meshes introduced in [7] ensure the uniform gap condition in some average sense.

In addition, the analysis associated with the spectral symbol is quite general (see [28, 22, 23, 10] and references therein) and hence it can be used for more general operators and/or more advanced/more precise discretizations, leading to matrices with larger bandwidth or even dense (as in the case of fractional operators [6]).

Further work is required to actually prove uniform spectral gap conditions of discretizations of the Laplacian, and not only in an averaged sense as we do in here. But the results in this paper confirm the expected spectral behaviour, showing that the techniques developed for the analysis of discrete spectra may be of use in the context of ensuring correct propagation properties for numerical waves.

2 Preliminaries on spectral symbols

Let us consider the $1-d$ elliptic boundary value problem,

$$\begin{cases} -\dfrac{\mathrm{d}}{\mathrm{d}x}\left(a(x)\dfrac{\mathrm{d}}{\mathrm{d}x}u(x)\right) = f(x), \qquad x \in (0,1), \\ u(0) = u(1) = 0. \end{cases} \tag{3}$$

We assume that $a = a(x)$ is measurable, bounded above and below by positive constants.

The classical variational theory ensures that, for every right hand side term $f \in \mathrm{H}^{-1}(0,1)$, there is a unique weak solution $u \in \mathrm{H}^1_0(0,1)$. Furthermore, if $a = a(x)$ is Lipschitz continuous and $f \in \mathrm{L}^2(0,1)$, the solution u is more regular, namely, $u \in \mathrm{H}^2(0,1)$.

The corresponding Sturm-Liouville eigenvalue problem is then as follows:

$$\begin{cases} -\dfrac{\partial}{\partial x}\left(a(x)\dfrac{\partial}{\partial x}\phi(x)\right) = \lambda\phi(x), \qquad x \in (0,1), \\ \phi(0) = \phi(1) = 0. \end{cases} \tag{4}$$

The associated eigenvalues $\{\lambda_j\}_{j\geq 1}$ constitute an increasing sequence of positive real numbers tending to infinity, and the corresponding eigenfunctions $\{\phi_j(x)\}_{j\geq 1}$ can be normalized to form an orthonormal basis of $\mathrm{L}^2(0,1)$.

When $a = a(x)$ is Lipschitz (in fact BV-regularity suffices, see [9]), the spectrum satisfies the following gap condition:

$$\mu_{j+1} - \mu_j \geq \gamma > 0, \forall j \geq 1, \tag{5}$$

where $\mu_j = \sqrt{\lambda_j}$, $\gamma > 0$ being the gap. Let us recall now the classical Ingham inequality ([16]).

Theorem 1. *Assume that the strictly increasing sequence $\{\mu_j\}_{j\in\mathbb{N}}$ of real numbers satisfies the gap condition* (5). *Then, for all $T > 2\pi/\gamma$ there exist two positive constants c_1 and c_2 depending only on γ and T such that*

$$c_1 \sum_{j=1}^{\infty} |a_j|^2 \leq \int_0^T \left| \sum_{j=1}^{\infty} a_k \mathrm{e}^{\mathrm{i}t\mu_j} \right|^2 \leq c_2 \sum_{j=1}^{\infty} |a_j|^2, \tag{6}$$

for every complex sequence $\{a_j\}_{j\in\mathbb{N}} \in l^2$*, where*

$$c_1 = c_1(T,\gamma) = \frac{2T}{\pi}\left(1 - \frac{4\pi^2}{T^2\gamma^2}\right) > 0, \qquad c_2 = c_2(T,\gamma) = \frac{8T}{\pi}\left(1 + \frac{4\pi^2}{T^2\gamma^2}\right) > 0$$

and l^2 *is the Hilbert space of square summable sequences,*

$$l^2 = \left\{ \{a_j\} \, : \, \|a_j\|_{l^2}^2 = \sum_{j=1}^{\infty} |a_j|^2 < \infty \right\}.$$

According to the above inequality (6), this gap condition suffices to ensure the boundary observability of the corresponding solutions of the associated wave equation in time $T > 2\pi/\gamma$.

We are interested in the spectrum of the corresponding finite-difference discretization schemes and their spectra, and, more precisely, on whether the gap condition is preserved uniformly for the corresponding spectra as the mesh-size tends to zero.

As mentioned above, the preservation of the spectral gap requires the design of suitable discretization grids. Strictly speaking, the tools we shall develop here do not allow to deal, strictly speaking, with the spectral gap but rather with an averaged version, but they allow to have a good understanding of the behaviour of the discrete spectra and its dependence on the grid behavior. The tools we shall develop here can also be applied for multi-dimensional problems (see [22, 23, 10]). However, as it is well-know, in higher dimensions, wave propagation can not be merely understood thorough the spectral gap, which, by the way, is not fulfilled for continuous multi-dimensional wave equations.

In this section we provide a brief description of the asymptotic distribution of the eigenvalues of the matrix of coefficients obtained by approximating the above elliptic PDE by finite-differences on a non-uniform grid described by a smooth mapping ϕ from $\overline{\Omega} = [0,1]$ onto $\overline{\Omega}$. In fact, as mentioned above, we restrict our attention to the $1-d$ case, even if a general theory, the theory of Generalized Locally Toeplitz (GLT) sequences, is available for multi-dimensional problems (see [28, 22, 23]).

