DISCONTINUOUS GALERKIN DISCRETIZATION OF THE HEAT EQUATION IN ANY DIMENSION: THE SPECTRAL SYMBOL*

PIETRO BENEDUSI†, CARLO GARONI‡, ROLF KRAUSE§, XIAOZHOU LI¶, AND STEFANO SERRA-CAPIZZANO‖

Abstract. The multidimensional heat equation, along with its more general version involving variable diffusion coefficients, is discretized by a discontinuous Galerkin (DG) method in time and a finite element (FE) method of arbitrary regularity in space. We show that the resulting space-time discretization matrices enjoy an asymptotic spectral distribution as the mesh fineness increases, and we determine the associated spectral symbol, i.e., the function that carefully describes the spectral distribution. The analysis of this paper is carried out in a stepwise fashion, without omitting details, and it is supported by several numerical experiments. It is preparatory to the development of specialized solvers for linear systems arising from the DG/FE approximation of the heat equation in the case of both constant and variable diffusion coefficients.

Key words. Spectral distribution, symbol, discontinuous Galerkin method, finite element method, B-splines, heat equation

AMS subject classifications. 15A18, 65M60, 41A15, 35K05, 15B05, 15A69

1. Introduction. Suppose a linear partial differential equation (PDE) is discretized by a linear numerical method characterized by a mesh fineness parameter n. In this situation, the computation of the numerical solution reduces to solving a linear system of the form $L_n\mathbf{u}_n = \mathbf{f}_n$, where the size of the matrix L_n increases with n. What is often observed in practice is that L_n enjoys an asymptotic spectral distribution as $n \to \infty$. More precisely, it often turns out that, for a large class of test functions F,

$$\lim_{n\to\infty} \frac{1}{d_n}\sum_{j=1}^{d_n} F(\lambda_j(L_n)) = \frac{1}{\mu_\ell(D)}\int_D \frac{\sum_{i=1}^{s} F(\lambda_i(\mathbf{f}(\mathbf{y})))}{s}\mathrm{d}\mathbf{y},$$

where d_n is the size of L_n, $\lambda_j(L_n)$, $j = 1, \ldots, d_n$, are the eigenvalues of L_n, μ_ℓ is the Lebesgue measure in $\mathbb{R}^\ell$, and $\lambda_i(\mathbf{f}(\mathbf{y}))$, $i = 1, \ldots, s$, are the eigenvalues of a certain matrix-valued function

$$\mathbf{f} : D \subset \mathbb{R}^\ell \to \mathbb{C}^{s\times s}.$$

We refer to $\mathbf{f}$ as the spectral symbol of the sequence $\{L_n\}_n$.

The spectral information carried by the symbol, which is detailed in Remark 2.2, is not only interesting from a theoretical viewpoint, but can also be used for practical

*Submitted to the editors June 18, 2017.

Funding: Carlo Garoni is a Marie-Curie fellow of the Italian INdAM (Istituto Nazionale di Alta Matematica) under grant agreement PCOFUND-GA-2012-600198.

†University of Italian Switzerland, Institute of Computational Science, Lugano, Switzerland (pietro.benedusi@usi.ch).

‡University of Italian Switzerland, Institute of Computational Science, Lugano, Switzerland, and Insubria University, Department of Science and High Technology, Como, Italy (carlo.garoni@usi.ch, carlo.garoni@uninsubria.it).

§University of Italian Switzerland, Institute of Computational Science, Lugano, Switzerland (rolf.krause@usi.ch).

¶University of Italian Switzerland, Institute of Computational Science, Lugano, Switzerland (xiaozhou.li@usi.ch).

‖Insubria University, Department of Science and High Technology, Como, Italy, and Uppsala University, Department of Information Technology, Division of Scientific Computing, Uppsala, Sweden (stefano.serrac@uninsubria.it, stefano.serra@it.uu.se).

purposes. For example, it is known that the convergence properties of mainstream iterative solvers, such as multigrid and preconditioned Krylov methods, strongly depend on the spectral features of the matrices to which they are applied. The symbol $\mathbf{f}$ can then be exploited to design efficient solvers of this kind for the matrix L_n, and to analyze/predict their performance. In this regard, we recall that noteworthy estimates on the superlinear convergence of the conjugate gradient method obtained by Beckermann and Kuijlaars in [1] are closely related to the asymptotic spectral distribution of the considered matrices. Furthermore, in the context of Galerkin and collocation isogeometric analysis (IgA) discretizations of elliptic boundary value problems, the symbol computed in a sequel of recent papers [7, 10, 11, 12, 13] was exploited in [5, 6, 8] to devise and analyze optimal and robust multigrid solvers for IgA linear systems.

In the present paper, we focus on the heat equation (3.1) defined over a rectangular space-time domain in $\mathbb{R}^{d+1}$, with $d \geq 1$ being an arbitrary positive integer. We consider for this equation a discontinuous Galerkin (DG) discretization in time and a finite element (FE) discretization of arbitrary regularity in space, as described in section 3. It is worth recalling that DG methods for the time integration of (ordinary) differential equations were proposed by Lesaint and Raviart [23] and applied to parabolic equations by Jamet [22]. They enjoy several appealing properties, such as the unconditional stability and the very high order of convergence [22, 23]. Moreover, when performing the time integration by a Galerkin method such as the DG method, the error does not grow significantly over time, so that a long-time integration is possible [9].

After proving in section 4 some key properties of the space discretization matrices arising from our DG/FE technique, in section 5 we determine the spectral symbol for the (normalized) space-time discretization matrices as a function of all the relevant parameters of the considered DG/FE approximation. Our main results are Theorems 5.1 and 5.2; note that in Theorem 5.2 we actually consider an even more general version of the standard heat equation (3.1), namely the heat equation with variable diffusion coefficients (5.8). Numerical experiments in support of the theoretical analysis are provided in section 6. We draw conclusions in section 7, where we also outline future lines of research. The study of this paper is motivated by our intention to exploit the spectral analysis carried out herein to design/analyze appropriate solvers for linear systems arising from the DG/FE discretization of both the heat equation (3.1) and its more general version (5.8) involving variable diffusion coefficients.