The task of finding the asymptotic eigenvalue distribution is motivated, as mentioned above, by wave propagation considerations, but is also related to, e.g. the fine (superlinear) convergence of the method of conjugate gradients (CG) [2] or to the study of spectral branches [15].

Here we are mainly interested in the preservation of spectral gaps of the continuous spectra, (5), [7].

A discretization of (4) for some sequence of stepsizes h tending to zero leads to a sequence of linear systems of the form $A_n x_n = b_n$, A_n being a symmetric positive definite matrix of order n, where of course n depends on h, and tends to ∞ for $h \to 0$.

Before proceeding further, let us recall the definition of a Toeplitz matrix.

Definition 1 (Toeplitz matrix). Let T be an $n \times n$ matrix. We say that T is *Toeplitz* if it takes the form

$$T = \begin{bmatrix} t_0 & t_{-1} & t_{-2} & \cdots & \cdots & t_{-(n-1)} \\ t_1 & t_0 & t_{-1} & \ddots & & \vdots \\ t_2 & t_1 & \ddots & \ddots & \ddots & \\ \vdots & \ddots & \ddots & \ddots & t_{-1} & t_{-2} \\ \vdots & & \ddots & t_1 & t_0 & t_{-1} \\ t_{n-1} & \cdots & \cdots & t_2 & t_1 & t_0 \end{bmatrix}.$$

It is fully specified by the vector $\mathbf{v} = \left[t_{n-1} \cdots t_1\ t_0\ t_{-1} \cdots t_{-(n-1)}\right]$.
If the (i,j) element of the matrix T is denoted by $T_{i,j}$, then we have

$$T_{i,j} = T_{i+1,j+1} = t_{i-j}.$$

Given a function $p : [-\pi,\pi] \to \mathbb{C}$ such that $p \in L^1([-\pi,\pi])$, then the nth Toeplitz matrix associated to p is defined as

$$T_{i,j} = p_{i-j},$$

where $\{p_k\}_{k\in\mathbb{N}}$ are the Fourier coefficients of p.

We are interested in studying the spectral symbol or the asymptotic spectrum, i.e., a function describing asymptotically all the eigenvalues of $\{A_n\}_{n\in\mathbb{N}}$.

Definition 2 (Spectral symbol). A sequence of Hermitian matrices $\{A_n\}_{n\in\mathbb{N}}$ of order n with spectrum $\Lambda(A_n) \subset \mathbb{R}$, is said to have a *spectral symbol* or an *asymptotic spectrum* given by some measure σ if for all functions $F \in C_c(\mathbb{R})$ (i.e., continuous with compact support) there holds

$$\lim_{n\to\infty} \frac{1}{n} \sum_{\lambda\in\Lambda(A_n)} F(\lambda) = \int F(\lambda)\, d\sigma(\lambda), \tag{7}$$

where each eigenvalue is counted according to its multiplicity (and hence σ is a probability measure). If the limit (7) exists and takes the form

$$\lim_{n\to\infty} \frac{1}{n} \sum_{\lambda\in\Lambda(A_n)} F(\lambda) = \int_D F(\omega(x,\theta)) \frac{dxd\theta}{m(D)} \tag{8}$$

with domain $(x,\theta) \in D = [0,1]\times[-\pi,\pi] \subset \mathbb{R}^2$ with Lebesgue measure $m(D) > 0$, the function ω will be referred to as the *spectral symbol* of (A_n).

The most classical example of sequence of matrices having an asymptotic spectrum is given by Hermitian Toeplitz matrices $A_n = (t_{i-j})_{i,j=1,\dots,n}$ obtained from the Fourier coefficients of the Lebesgue integrable generating function $\omega(\theta) = \sum_{j\in\mathbb{Z}} t_j e^{\mathrm{i}j\theta}$, $\mathrm{i}^2 = -1$, see for instance [29] and references therein. Here the spectral symbol coincides with the generating function. In this case, for simplicity, we may consider $D = (-\pi,\pi)$, since the symbol is independent of x.

A much wider class of matrix sequences, where every sequence has its own spectral symbol, has been constructed in [28, 22, 23]. This is the class of GLT sequences which behaves as a large algebra containing the Toeplitz sequences generated by Lebesgue integrable functions and virtually any discretization by local methods of integral and partial differential equations (see [22, 3, 1, 5, 6] and references therein).

As already mentioned, we focus our attention on the $1-d$ case, addressing the spectral gap problem arising in the analysis of the propagation of numerical waves, as described in the the preceding Section 1.

The resulting linear systems are of tridiagonal type with coefficient matrices $\{A_n\}_{n\in\mathbb{N}}$ having entries which are samples of a:

$$A_n = \begin{bmatrix} a_{\frac{1}{2}}+a_{\frac{3}{2}} & -a_{\frac{3}{2}} & & & \\ -a_{\frac{3}{2}} & a_{\frac{3}{2}}+a_{\frac{5}{2}} & -a_{\frac{5}{2}} & & \\ & -a_{\frac{5}{2}} & \ddots & \ddots & \\ & & \ddots & \ddots & -a_{\frac{2n-1}{2}} \\ & & & -a_{\frac{2n-1}{2}} & a_{\frac{2n-1}{2}}+a_{\frac{2n-1}{2}} \end{bmatrix}, \tag{9}$$

with $a_t = a(ht)$, $h = (n+1)^{-1}$.

When $a(x)\equiv 1$, the matrix A_n reduces to the Toeplitz matrix $T_n(p)$ of size n,

$$T_n(p) = \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & \ddots & \ddots & \\ & & \ddots & \ddots & -1 \\ & & & -1 & 2 \end{bmatrix}, \tag{10}$$

characterized by the vector $\mathbf{v} = \begin{bmatrix}-1 & 2 & -1\end{bmatrix}$ and generated by $p(\theta) = 2-2\cos(\theta)$.

Note that the numbers $-1, 2, -1$ are the (nonzero) Fourier coefficients c_1, c_0, c_{-1} of p and represent also the stencil of the considered finite-difference formula. If the stencil is modified (for instance in order to obtain more precise discretization schemes), then we obtain Toeplitz matrices generated by a new function p having Fourier coefficients given by the entries of this new stencil [24].

A well-known fact from the theory of Toeplitz matrices is that

$$\{T_n(p)\}_{n\in\mathbb{N}}$$

has an asymptotic spectrum given by $\omega(\theta) = p(\theta)$ with $D = [-\pi,\pi]$, see for instance the seminal work by Szegö [14]. Note that, in the present case, again, the symbol does not depend on x.

In the more general case of variable coefficients, it follows from the Locally Toeplitz analysis of [28] that the matrix sequence $\{A_n\}_{n\in\mathbb{N}}$ of (9) has an asymptotic spectrum given by the symbol

$$\omega(x,\theta) = a(x)p(\theta)$$

with $D = (0,1)\times[-\pi,\pi]$ (see also [21]).

This result is in some sense natural since the samplings of a move along the diagonals of A_n smoothly (if a is smooth) and therefore also the algebraic structure of A_n looks like a Toeplitz matrix if we restrict the attention to a local portion, and this nice algebraic behavior has a natural counterpart in the global spectral behavior. As in the constant coefficient case, the change of the discretization scheme, i.e., of the stencil, will change only the polynomial p in the symbol, its degree, and hence the bandwidth of the resulting matrix [24]. Of course similar results holds if other approximation methods are employed, such as the isogeometric analysis [5] or the finite-elements [3].

Finally, we observe that the matrices $\{A_n\}_{n\in\mathbb{N}}$ are essentially of the same type as those which one encounters when dealing with sequences of orthogonal polynomials with varying coefficients. Here again Locally Toeplitz tools have been used for finding the distribution of the zeros of the considered orthogonal polynomials under very weak assumptions (only measurability) on the regularity of the coefficients [17].

A further variation, which plays a role in the spectral gap problem under consideration is the use of non-equispaced grids. Indeed, if the new grid of size n is obtained as the image under a continuously differentiable map $\phi : [0,1] \mapsto [0,1]$ of a uniform grid of the same size n or if the new grid can be approximated in this way (see e.g. [25, Definition 4.6]), then the corresponding matrix sequence $\{A_n\}_{n\in\mathbb{N}}$ has an asymptotic spectrum described by the symbol

$$\omega(x,\theta) = \frac{a(\phi(x))}{[\phi'(x)]^2}p(\theta) \quad \text{with} \quad D = (0,1)\times[-\pi,\pi], \tag{11}$$

where the formula shows three players: the differential operator with its weight, the finite-difference formula with its stencil, and the non-uniform gridding represented compactly by the function ϕ.

3 Analysis of the average spectral gap

Using centered finite-differences of order 2 precision on a non-uniform grid ϕ for the approximation of (3), which is bijective and C^1 from $[0,1]$ onto $[0,1]$, one obtains a matrix A_n and, from the GLT theory [22], as stated in Section 2, the corresponding distribution function for $\{A_n\}_{n\in\mathbb{N}}$ is $\omega(x,\theta)$ given in (11). Therefore, if n denotes the size of A_n, then for any continuous function F with bounded support, we have

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}F(\lambda_j(A_n)) = \frac{1}{\pi}\int_{[0,1]}dx\int_{[0,\pi]}F(\omega(x,\theta))\,d\theta, \tag{12}$$

with

$$\omega(x,\theta) = \frac{a(\phi(x))}{[\phi'(x)]^2} p(\theta).$$

Now we give conditions on ϕ such that the square root of the eigenvalues $\mu_j = \sqrt{\lambda_j(A_n)}_{j=1,\ldots,n}$, ordered in a increasing way grow linearly, so that the gap property is preserved. Obviously, each μ_j also depends on the size of the matrix A_n, and/or the mesh-size h, but this dependence will not be made explicit in the notation for the sake of simplicity. Note however that, the discrete versions of λ_j and μ_j as defined above, do not approximate the corresponding spectra of the continuous problem. This is so since the matrices A_n as defined above need to be amplified by a factor h^2 so that they approximate the differential operator $-\partial_x(a(x)\partial_x\cdot)$.