2. Preliminaries.

2.1. Multi-index notation. Throughout this paper, we will systematically use the multi-index notation. A multi-index $\boldsymbol{m} \in \mathbb{Z}^d$, also called a d-index, is simply a (row) vector in $\mathbb{Z}^d$; its components are denoted by $m_1, \ldots, m_d$. We denote by $\mathbf{0}, \mathbf{1}, \mathbf{2}$, etc., the vectors consisting of all zeros, all ones, all twos, etc. (their size will be clear from the context). For any d-index $\boldsymbol{m}$, we set $P(\boldsymbol{m}) = \prod_{i=1}^d m_i$ and we write $\boldsymbol{m} \to \infty$ to indicate that $\min(\boldsymbol{m}) \to \infty$. Inequalities between multi-indices must be interpreted in the componentwise sense. For example, $\boldsymbol{j} \leq \boldsymbol{k}$ means that $j_i \leq k_i$ for every i. If $\boldsymbol{j}, \boldsymbol{k}$ are d-indices such that $\boldsymbol{j} \leq \boldsymbol{k}$, the multi-index range $\boldsymbol{j}, \ldots, \boldsymbol{k}$ is the set $\{\boldsymbol{i} \in \mathbb{Z}^d : \ \boldsymbol{j} \leq \boldsymbol{i} \leq \boldsymbol{k}\}$. We assume for this set the standard lexicographic ordering:

$$\left[\ldots \left[\left[(i_1, \ldots, i_d) \right]_{i_d = j_d, \ldots, k_d} \right]_{i_{d-1} = j_{d-1}, \ldots, k_{d-1}} \ldots \right]_{i_1 = j_1, \ldots, k_1} . \tag{2.1}$$

For instance, in the case $d = 2$ this ordering is

$$(j_1, j_2),\ (j_1, j_2+1),\ \ldots,\ (j_1, k_2),\ (j_1+1, j_2),\ (j_1+1, j_2+1),\ \ldots,\ (j_1+1, k_2),$$
$$\ldots,\ \ldots,\ \ldots,\ (k_1, j_2),\ (k_1, j_2+1),\ \ldots,\ (k_1, k_2).$$

When a d-index $\boldsymbol{i}$ varies in a multi-index range $\boldsymbol{j}, \ldots, \boldsymbol{k}$ (this is often written as $\boldsymbol{i} = \boldsymbol{j}, \ldots, \boldsymbol{k}$), it is always assumed that $\boldsymbol{i}$ varies from $\boldsymbol{j}$ to $\boldsymbol{k}$ following the specific ordering (2.1). In particular, if $\boldsymbol{m} \in \mathbb{N}^d$ and $\mathbf{x} = [x_{\boldsymbol{i}}]_{\boldsymbol{i}=\mathbf{1}}^{\boldsymbol{m}}$ then $\mathbf{x}$ is a vector of length $P(\boldsymbol{m})$ whose components $x_{\boldsymbol{i}}$, $\boldsymbol{i} = \mathbf{1}, \ldots, \boldsymbol{m}$, are ordered in accordance with (2.1): the first component is $x_{\mathbf{1}} = x_{(1,\ldots,1,1)}$, the second component is $x_{(1,\ldots,1,2)}$, and so on until the last component, which is $x_{\boldsymbol{m}} = x_{(m_1,\ldots,m_d)}$. Similarly, if $X = [x_{\boldsymbol{ij}}]_{\boldsymbol{i},\boldsymbol{j}=\mathbf{1}}^{\boldsymbol{m}}$ then X is a $P(\boldsymbol{m}) \times P(\boldsymbol{m})$ matrix whose entries are indexed by two d-indices $\boldsymbol{i}, \boldsymbol{j}$, both varying from $\mathbf{1}$ to $\boldsymbol{m}$ according to the lexicographic ordering (2.1). The symbol $\sum_{\boldsymbol{i}=\boldsymbol{j}}^{\boldsymbol{k}}$ denotes the summation over all multi-indices $\boldsymbol{i} = \boldsymbol{j}, \ldots, \boldsymbol{k}$. Operations involving multi-indices that do not have a meaning when considering multi-indices like usual vectors must always be interpreted in the componentwise sense. For example, $\boldsymbol{jk} = (j_1k_1, \ldots, j_dk_d)$, $\boldsymbol{j}/\boldsymbol{k} = (j_1/k_1, \ldots, j_d/k_d)$, etc.

2.2. Matrix norms. For all $X \in \mathbb{C}^{m\times m}$ the eigenvalues and singular values of X are denoted by $\lambda_j(X)$, $j = 1, \ldots, m$, and $\sigma_j(X)$, $j = 1, \ldots, m$, respectively. The conjugate transpose of X is denoted by X^*. The identity matrix and the zero matrix of order m are denoted by I_m and O_m, respectively. The ∞-norm and the 2-norm (spectral norm) of both vectors and matrices are denoted by $\|\cdot\|_\infty$ and $\|\cdot\|$, respectively. We recall that

$$\|X\| \le \sqrt{\|X\|_\infty \|X^T\|_\infty}, \quad \forall\, X \in \mathbb{C}^{m\times m}; \tag{2.2}$$

see, e.g., [20, section 2.3]. For $X \in \mathbb{C}^{m\times m}$, let $\|X\|_1$ be the trace-norm (or Schatten 1-norm) of X, i.e., the sum of all the singular values of X; see [2]. Since $\operatorname{rank}(X)$ is the number of nonzero singular values of X and $\|X\|$ is the maximal singular value of X, we have

$$\|X\|_1 \le \operatorname{rank}(X)\|X\| \le m\|X\|, \quad \forall\, X \in \mathbb{C}^{m\times m}. \tag{2.3}$$

2.3. Tensor products. If X, Y are matrices of any dimension, say $X \in \mathbb{C}^{m_1\times m_2}$ and $Y \in \mathbb{C}^{\ell_1\times \ell_2}$, the tensor (Kronecker) product of X and Y is the $m_1\ell_1 \times m_2\ell_2$ matrix defined by

$$X \otimes Y = \big[x_{ij}Y\big]_{\substack{i=1,\ldots,m_1\\ j=1,\ldots,m_2}} = \begin{bmatrix} x_{11}Y & \cdots & x_{1m_2}Y \\ \vdots & & \vdots \\ x_{m_11}Y & \cdots & x_{m_1m_2}Y \end{bmatrix}.$$

Tensor products possess a lot of nice algebraic properties. One of them is the associativity, which allows one to omit parentheses in expressions like $X_1 \otimes X_2 \otimes \cdots \otimes X_d$. Another property is the bilinearity: for each matrix X, the application $Y \mapsto X \otimes Y$ is linear on $\mathbb{C}^{\ell_1\times\ell_2}$ for all $\ell_1, \ell_2 \in \mathbb{N}$; and for each matrix Y, the application $X \mapsto X \otimes Y$ is linear on $\mathbb{C}^{m_1\times m_2}$ for all $m_1, m_2 \in \mathbb{N}$. If X_1, X_2 can be multiplied and Y_1, Y_2 can be multiplied, then