Definition 3. We define the gap $\delta_j = \mu_{j+1} - \mu_j$ (which also depends on n and/or h). Having the property of *preserving the spectral gap* means that there exists a positive constant $\gamma > 0$, independent of j and n such that

$$\delta_j \geq \frac{\gamma}{n}$$

for every j and n.

Observe that the factor n on the denominator on the right hand side term of the inequality gap is due to the need of normalization, as mentioned above.

The tools of the previous section do not allow to handle this gap property. However, they allow to deal with the following notion of '*average spectral gap*'.

Definition 4. For every $[a,b] \subset \mathrm{Rg}(\sqrt{\omega(\cdot,\cdot)})$ with $a < b$ define

$$\delta_n[a,b] = \frac{1}{N_n[a,b]} \sum_{\mu_j \in [a,b]} \delta_j, \tag{13}$$

where $N_n[a,b]$ is the cardinality of the μ_j belonging to $[a,b]$, that is

$$N_n[a,b] = |\{j : \ \mu_j \in [a,b]\}|.$$

We say that the '*average spectral gap*' is preserved on $[a,b]$ if

$$L = \inf_{a<b} \liminf \delta_n[a,b] \frac{n}{\pi} > 0. \tag{14}$$

Note that this average spectral gap is related to the classical one in Ingham's inequality through the relation

$$\gamma = L\pi.$$

The problem of the spectral gap, in terms of the GLT symbol, can be rephrased as follows.

1. Let consider $s(x,\theta) = \sqrt{\omega(x,\theta)}$. By (11), we have

$$s(x,\theta) = 2\sqrt{a(\phi(x))}\sin(\theta/2)/\phi'(x);$$

2. Let $\psi_s(y)$ be the function $\psi_s(y) = \mu\left((x,\theta) \in [0,1]\times[0,\pi] : s(x,\theta) \leq y\right)$ defined on the range of s and taking values in $[0,\pi]$: the function $\psi_s : \mathrm{Rg}(s) \to [0,\pi]$ is such that $\mathrm{Rg}(s) = \mathbb{R}_0^+$ if ϕ' vanishes at some point in $[0,1]$;
3. Consider the inverse function $\psi_s^{(-1)} : [0,\pi] \to \mathbb{R}_0^+$ and observe that

$$\left(\psi_s^{(-1)}\right)'(y_0) = \frac{1}{(\psi_s)'(x_0)}, \quad \psi_s(x_0) = y_0;$$

4. The desired condition on the spectral gap is

$$\inf\left(\psi_s^{(-1)}\right)' > 0,$$

i.e.,

$$\sup(\psi_s)' < \infty.$$

The optimal case would be when $(\psi_s)'$ identically constant so that the quantities $\sqrt{\lambda_j(A_n)}$ form, up to some negligible error, a line.

Remark 1.
Concerning (12), since $\psi_\omega^{(-1)}$ is a rearrangement of ω, we have (see [4], [Theorem 3.4])

$$\frac{1}{\pi}\int_{[0,1]} dx \int_{[0,\pi]} F(\omega(x,\theta))\, d\theta = \frac{1}{\pi}\int_0^{\pi} F(\psi_\omega^{(-1)}(y))dy$$

which could simplify a lot the understanding of (12) if $\psi_\omega^{(-1)}$ is identified.

Let us now make some considerations in order to obtain the desired condition to preserve the spectral gap, namely $\sup(\psi_s)' < \infty$, as observed in the preceding point 4.

First, since the new griding ϕ is actually a reparametrization of the unit interval $[0,1]$ and, in light of the definition of ψ_s, in general, we can relax the hypothesis on ϕ, considering ϕ to be just Lipschitz continuous (rather then differentiable) and monotonic increasing with $\phi(0) = 0$ and $\phi(1) = 1$.

Moreover, we restrict ourselves to the class of functions $\{a : [0,1] \to (0,+\infty) \mid a \in C([0,1])\}$. Then without a big effort it is possible to show that for every continuous, positive function

$$\hat{\phi}' : [-1,1] \to [0,+\infty)$$

not identically zero, and for every continuous extension $\hat{a}$ of a to the interval $[-1,1]$, the following Cauchy problem

$$\begin{cases} \phi'(x) = \beta\sqrt{\hat{a}(\phi(x))}\hat{\phi}'(x), & \beta > 0, \\ \phi(0) = 0, \end{cases} \tag{15}$$

admits a non identically vanishing solution ϕ on the unit interval $[-1,1]$ such that $\phi(1) = 1$, with

$$\beta = \frac{\int_0^1 \left(\sqrt{a(t)}\right)^{-1} dt}{\int_0^1 \hat{\phi}'(x)dx}.$$

In other words, a solution ϕ of (15) is such that $\phi_{|[0,1]}$ is a reparametrization of the unit interval, and therefore with that choice of the functional class for the diffusion coefficient $a(x)$, without loss of generality, we can concentrate ourselves only on the case of constant diffusion coefficient $a(x) \equiv 1$, with $s(x,\theta) = 2\sin(\theta/2)/\phi'(x)$.

We can state now the following result.

Proposition 1. *Let us fix the diffusivity coefficient $a(x) \equiv 1$, and let $\phi \in C^2([0,1])$ be a reparametrization of the unit interval such that $\phi'' > 0$ (respectively, $\phi'' < 0$) on $[0,1]$.*

Then $\sup(\psi_s)' < \infty$. In particular, if $\phi'(0) = 0$ (respectively $\phi'(1) = 0$), then $\mathrm{Rg}(\psi_s) = \mathbb{R}_0^+$ and $\psi_s'(y) \to 0$ as $y \to +\infty$.

Proof. Suppose $\phi'' > 0$ and $\phi'(0) > 0$. It is not difficult to see that

$$\begin{aligned}
\psi_s(y) &= \mu\left\{(x,\theta) \in [0,1]\times[0,\pi] : \frac{2\sin(\theta/2)}{\phi'(x)} \le y\right\} \\
&= \int_0^1 \min\left\{2\arcsin\left(\min\left\{\phi'(x)\frac{y}{2};1\right\}\right);\pi\right\} dx \\
&= 2\int_0^1 \arcsin\left(\min\left\{\phi'(x)\frac{y}{2};1\right\}\right) dx \\
&= 2\int_0^{\min\left\{(\phi')^{(-1)}\left(\frac{2}{y}\right);1\right\}} \arcsin\left(\phi'(x)\frac{y}{2}\right) dx + \pi\left(1 - \min\left\{(\phi')^{(-1)}\left(\frac{2}{y}\right);1\right\}\right),
\end{aligned}$$

namely

$$\psi_s(y) = \begin{cases} 2\int_0^1 \arcsin\left(\phi'(x)\frac{y}{2}\right) dx & \text{for } y \in [0, \frac{2}{\phi'(1)}], \\ \pi\left(1 - (\phi')^{(-1)}\left(\frac{2}{y}\right)\right) + 2\int_0^{(\phi')^{(-1)}\left(\frac{2}{y}\right)} \arcsin\left(\phi'(x)\frac{y}{2}\right) dx & \text{for } y \in [\frac{2}{\phi'(1)}, \frac{2}{\phi'(0)}]. \end{cases}$$

Let us observe that

$$\begin{aligned}
\int_0^1 \frac{\phi'(x)}{\sqrt{1-\left(\frac{\phi'(x)}{\phi'(1)}\right)^2}} dx &= \int_0^1 \frac{\sqrt{\phi'(1)}\phi'(x)}{\sqrt{1+\frac{\phi'(x)}{\phi'(1)}}} \frac{1}{\sqrt{\phi'(1)-\phi'(x)}} \\
&\le \frac{(\phi'(1))^{3/2}}{\min_{[0,1]}\{\phi''(\xi)\}\sqrt{1+\frac{\phi'(0)}{\phi'(1)}}} \int_0^1 \frac{1}{\sqrt{1-x}} < \infty. \qquad (16)
\end{aligned}$$

Then, for $y \in \left[0, \frac{2}{\phi'(1)}\right]$,

$$\psi_s'(y) = \int_0^1 \frac{\phi'(x)}{\sqrt{1-\left(\frac{y\phi'(x)}{2}\right)^2}}dx \leq \int_0^1 \frac{\phi'(x)}{\sqrt{1-\left(\frac{\phi'(x)}{\phi'(1)}\right)^2}}dx < \infty, \tag{17}$$

by (16), and for $y \in \left[\frac{2}{\phi'(1)}, \frac{2}{\phi'(0)}\right]$ we have that

$$\psi_s'(y) = \int_0^{(\phi')^{(-1)}\left(\frac{2}{y}\right)} \frac{\phi'(x)}{\sqrt{1-\left(\frac{y\phi'(x)}{2}\right)^2}}dx = \left(\frac{\phi'(t)}{\phi'(1)}\right)^2 \int_{w_0}^1 \frac{g(w)\phi'(w)}{\sqrt{1-\left(\frac{\phi'(w)}{\phi'(1)}\right)^2}}dw,$$

where $t = (\phi')^{(-1)}\left(\frac{2}{y}\right)$ and

$$w = (\phi')^{(-1)}\left(\frac{\phi'(1)}{\phi'(t)}\phi'(x)\right), \qquad w_0 = (\phi')^{(-1)}\left(\frac{\phi'(0)\phi'(1)}{\phi'(t)}\right) \in [0,1],$$

$$g(w) = \frac{\phi''(w)}{\phi''\left((\phi')^{(-1)}\left(\frac{\phi'(t)\phi'(w)}{\phi'(1)}\right)\right)}.$$

It is straightforward now to see that

$$\int_0^{(\phi')^{-1}\left(\frac{2}{y}\right)} \frac{\phi'(x)}{\sqrt{1-\left(\frac{y\phi'(x)}{2}\right)^2}}dx \leq \frac{4\max_{[0,1]}\{\phi''(w)\}}{(y\phi'(1))^2\min_{[0,1]}\{\phi''(w)\}} \int_0^1 \frac{\phi'(w)}{\sqrt{1-\left(\frac{\phi'(w)}{\phi'(1)}\right)^2}}dw, \tag{18}$$

where the right hand side is bounded again by (16). Then, combining (17) and (18), we obtain the boundedness of ψ_s'. In particular, if we let $\phi'(0) \to 0$, from the above inequality we can infer that $\psi_s'(y) \to 0$ if $y \to 2/\phi'(0)$. The thesis then follows. The case $\phi'' < 0$ can be proved by the means of slight modifications of the previous argument.