$$(X_1 \otimes Y_1)(X_2 \otimes Y_2) = (X_1X_2) \otimes (Y_1Y_2). \tag{2.4}$$

For all matrices X, Y, we have $(X \otimes Y)^* = X^* \otimes Y^*$ and $(X \otimes Y)^T = X^T \otimes Y^T$. In particular, if X, Y are Hermitian (resp., symmetric) then $X \otimes Y$ is also Hermitian (resp., symmetric). If $X \in \mathbb{C}^{m\times m}$ and $Y \in \mathbb{C}^{\ell\times\ell}$, the eigenvalues and singular values of $X \otimes Y$ are, respectively, $\{\lambda_i(X)\lambda_j(Y) : \ i = 1, \dots, m, \ j = 1, \dots, \ell\}$ and $\{\sigma_i(X)\sigma_j(Y) : i = 1, \dots, m, \ j = 1, \dots, \ell\}$; see, e.g., [14, Exercise 2.5]. In particular, for all $X \in \mathbb{C}^{m\times m}$ and $Y \in \mathbb{C}^{\ell\times\ell}$, we have

$$\|X \otimes Y\| = \|X\| \, \|Y\|. \tag{2.5}$$

If $X_\ell \in \mathbb{C}^{m_\ell \times m_\ell}$ for $\ell = 1, \dots, d$, then

$$(X_1 \otimes X_2 \otimes \cdots \otimes X_d)_{\boldsymbol{ij}} = (X_1)_{i_1 j_1}(X_2)_{i_2 j_2} \cdots (X_d)_{i_d j_d}, \quad \boldsymbol{i}, \boldsymbol{j} = \boldsymbol{1}, \dots, \boldsymbol{m}, \tag{2.6}$$

where $\boldsymbol{m} = (m_1, m_2, \dots, m_d)$. For every $\boldsymbol{m} = (m_1, m_2) \in \mathbb{N}^2$ there exists a permutation matrix $\Pi_{\boldsymbol{m}}$ of size $m_1 m_2$ such that

$$X_2 \otimes X_1 = \Pi_{\boldsymbol{m}}(X_1 \otimes X_2)\Pi_{\boldsymbol{m}}^T \tag{2.7}$$

for all matrices $X_1 \in \mathbb{C}^{m_1\times m_1}$ and $X_2 \in \mathbb{C}^{m_2\times m_2}$; see, e.g., [17, Lemma 1].

2.4. Spectral distribution and spectral symbol. We say that a matrix-valued function $\mathbf{f} : D \to \mathbb{C}^{s\times s}$, defined on a measurable set $D \subseteq \mathbb{R}^\ell$, is measurable (resp., is continuous, belongs to $L^p(D)$) if its components $f_{ij} : D \to \mathbb{C}, \ i, j = 1, \dots, s$, are measurable (resp., are continuous, belong to $L^p(D)$). Moreover, we say that $\mathbf{f}$ is Hermitian (resp., symmetric) if $\mathbf{f}(\mathbf{y})$ is Hermitian (resp., symmetric) for all $\mathbf{y} \in D$. We denote by μ_ℓ the Lebesgue measure in $\mathbb{R}^\ell$ and by $C_c(\mathbb{R})$ (resp., $C_c(\mathbb{C})$) the set of continuous complex-valued functions with compact support defined over $\mathbb{R}$ (resp., $\mathbb{C}$).

Definition 2.1. *Let $\{X_n\}_n$ be a sequence of matrices, with X_n of size d_n tending to infinity, and let $\mathbf{f} : D \to \mathbb{C}^{s\times s}$ be a measurable matrix-valued function defined on a set $D \subset \mathbb{R}^\ell$ with $0 < \mu_\ell(D) < \infty$. We say that $\{X_n\}_n$ has an asymptotic spectral distribution described by $\mathbf{f}$, and we write $\{X_n\}_n \sim_\lambda \mathbf{f}$, if*

$$\lim_{n\to\infty} \frac{1}{d_n} \sum_{j=1}^{d_n} F(\lambda_j(X_n)) = \frac{1}{\mu_\ell(D)} \int_D \frac{\sum_{i=1}^s F\left(\lambda_i(\mathbf{f}(\mathbf{y}))\right)}{s} \mathrm{d}\mathbf{y}, \quad \forall\, F \in C_c(\mathbb{C}).$$

In this case, $\mathbf{f}$ is referred to as the spectral symbol of the sequence $\{X_n\}_n$.

Whenever we write a spectral distribution relation such as $\{X_n\}_n \sim_\lambda \mathbf{f}$, it is understood that $\{X_n\}_n$ and $\mathbf{f}$ are as in Definition 2.1.

Remark 2.2. *The informal meaning behind Definition 2.1 is the following: assuming that $\mathbf{f}$ possesses s Riemann-integrable eigenvalue functions $\lambda_i(\mathbf{f}(\mathbf{y}))$, $i = 1, \dots, s$, the eigenvalues of X_n, except possibly for $o(d_n)$ outliers, can be subdivided into s different subsets of approximately the same cardinality; and the eigenvalues belonging to the ith subset are approximately equal to the samples of the ith eigenvalue function $\lambda_i(\mathbf{f}(\mathbf{y}))$ over a uniform grid in the domain D. For instance, if $\ell = 1$, $d_n = ns$, and $D = [a, b]$, then, assuming we have no outliers, the eigenvalues of X_n are approximately equal to*

$$\lambda_i\Big(\mathbf{f}\Big(a + j\frac{b-a}{n}\Big)\Big), \quad j = 1, \dots, n, \quad i = 1, \dots, s,$$

for n large enough; similarly, if $\ell = 2$, $d_n = n^2 s$, and $D = [a_1, b_1] \times [a_2, b_2]$, then, assuming we have no outliers, the eigenvalues of X_n are approximately equal to

$$\lambda_i\Big(\mathbf{f}\Big(a_1 + j_1\frac{b_1 - a_1}{n}, a_2 + j_2\frac{b_2 - a_2}{n}\Big)\Big), \quad j_1, j_2 = 1, \dots, n, \quad i = 1, \dots, s,$$

for n large enough; and so on for $\ell \geq 3$.

REMARK 2.3. *Let $D = [a_1, b_1] \times \cdots \times [a_\ell, b_\ell] \subset \mathbb{R}^\ell$ and let $\mathbf{f} : D \to \mathbb{C}^{s\times s}$ be a measurable function possessing s real-valued Riemann-integrable eigenvalue functions $\lambda_i(\mathbf{f}(\mathbf{y}))$, $i = 1, \ldots, s$. Compute for each $r \in \mathbb{N}$ the uniform samples*

$$\lambda_i\Big(\mathbf{f}\Big(a_1 + j_1\,\frac{b_1 - a_1}{r}, \ldots, a_\ell + j_\ell\,\frac{b_\ell - a_\ell}{r}\Big)\Big), \quad j_1, \ldots, j_\ell = 1, \ldots, r, \quad i = 1, \ldots, s,$$

sort them in non-decreasing order and put them in a vector $(\varsigma_1, \varsigma_2, \ldots, \varsigma_{sr^\ell})$. Let $\kappa_r : [0, 1] \to \mathbb{R}$ be the piecewise linear non-decreasing function that interpolates the samples $(\varsigma_0 = \varsigma_1, \varsigma_1, \varsigma_2, \ldots, \varsigma_{sr^\ell})$ over the nodes $(0, \frac{1}{sr^\ell}, \frac{2}{sr^\ell}, \ldots, 1)$, i.e.,

$$\begin{cases} \kappa_r\Big(\dfrac{i}{sr^\ell}\Big) = \varsigma_i, \quad i = 0, \ldots, sr^\ell, \\ \kappa_r \text{ linear on } \Big[\dfrac{i}{sr^\ell}, \dfrac{i+1}{sr^\ell}\Big] \text{ for } i = 0, \ldots, sr^\ell - 1. \end{cases}$$