Corollary 1. *Let $a(x) > 0$ be a continuous positive function on $[0,1]$. Then there exists a reparametrization ϕ of the unit interval such that $\sup(\psi_s)' < \infty$.*

Proof. It is a consequence of the above Proposition and our preliminary considerations. Indeed, let $\hat{\phi}$ be a reparametrization of the unit interval that satisfies the hypothesis of Proposition 1 and extend it C^2-continuously to the interval $[-1,1]$. Let Φ be a solution of (15) and define $\phi := \Phi_{|[0,1]}$. Then for every $x \in [0,1]$ it holds that

$$\hat{s}(x,\theta) = \frac{2\sqrt{a(\phi(x))}\sin(\theta/2)}{\phi'(x)} = \frac{2\sin(\theta/2)}{\beta\hat{\phi}(x)},$$

and the thesis follows from the proof of the previous Proposition 1.

A couple of immediate examples of such reparametrization functions ϕ are

$$\phi_1(x) = \frac{e^x - 1}{e-1}, \qquad \phi_2(x) = \frac{\log(1+x)}{\log(2)}, \qquad \phi_3(x) = x^2. \tag{19}$$

It is easy to check that they satisfy the hypothesis of Proposition 1, in particular they are characterized by $\phi_1'' > 0$, $\phi_2'' < 0$, $\phi_3'(0) = 0$. In the following Figure 1 we compare the distributions $\psi_s^{(-1)}$, with constant diffusivity $a(x) \equiv 1$, that arise from the uniform grid $id(x) = x$ and from the different reparametrizations ϕ_1 and ϕ_2 introduced above.

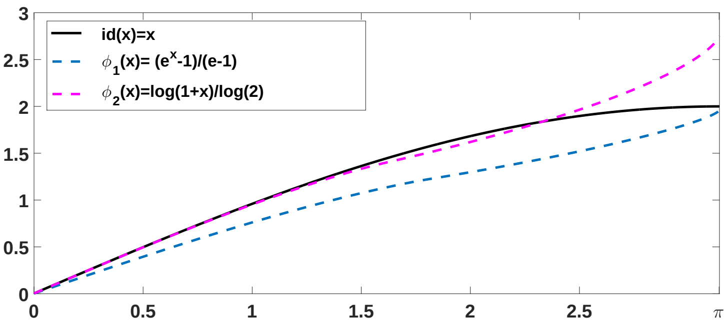


Fig. 1 Spectra for different $\psi_s^{(-1)}$ distributions.

In Table 1 we report instead the minimum of the difference quotients of the distribution $\psi_s^{(-1)}$ for different choices of the reparametrization ϕ of the unit interval, namely

$$\min_{k=1,\cdots,n-1} \left\{ \frac{\psi_{s,\phi}^{(-1)}(\alpha_{i+1}) - \psi_{s,\phi}^{(-1)}(\alpha_i)}{\alpha_{i+1} - \alpha_i} \right\},$$

where $\{\alpha_j\}_{j=1}^n$ is a uniform grid on $[0,\pi]$ of size n. It is possible to observe that the minimum in the case of uniform grid id is several orders of magnitude smaller than the minima of the non-uniform grids ϕ_i. The latter ones appear to be bounded away from 0 since if we increase they decrease at a lower rate and tend to stabilize. This is in accordance to Proposition 1.

Grid reparametrization	minimum difference quotient for various n			
	1×10^3	3×10^3	6×10^3	1×10^4
$id(x) = x$	2,4723e-6	2,7434e-7	6,8562e-8	2,4679e-8
$\phi_1(x) = (e^x-1)/(e-1)$	1,6697	1,6626	1,4637	0,9968
$\phi_2(x) = \log(1+x)/\log(2)$	1,6567	1,6531	1,6340	1,4393
$\phi_3(x) = x^2$	1.5227	1,9992	1,8833	1,7840

Table 1 The minimum difference quotient of $\psi_s^{(-1)}$ for different reparametrization choices at increasing grid sizes n.

To double-check numerically Corollary 1, let us consider the diffusion coefficient $a(x) = (1+x)^2$. From our previous considerations, if we choose the reparametrization

$$\phi(x) = e^{\frac{\log(2)(e^x-1)}{e-1}} - 1,$$

it is easy to see that

$$\frac{\sqrt{a(\phi(x))}}{\phi'(x)} = \frac{e-1}{\log(2)e^x} = \frac{1}{\log(2)\phi_1'(x)} = \frac{1}{\hat{\phi}'(x)},$$

where we put $\hat{\phi}'(x) = \log(2)\phi_1'(x)$, with ϕ_1' as in (19), and therefore

$$\hat{s}(x,\theta) = \frac{2\sin(\theta/2)}{\hat{\phi}'(x)}. \tag{20}$$

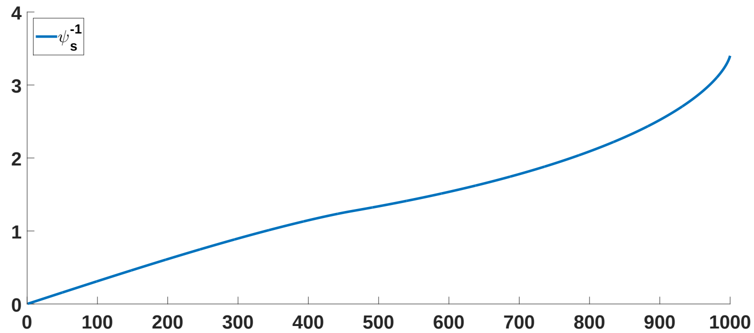
In Figure 2 we report the distribution $\psi_{\hat{s}}^{(-1)}$ and the distribution of the eigenvalues of the discretized operator $\frac{1}{(n+1)^2}A_{\phi,n}$, where

$$A_{\phi,n} = \mathrm{tridiag}_n\left[-\frac{2a_{j,1}}{h_{j+1}+h_j} \;\; \frac{2\left(a_{j,1}+a_{j,2}\right)}{h_{j+1}+h_j} \;\; -\frac{2a_{j,2}}{h_{j+1}+h_j}\right] \tag{21}$$

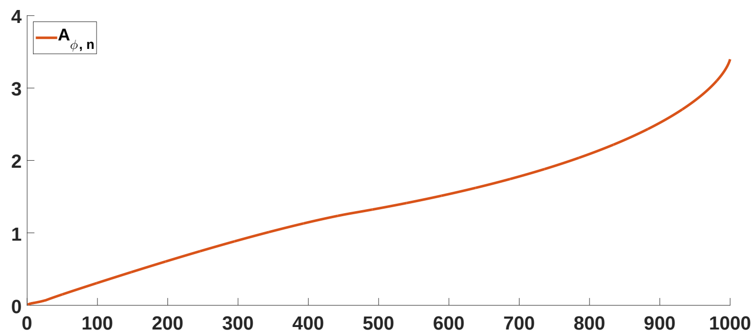
$$h_j = \phi(x_j) - \phi(x_{j-1}), \quad a_{j,1} := \frac{a\left(\phi(x_j) - \frac{h_j}{2}\right)}{h_j}, \quad a_{j,2} := \frac{a\left(\phi(x_j) + \frac{h_{j+1}}{2}\right)}{h_{j+1}},$$

$$x_j = \frac{j}{n+1}, \quad j = 1, \cdots, n+1.$$

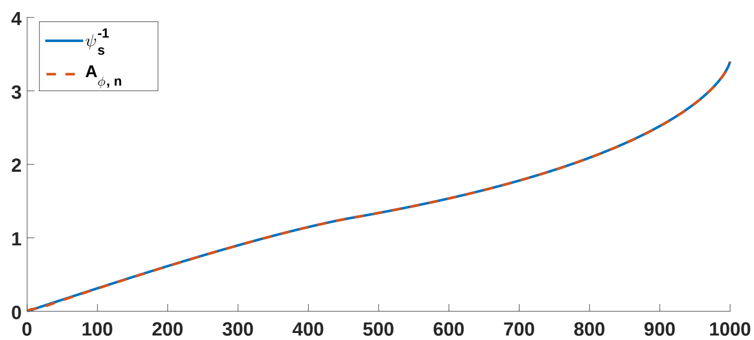
As it can be seen, the two distributions are equivalent up to small errors.



(a) $\psi_{\hat{s}}^{(-1)}$ for $\hat{s}(x,\theta)$ as in (20).



(b) Distribution of the eigenvalues of the scaled matrix $A_{\phi,n}$ as in (21) with $n = 1000$.



(c) Overlapping of the two distributions' graphics.

Fig. 2 Graphic comparison between different distributions.

4 Shishkin meshes and spectral gap

Here we have introduced non-uniform meshes in order to preserve the spectral gap condition at the discrete level. But there are some other, more classical PDE contexts in which the use of non-uniform meshes plays a key role. This occurs, for instance, when resolving boundary layers for convection-diffusion problem. We explore the behaviour of Shishkin meshes [26], designed to deal with boundary layers [27][pp. 465-466], from the point of view of the spectral gap.

Let us first recall the construction of Shiskhin meshes.

Take $r = (2/\alpha)\varepsilon \log(n)$ and set

$$\sigma = \min\{1/2, (2/\alpha)\varepsilon \log(n)\}.$$

In theory, we could choose any pair of positive parameters α and ε; in practice the Shishkin meshes are of interest if the resulting σ is much smaller than $1/2$ in order to deal with a boundary layer at the right boundary.