Suppose κ_r converges in measure over $[0, 1]$ to some function κ as $r \to \infty$ (this is always the case in real-world applications). Then,

$$\int_0^1 F(\kappa(y))\mathrm{d}y = \frac{1}{\mu_\ell(D)} \int_D \frac{\sum_{i=1}^s F(\lambda_i(\mathbf{f}(\mathbf{y})))}{s}\mathrm{d}\mathbf{y}, \quad \forall\, F \in C_c(\mathbb{C}). \tag{2.8}$$

This result can be proved by adapting the argument used in [14, solution of Exercise 3.1]. The function κ is referred to as the canonical rearranged version of $\mathbf{f}$. What is interesting about κ is that, by (2.8), if $\{X_n\}_n \sim_\lambda \mathbf{f}$ then $\{X_n\}_n \sim_\lambda \kappa$, i.e., if $\mathbf{f}$ is a spectral symbol of $\{X_n\}_n$ then κ is a spectral symbol of $\{X_n\}_n$ as well. Moreover, κ is a univariate scalar function and hence it is much easier to handle than $\mathbf{f}$.

Two very useful tools for determining spectral distributions are the following; see [16, Theorem 3.3] for the first one and [24, Theorem 4.3] for the second one.

THEOREM 2.4. *Let $\{X_n\}_n, \{Y_n\}_n$ be sequences of matrices, with $X_n, Y_n \in \mathbb{C}^{d_n\times d_n}$ and d_n tending to infinity as $n \to \infty$, and assume the following.*
1. *Every X_n is Hermitian and $\{X_n\}_n \sim_\lambda \mathbf{f}$.*
2. *$\|X_n\|, \|Y_n\| \leq C$ for all n, with C a constant independent of n.*
3. *$\|Y_n\|_1 = o(d_n)$ as $n \to \infty$.*

Then $\{X_n + Y_n\}_n \sim_\lambda \mathbf{f}$.

THEOREM 2.5. *Let $\{X_n\}_n$ be a sequence of Hermitian matrices, with $X_n \in \mathbb{C}^{d_n\times d_n}$ and d_n tending to infinity as $n \to \infty$, and let $\{P_n\}_n$ be a sequence of matrices, with $P_n \in \mathbb{C}^{d_n\times\delta_n}$ such that $P_n^* P_n = I_{\delta_n}$ and $\delta_n \leq d_n$ such that $\delta_n/d_n \to 1$ as $n \to \infty$. Then,*

$$\{X_n\}_n \sim_\lambda \mathbf{f} \quad \Longleftrightarrow \quad \{P_n^* X_n P_n\}_n \sim_\lambda \mathbf{f}.$$

Another result of interest herein is stated and proved in the next lemma. Throughout this paper, for any $\boldsymbol{s} \in \mathbb{N}^d$ and any functions $\mathbf{f}_i : D_i \to \mathbb{C}^{s_i\times s_i}$, $i = 1, \ldots, d$, the tensor-product function $\mathbf{f}_1 \otimes \cdots \otimes \mathbf{f}_d : D_1 \times \cdots \times D_d \to \mathbb{C}^{P(\boldsymbol{s})\times P(\boldsymbol{s})}$ is defined as

$$(\mathbf{f}_1 \otimes \cdots \otimes \mathbf{f}_d)(\zeta_1, \ldots, \zeta_d) = \mathbf{f}_1(\zeta_1) \otimes \cdots \otimes \mathbf{f}_d(\zeta_d), \quad (\zeta_1, \ldots, \zeta_d) \in D_1 \times \cdots \times D_d.$$

LEMMA 2.6. *Let $\{X_n\}_n, \{Y_n\}_n$ be sequences of Hermitian matrices, with $X_n \in \mathbb{C}^{d_n\times d_n}$, $Y_n \in \mathbb{C}^{\delta_n\times\delta_n}$, and both d_n and δ_n tending to infinity as $n \to \infty$. Assume $\|X_n\|, \|Y_n\| \leq C$ for all n and for some constant C independent of n. Let $\mathbf{f} : D \subseteq$*

$\mathbb{R}^\ell \to \mathbb{C}^{r\times r}$ and $\mathbf{g} : E \subseteq \mathbb{R}^\ell \to \mathbb{C}^{s\times s}$ be measurable Hermitian matrix-valued functions, with $0 < \mu_\ell(D) < \infty$ and $0 < \mu_\ell(E) < \infty$. Then,

$$\{X_n\}_n \sim_\lambda \mathbf{f}, \quad \{Y_n\}_n \sim_\lambda \mathbf{g} \quad \Longrightarrow \quad \{X_n \otimes Y_n\}_n \sim_\lambda \mathbf{f} \otimes \mathbf{g}.$$

Proof. We have to show that, for all $F \in C_c(\mathbb{C})$,

$$\begin{aligned} &\lim_{n\to\infty} \frac{1}{d_n\delta_n} \sum_{i=1}^{d_n}\sum_{j=1}^{\delta_n} F(\lambda_i(X_n)\lambda_j(Y_n)) \\ &= \frac{1}{\mu_\ell(D)\mu_\ell(E)} \int_D\int_E \frac{\sum_{i=1}^r\sum_{j=1}^s F(\lambda_i(\mathbf{f}(\mathbf{x}))\lambda_j(\mathbf{g}(\mathbf{y})))}{rs} \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}. \end{aligned} \tag{2.9}$$

Actually, since all the eigenvalues $\lambda_i(X_n)$, $\lambda_j(Y_n)$, $\lambda_i(\mathbf{f}(\mathbf{x}))$, $\lambda_j(\mathbf{g}(\mathbf{y}))$ are real, it suffices to prove (2.9) for all real-valued functions $F \in C_c(\mathbb{R})$. Throughout the proof, the letter C will denote a generic constant independent of n. Since $\|X_n\|, \|Y_n\| \le C$, we have

$$\lambda_1(X_n), \dots, \lambda_{d_n}(X_n), \lambda_1(Y_n), \dots, \lambda_{\delta_n}(Y_n) \in [-C, C],$$

and consequently, by [19, Theorem 4.2], we also have

$$\lambda_1(\mathbf{f}(\mathbf{x})), \dots, \lambda_r(\mathbf{f}(\mathbf{x})), \lambda_1(\mathbf{g}(\mathbf{y})), \dots, \lambda_s(\mathbf{g}(\mathbf{y})) \in [-C, C] \text{ almost everywhere.}$$