Now let $n = 2m$ be an even integer and divide each of $[0,1-\sigma]$ and $[1-\sigma,1]$ by an equidistant mesh with m subintervals. The resulting global mesh is a Shishkin mesh, see the next Figure 3.

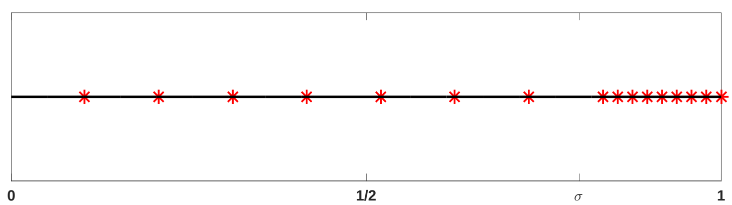


Fig. 3 Example of Shishkin mesh.

It is worth to state a simple but crucial observation.

Remark 2. Due to the previous definition of σ, the Shishkin meshes reduce to the uniform ones if $(2/\alpha)\varepsilon \log(n)$ is larger than $1/2$. Since α is a fixed constant, in an asymptotic sense, that is, for $n \to \infty$, we are forced to "imagine" that our small ε (usually used in connection with layers and so small that $\varepsilon \log(n)$ is small) is part of a sequence ε_h, $h = 1/n$, such that $\varepsilon_h log(n) \to 0$ when $n \to \infty$.

In the setting of Remark 2, when using Shishkin meshes, the resulting finite-difference matrix A_n^S has a very strong structure, and can be decomposed as ablock diagonal matrix B_n^S plus a symmetric rank 2 correction. Therefore, by the Cauchy interlacing Theorem, the spectral distribution symbols (if any) of the two sequences A_n^S and B_n^S are the same.

Furthermore the block diagonal matrix B_n^S has a strong structure since it can be written as

$$X_n^1 \oplus X_n^2, \;\; X_n^1 = \alpha_h T_m(2-2\cos(\theta)), \;\; X_n^2 = \beta_h T_m(2-2\cos(\theta)), \quad n = 2m,$$

so that the spectrum of B_n^S is the union of the spectra of $X_n^1 = \alpha_h T_m(2-2\cos(\theta))$ and of $X_n^2 = \beta_h T_m(2-2\cos(\theta))$.

If we normalize the matrix in such a way that $\beta_h = 1$, then $\alpha_h \to 0$, as $h \to 0$, due the conditions imposed by the Shishkin gridding. As a consequence the gap property is not satisfied because of $X_n^2 = T_m(2-2\cos(\theta))$ so that the usual problem at high frequencies appear. However, things are much worse because m eigenvalues cluster at zero since $X_n^1 = \alpha_h T_m(2-2\cos(\theta))$ and $\alpha_h \to 0$, as $h \to 0$.

More precisely,

$$\sigma = \frac{2}{\alpha}\varepsilon\log(n) \ll 1, \quad h = \frac{1}{n},$$
$$H_1^2 = \alpha_h^{-1} = \frac{4(1-\sigma)^2}{n^2},$$
$$H_2^2 = \beta_h^{-1} = \frac{4\sigma^2}{n^2},$$

so that

$$\frac{\alpha_h}{\beta_h} = \frac{\sigma^2}{(1-\alpha)^2} = \frac{4}{\alpha^2(1-\alpha)^2}\varepsilon^2\log^2(n) \ll 1.$$

Conversely, if we normalize putting $\alpha_h = 1$, then we will encounter a “symmetric pathological” situation with the usual problem at high frequencies due to X_n^1 plus a branch at infinity, due to the other matrix X_n^2.

This shows that some specific meshes, as Shiskhin ones, that have been traditionally designed to deal with boundary layer problems, do not behave well from the perspective of the spectral gap.

5 Conclusions, open problems and further comments

In this note we have described how the symbolic calculus of spectra allows to deal with the averaged spectral gap condition, related to the boundary observability of discrete waves. Note however that, as described in [7], the intention of uniform observability inequalities for discrete waves needs of a uniform spectral gap condition to fulfilled, not only in an average sense, but for each pair of eigenvalues. In those circumstances, Ingham’s inequalities reduces the problem to the uniform observability of the discrete eigenvectors. This is an issue that has not been considered in the present paper, even if it should be observed that if the gap is not preserved in average then it cannot be preserved for each pair, which implies that our study delivers necessary conditions.

A complete understanding of the observability properties of finite-difference discretizations of $1-d$ waves would require of further developments in these two directions. Namely, in the analysis of the strong gap and not only the averaged one, and the boundary observation of discrete eigenvectors.

Finally we just notice that the spectral machinery used in this paper and the tools for handling/designing the symbol are available for differential operators defined on general domains in several dimensions (see [23] and references therein): this gives the hope of treating analogous observability problems in dimension larger than one.

Acknowledgments

We are grateful to Professor Enrique Zuazua for the time he dedicated to us and to this work, for discussions and illuminating advices. Without his help this paper would have never came to light.

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