We start with proving (2.9) in the case where $F(y) = y^N$ is a monomial over $[-C, C]$. In this case the proof can be done by direct computation, due to the separability property $F(xy) = F(x)F(y)$ and the hypotheses $\{X_n\}_n \sim_\lambda \mathbf{f}$ and $\{Y_n\}_n \sim_\lambda \mathbf{g}$:

$$\begin{aligned} &\lim_{n\to\infty} \frac{1}{d_n\delta_n} \sum_{i=1}^{d_n}\sum_{j=1}^{\delta_n} F(\lambda_i(X_n)\lambda_j(Y_n)) = \lim_{n\to\infty} \frac{1}{d_n} \sum_{i=1}^{d_n} F(\lambda_i(X_n)) \frac{1}{\delta_n} \sum_{j=1}^{\delta_n} F(\lambda_j(Y_n)) \\ &= \frac{1}{\mu_\ell(D)} \int_D \frac{\sum_{i=1}^r F(\lambda_i(\mathbf{f}(\mathbf{x})))}{r} \mathrm{d}\mathbf{x} \, \frac{1}{\mu_\ell(E)} \int_E \frac{\sum_{j=1}^s F(\lambda_j(\mathbf{g}(\mathbf{y})))}{s} \mathrm{d}\mathbf{y} \\ &= \frac{1}{\mu_\ell(D)\mu_\ell(E)} \int_D\int_E \frac{\sum_{i=1}^r\sum_{j=1}^s F(\lambda_i(\mathbf{f}(\mathbf{x}))\lambda_j(\mathbf{g}(\mathbf{y})))}{rs} \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}. \end{aligned}$$

By linearity, (2.9) holds for all functions F such that $F(y)$ is a polynomial over $[-C, C]$. Thus, (2.9) holds for all real-valued $F \in C_c(\mathbb{R})$ because, by the Weierstrass approximation theorem, for any such F and any $\varepsilon > 0$ we can find a polynomial p_ε such that $\|F - p_\varepsilon\|_\infty \le \varepsilon$. ∎

2.5. Multilevel block Toeplitz matrices. Given $\boldsymbol{m} \in \mathbb{N}^d$, a matrix of the form

$$[A_{\boldsymbol{i}-\boldsymbol{j}}]_{\boldsymbol{i},\boldsymbol{j}=\boldsymbol{1}}^{\boldsymbol{m}} \in \mathbb{C}^{P(\boldsymbol{m})s\times P(\boldsymbol{m})s}, \tag{2.10}$$

with blocks $A_{\boldsymbol{k}} \in \mathbb{C}^{s\times s}$, $\boldsymbol{k} = -(\boldsymbol{m}-\boldsymbol{1}), \dots, \boldsymbol{m}-\boldsymbol{1}$, is called a multilevel block Toeplitz matrix, or, more precisely, a d-level block Toeplitz matrix. Given a function $\mathbf{f} : [-\pi, \pi]^d \to \mathbb{C}^{s\times s}$ in $L^1([-\pi, \pi]^d)$, we denote its Fourier coefficients by

$$\mathbf{f}_{\boldsymbol{k}} = \frac{1}{(2\pi)^d} \int_{[-\pi,\pi]^d} \mathbf{f}(\boldsymbol{\theta}) \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{\theta}} \mathrm{d}\boldsymbol{\theta} \in \mathbb{C}^{s\times s}, \quad \boldsymbol{k} \in \mathbb{Z}^d, \tag{2.11}$$

where the integrals are computed componentwise and $\boldsymbol{k}\cdot\boldsymbol{\theta} = k_1\theta_1 + \ldots + k_d\theta_d$. For every $\boldsymbol{m}\in\mathbb{N}^d$, the $\boldsymbol{m}$th Toeplitz matrix associated with $\mathbf{f}$ is defined as

$$T_{\boldsymbol{m}}(\mathbf{f}) = [\mathbf{f}_{\boldsymbol{i}-\boldsymbol{j}}]_{\boldsymbol{i},\boldsymbol{j}=\mathbf{1}}^{\boldsymbol{m}}. \tag{2.12}$$

We call $\{T_{\boldsymbol{m}}(\mathbf{f})\}_{\boldsymbol{m}\in\mathbb{N}^d}$ the family of (multilevel block) Toeplitz matrices associated with $\mathbf{f}$, which in turn is called the generating function of $\{T_{\boldsymbol{m}}(\mathbf{f})\}_{\boldsymbol{m}\in\mathbb{N}^d}$.

Let $L^1([-\pi,\pi]^d,\mathbb{C}^{s\times s})$ be the space consisting of all matrix-valued functions $\mathbf{f}:[-\pi,\pi]^d\to\mathbb{C}^{s\times s}$ belonging to $L^1([-\pi,\pi]^d)$. For every $s\geq 1$ and $\boldsymbol{m}\in\mathbb{N}^d$, the map $T_{\boldsymbol{m}}(\cdot): L^1([-\pi,\pi]^d,\mathbb{C}^{s\times s})\to\mathbb{C}^{P(\boldsymbol{m})s\times P(\boldsymbol{m})s}$ is linear, i.e.,

$$T_{\boldsymbol{m}}(\alpha\mathbf{f}+\beta\mathbf{g}) = \alpha T_{\boldsymbol{m}}(\mathbf{f}) + \beta T_{\boldsymbol{m}}(\mathbf{g})$$

for all $\alpha,\beta\in\mathbb{C}$ and $\mathbf{f},\mathbf{g}\in L^1([-\pi,\pi]^d,\mathbb{C}^{s\times s})$. Moreover, if $\mathbf{f}\in L^1([-\pi,\pi]^d,\mathbb{C}^{s\times s})$ is a Hermitian matrix-valued function, then all the matrices $T_{\boldsymbol{m}}(\mathbf{f})$ are Hermitian.

Theorem 2.7 is a fundamental result concerning multilevel block Toeplitz matrices generated by Hermitian matrix-valued functions; its proof can be found in [31].

THEOREM 2.7. *Let $\mathbf{f}:[-\pi,\pi]^d\to\mathbb{C}^{s\times s}$ be a Hermitian matrix-valued function in $L^1([-\pi,\pi]^d)$. Then $\{T_{\boldsymbol{m}}(f)\}_n\sim_\lambda\mathbf{f}$ for all sequences of multi-indices $\{\boldsymbol{m}=\boldsymbol{m}(n)\}_n$ such that $\boldsymbol{m}\to\infty$ as $n\to\infty$.*

Besides Theorem 2.7, we shall need the following lemma [17, Lemma 4]. Any matrix-valued function of the form $\mathbf{p}(\theta)=\sum_{r=-M}^{M}P_r\mathrm{e}^{\mathrm{i}r\theta}$, with $P_r\in\mathbb{C}^{s\times s}$ for all r, is referred to as a matrix-valued trigonometric polynomial.

LEMMA 2.8. *For every $\boldsymbol{m},\boldsymbol{s}\in\mathbb{N}^d$ there exists a permutation matrix $\Gamma_{\boldsymbol{m},\boldsymbol{s}}$ of size $P(\boldsymbol{m}\boldsymbol{s})$ such that*

$$T_{m_1}(\mathbf{p}_1)\otimes\cdots\otimes T_{m_d}(\mathbf{p}_d) = \Gamma_{\boldsymbol{m},\boldsymbol{s}}(T_{\boldsymbol{m}}(\mathbf{p}_1\otimes\cdots\otimes\mathbf{p}_d))\Gamma_{\boldsymbol{m},\boldsymbol{s}}^T$$

for all matrix-valued trigonometric polynomials $\mathbf{p}_j:[-\pi,\pi]\to\mathbb{C}^{s_j\times s_j}$, $j=1,\ldots,d$.

3. Problem setting and discretization.

Consider the heat equation

$$\begin{cases}\partial_t u(t,\mathbf{x}) - \Delta u(t,\mathbf{x}) = f(t,\mathbf{x}), & (t,\mathbf{x})\in(0,\mathrm{T})\times(0,1)^d,\\ u(t,\mathbf{x}) = 0, & (t,\mathbf{x})\in(0,\mathrm{T})\times\partial((0,1)^d),\\ u(0,\mathbf{x}) = 0, & \mathbf{x}\in(0,1)^d.\end{cases} \tag{3.1}$$

We are imposing homogeneous Dirichlet initial/boundary conditions both for simplicity and because the case of inhomogeneous Dirichlet initial/boundary conditions reduces to the homogeneous case by considering a lifting of the boundary data; see [25] for more on this subject. We stress that the spatial domain $(0,1)^d$ may be replaced by any other rectangular domain in $\mathbb{R}^d$ without affecting the essence of this paper.

To approximate the solution $u(t,\mathbf{x})$ of the differential problem (3.1), we use a q-degree DG discretization in time and a $\boldsymbol{p}$-degree $C^{\boldsymbol{k}}$ FE discretization in space, with $0\leq k_i\leq p_i-1$ for all $i=1,\ldots,d$. For the sake of completeness, this numerical technique is described here in some detail. For more on DG methods we refer the reader to [4, 21, 22, 23, 26].

3.1. Weak form.

Consider a partition in time $0=t_0<t_1<\cdots<t_N=\mathrm{T}$ and define the mth space-time slab $\mathcal{E}^m=[t_m,t_{m+1}]\times[0,1]^d$ for $m=0,\ldots,N-1$. Assuming the solution $u(t,\mathbf{x})$ is sufficiently regular over $[0,\mathrm{T}]\times[0,1]^d$, we multiply the PDE in (3.1) by a sufficiently regular test function $v(t,\mathbf{x})$ satisfying the same

boundary conditions as $u(t,\mathbf{x})$, i.e., $v(t,\mathbf{x})=0$ for $(t,\mathbf{x})\in(0,\mathrm{T})\times\partial((0,1)^d)$, and we integrate over $\mathcal{E}^m$:

$$\begin{aligned}
&\int_{\mathcal{E}^m}\big[\partial_t u(t,\mathbf{x})-\Delta u(t,\mathbf{x})\big]v(t,\mathbf{x})\mathrm{d}t\mathrm{d}\mathbf{x}=\int_{\mathcal{E}^m}f(t,\mathbf{x})v(t,\mathbf{x})\mathrm{d}t\mathrm{d}\mathbf{x}\\
&\iff \int_{[0,1]^d}\mathrm{d}\mathbf{x}\int_{t_m}^{t_{m+1}}\partial_t u(t,\mathbf{x})v(t,\mathbf{x})\mathrm{d}t-\int_{t_m}^{t_{m+1}}\mathrm{d}t\int_{[0,1]^d}\Delta u(t,\mathbf{x})v(t,\mathbf{x})\mathrm{d}\mathbf{x}\\
&\qquad=\int_{\mathcal{E}^m}f(t,\mathbf{x})v(t,\mathbf{x})\mathrm{d}t\mathrm{d}\mathbf{x}\\
&\iff \int_{[0,1]^d}\mathrm{d}\mathbf{x}\Big[u(t,\mathbf{x})v(t,\mathbf{x})\big|_{t_m}^{t_{m+1}}-\int_{t_m}^{t_{m+1}}u(t,\mathbf{x})\partial_t v(t,\mathbf{x})\mathrm{d}t\Big]\\
&\qquad-\int_{t_m}^{t_{m+1}}\mathrm{d}t\Big[\underbrace{\int_{\partial([0,1]^d)}v(t,\mathbf{x})\nabla u(t,\mathbf{x})\cdot\mathbf{n}(\mathbf{x})\mathrm{d}\sigma(\mathbf{x})}_{=0}-\int_{[0,1]^d}\nabla u(t,\mathbf{x})\cdot\nabla v(t,\mathbf{x})\mathrm{d}\mathbf{x}\Big]\\
&\qquad=\int_{\mathcal{E}^m}f(t,\mathbf{x})v(t,\mathbf{x})\mathrm{d}t\mathrm{d}\mathbf{x}.
\end{aligned}$$

This means that, for every $m=0,\dots,N-1$ and every sufficiently regular test function $v(t,\mathbf{x})$ satisfying $v(t,\mathbf{x})=0$ for $(t,\mathbf{x})\in(0,\mathrm{T})\times\partial((0,1)^d)$, the solution $u(t,\mathbf{x})$ satisfies

$$a_m(u,v)=F_m(v), \tag{3.2}$$

where

$$\begin{aligned}
a_m(u,v)=&-\int_{\mathcal{E}^m}u(t,\mathbf{x})\partial_t v(t,\mathbf{x})\mathrm{d}t\mathrm{d}\mathbf{x}+\int_{\mathcal{E}^m}\nabla u(t,\mathbf{x})\cdot\nabla v(t,\mathbf{x})\mathrm{d}t\mathrm{d}\mathbf{x}\\
&+\int_{[0,1]^d}\big[u(t_{m+1}^-,\mathbf{x})v(t_{m+1}^-,\mathbf{x})-u(t_m^+,\mathbf{x})v(t_m^+,\mathbf{x})\big]\mathrm{d}\mathbf{x},
\end{aligned}\tag{3.3}$$

$$F_m(v)=\int_{\mathcal{E}^m}f(t,\mathbf{x})v(t,\mathbf{x})\mathrm{d}t\mathrm{d}\mathbf{x}. \tag{3.4}$$

Here, the symbols $w(\tau^-,\mathbf{x})$ and $w(\tau^+,\mathbf{x})$ stand for the limits $\lim_{t\to\tau^-}w(t,\mathbf{x})$ and $\lim_{t\to\tau^+}w(t,\mathbf{x})$, respectively. For $m=0$ it is assumed that $u(t_0^+,\mathbf{x})=u(t_0,\mathbf{x})=0$ according to the initial condition in (3.1).

3.2. Space-time discretization. Let $N\in\mathbb{N}$ and $\boldsymbol{n}\in\mathbb{N}^d$, and consider uniform partitions in time and space:

$$\begin{aligned}
&t_i=i\Delta t, && i=0,\dots,N, && \Delta t=\mathrm{T}/N,\\
&\mathbf{x}_{\boldsymbol{i}}=\boldsymbol{i}\Delta\mathbf{x}=(i_1\Delta x_1,\dots,i_d\Delta x_d), && \boldsymbol{i}=\mathbf{0},\dots,\boldsymbol{n}, && \Delta\mathbf{x}=(\Delta x_1,\dots,\Delta x_d)\\
& && && \qquad=(1/n_1,\dots,1/n_d)=\mathbf{1}/\boldsymbol{n}.
\end{aligned}$$

Define the q-degree DG approximation space and the $\boldsymbol{p}$-degree $C^{\boldsymbol{k}}$ FE approximation space as follows:

$$\begin{aligned}
\mathscr{W}_{N,[q]}&=\big\{w:\ w|_{[t_m,t_{m+1}]}\in\mathbb{P}_q\text{ for all }m=0,\dots,N-1\big\},\\
\mathscr{W}_{\boldsymbol{n},[\boldsymbol{p},\boldsymbol{k}]}&=\mathscr{W}_{n_1,[p_1,k_1]}\otimes\cdots\otimes\mathscr{W}_{n_d,[p_d,k_d]}\\
&=\mathrm{span}(w_1\otimes\cdots\otimes w_d:\ w_i\in\mathscr{W}_{n_i,[p_i,k_i]}\text{ for all }i=1,\dots,d),
\end{aligned}$$

where $\mathbb{P}_q$ is the space of polynomials of degree less than or equal to q and, for all $p, n \in \mathbb{N}$ and $0 \le k \le p-1$, the space $\mathscr{W}_{n,[p,k]}$ is defined as

$$\begin{aligned}\mathscr{W}_{n,[p,k]} = \big\{ & w \in C^k([0,1]) : \\ & w|_{\left[\frac{i}{n}, \frac{i+1}{n}\right]} \in \mathbb{P}_p \text{ for all } i = 0, \dots, n-1, \;\; w(0) = w(1) = 0 \big\}.\end{aligned} \tag{3.5}$$

Note that the generic element $w \in \mathscr{W}_{N,[q]}$ is not a function from $[0, \mathrm{T}]$ to $\mathbb{R}$ in the true sense of this word, because it takes two values at the points t_m, $m = 1, \dots, N-1$. However, for simplicity we will refer to each $w \in \mathscr{W}_{N,[q]}$ as a function without further specifications. It can be shown that

$$\dim(\mathscr{W}_{n,[p,k]}) = n(p-k) + k - 1$$

and

$$\begin{aligned}\overline{N} &= \dim(\mathscr{W}_{N,[q]}) = N(q+1), \\ \overline{n} &= \dim(\mathscr{W}_{\boldsymbol{n},[\boldsymbol{p},\boldsymbol{k}]}) = \prod_{i=1}^{d} \dim(\mathscr{W}_{n_i,[p_i,k_i]}) = P(\boldsymbol{n}(\boldsymbol{p}-\boldsymbol{k}) + \boldsymbol{k} - \boldsymbol{1}).\end{aligned}$$

Let $\{\phi_1, \dots, \phi_{\overline{N}}\}$ be a basis for $\mathscr{W}_{N,[q]}$, let $\{\varphi_1, \dots, \varphi_{\overline{n}}\}$ be a basis for $\mathscr{W}_{\boldsymbol{n},[\boldsymbol{p},\boldsymbol{k}]}$, and set

$$\mathscr{W} = \mathscr{W}_{N,[q]} \otimes \mathscr{W}_{\boldsymbol{n},[\boldsymbol{p},\boldsymbol{k}]} = \mathrm{span}(\psi_{\boldsymbol{j}} = \phi_{j_1} \otimes \varphi_{j_2} : \;\; \boldsymbol{j} = \boldsymbol{1}, \dots, \boldsymbol{N}), \quad \boldsymbol{N} = (\overline{N}, \overline{n}).$$

We look for an approximation $u_{\mathscr{W}}(t, \mathbf{x})$ of the solution $u(t, \mathbf{x})$ by solving the following discrete problem: find $u_{\mathscr{W}} \in \mathscr{W}$ such that, for all $m = 0, \dots, N-1$ and all $v \in \mathscr{W}$,

$$a_m(u_{\mathscr{W}}, v) = F_m(v), \tag{3.6}$$

where $a_m(u, v)$ and $F_m(v)$ are given by (3.3) and (3.4), respectively.

It should be noted, however, that, due to the structure of the DG approximation space $\mathscr{W}_{N,[q]}$, the solution of (3.6) for $m = 1$ is completely independent of the solution of (3.6) for $m = 0$. Similarly, the solution of (3.6) for $m = 2$ is completely independent of the solution of (3.6) for $m = 1$, and so on. In particular, the information provided by the initial condition $u(0, \mathbf{x})$ is present only until $t = t_1^-$ and it is lost for $t > t_1$. To avoid this decoupling of the various problems (3.6) corresponding to different indices m, as well as to avoid the loss of information carried by the initial condition, we impose that the initial condition of problem (3.6) for $m = 1$ is given by $u_{\mathscr{W}}(t_1^-, \mathbf{x})$, which is obtained by solving (3.6) for $m = 0$. More generally, we impose that the initial condition of problem (3.6) for $m = 1, \dots, N-1$ is given by $u_{\mathscr{W}}(t_m^-, \mathbf{x})$, which is obtained by solving (3.6) for the previous index $m-1$. Of course, the initial condition for $m = 0$ is $u(0, \mathbf{x})$. In conclusion, we replace $a_m(u_{\mathscr{W}}, v)$ in (3.6) with $\hat{a}_m(u_{\mathscr{W}}, v)$, where

$$\begin{aligned}\hat{a}_0(u_{\mathscr{W}}, v) = a_0(u_{\mathscr{W}}, v) = & -\int_{\mathcal{E}^0} u_{\mathscr{W}}(t, \mathbf{x}) \partial_t v(t, \mathbf{x}) \mathrm{d}t \mathrm{d}\mathbf{x} + \int_{\mathcal{E}^0} \nabla u_{\mathscr{W}}(t, \mathbf{x}) \cdot \nabla v(t, \mathbf{x}) \mathrm{d}t \mathrm{d}\mathbf{x} \\ & + \int_{[0,1]^d} \big[u_{\mathscr{W}}(t_1^-, \mathbf{x}) v(t_1^-, \mathbf{x}) - \underbrace{u(0, \mathbf{x})}_{=0} v(t_0^+, \mathbf{x}) \big] \mathrm{d}\mathbf{x}\end{aligned} \tag{3.7}$$

and

$$\begin{aligned}\hat{a}_m(u_{\mathscr{W}}, v) = & -\int_{\mathcal{E}^m} u_{\mathscr{W}}(t, \mathbf{x}) \partial_t v(t, \mathbf{x}) \mathrm{d}t \mathrm{d}\mathbf{x} + \int_{\mathcal{E}^m} \nabla u_{\mathscr{W}}(t, \mathbf{x}) \cdot \nabla v(t, \mathbf{x}) \mathrm{d}t \mathrm{d}\mathbf{x} \\ & + \int_{[0,1]^d} \big[u_{\mathscr{W}}(t_{m+1}^-, \mathbf{x}) v(t_{m+1}^-, \mathbf{x}) - u_{\mathscr{W}}(t_m^-, \mathbf{x}) v(t_m^+, \mathbf{x}) \big] \mathrm{d}\mathbf{x}\end{aligned} \tag{3.8}$$

for $m = 1, \dots, N-1$. Then, we look for an approximation $u_{\mathscr{W}}(t, \mathbf{x})$ of the solution $u(t, \mathbf{x})$ by solving the following discrete problem: find $u_{\mathscr{W}} \in \mathscr{W}$ such that, for all $m = 0, \dots, N-1$ and all $v \in \mathscr{W}$,

$$\hat{a}_m(u_{\mathscr{W}}, v) = F_m(v), \tag{3.9}$$

where $\hat{a}_m(u, v)$ and $F_m(v)$ are given by (3.7)–(3.8) and (3.4), respectively.

Considering that $\{\psi_{\boldsymbol{j}} : \boldsymbol{j} = \mathbf{1}, \dots, \boldsymbol{N}\}$ is a basis for $\mathscr{W}$, we have $u_{\mathscr{W}} = \sum_{\boldsymbol{j}=\mathbf{1}}^{\boldsymbol{N}} u_{\boldsymbol{j}} \psi_{\boldsymbol{j}}$ for a unique vector $\mathbf{u} = [u_{\boldsymbol{j}}]_{\boldsymbol{j}=\mathbf{1}}^{\boldsymbol{N}}$ and, by linearity, the computation of $u_{\mathscr{W}}$ reduces to finding $\mathbf{u}$ such that, for all $m = 0, \dots, N-1$,

$$A_m \mathbf{u} = \mathbf{F}_m, \tag{3.10}$$

where

$$\mathbf{F}_m = [F_m(\psi_{\boldsymbol{i}})]_{\boldsymbol{i}=\mathbf{1}}^{\boldsymbol{N}}, \tag{3.11}$$

$$A_m = [\hat{a}_m(\psi_{\boldsymbol{j}}, \psi_{\boldsymbol{i}})]_{\boldsymbol{i},\boldsymbol{j}=\mathbf{1}}^{\boldsymbol{N}}. \tag{3.12}$$

3.3. Choice of the bases in time and space. Fix a basis $\{\ell_{1,[q]}, \dots, \ell_{q+1,[q]}\}$ for the polynomial space $\mathbb{P}_q$, and let

$$\hat{\phi}_{s,[q]}(\tau) = \begin{cases} \ell_{s,[q]}(\tau), & \text{if } \tau \in [-1, 1], \\ 0, & \text{otherwise}, \end{cases} \qquad s = 1, \dots, q+1. \tag{3.13}$$

We refer to $\{\hat{\phi}_{1,[q]}, \dots, \hat{\phi}_{q+1,[q]}\}$ and $[-1, 1]$ as the reference basis in time and the reference interval in time, respectively.

The basis $\{\phi_1, \dots, \phi_{\overline{N}}\}$ for $\mathscr{W}_{N,[q]}$ is defined as follows:

$$\begin{aligned} \phi_{(q+1)(r-1)+s}(t) &= \hat{\phi}_{s,[q]}\left(\frac{2t - (t_r + t_{r-1})}{t_r - t_{r-1}}\right) \\ &= \hat{\phi}_{s,[q]}\left(\frac{2t - (2r-1)\Delta t}{\Delta t}\right), \quad r = 1, \dots, N, \quad s = 1, \dots, q+1. \end{aligned} \tag{3.14}$$

Note that $\phi_{(q+1)(r-1)+s}$ is identically zero outside $[t_{r-1}, t_r]$. In the context of (nodal) DG methods, $\ell_{1,[q]}, \dots, \ell_{q+1,[q]}$ are often chosen as the Lagrange polynomials associated with $q+1$ fixed points $\{\tau_1, \dots, \tau_{q+1}\} \subseteq [-1, 1]$, such as, for example, the Gauss–Lobatto or the right Gauss–Radau nodes in $[-1, 1]$; see, e.g., [21]. Nevertheless, other choices are also allowed, and since the analysis of this paper is not affected by the specific choice of the reference basis, we will not make any specific assumptions on $\ell_{1,[q]}, \dots, \ell_{q+1,[q]}$.

For $p, n \in \mathbb{N}$ and $0 \le k \le p-1$, let $B_{1,[p,k]}, \dots, B_{n(p-k)+k+1,[p,k]}$ be the B-splines of degree p and smoothness C^k defined on the knot sequence

$$\begin{aligned} &\{\xi_1, \dots, \xi_{n(p-k)+p+k+2}\} \\ &= \Big\{\underbrace{0, \dots, 0}_{p+1}, \underbrace{\frac{1}{n}, \dots, \frac{1}{n}}_{p-k}, \underbrace{\frac{2}{n}, \dots, \frac{2}{n}}_{p-k}, \dots, \underbrace{\frac{n-1}{n}, \dots, \frac{n-1}{n}}_{p-k}, \underbrace{1, \dots, 1}_{p+1}\Big\}. \end{aligned} \tag{3.15}$$

A few properties of the functions $B_{1,[p,k]}, \dots, B_{n(p-k)+k+1,[p,k]}$ that we shall use in this paper are listed below.

- Local support property: the support of the ith B-spline is given by

$$\operatorname{supp}(B_{i,[p,k]}) = [\xi_i, \xi_{i+p+1}], \quad i = 1, \dots, n(p-k)+k+1. \tag{3.16}$$

- Vanishment on the boundary: except for the first and the last one, all the other B-splines vanish on the boundary of $[0,1]$, i.e.,

$$B_{i,[p,k]}(0) = B_{i,[p,k]}(1) = 0, \quad i = 2, \dots, n(p-k)+k. \tag{3.17}$$

- Basis property: $\{B_{1,[p,k]}, \dots, B_{n(p-k)+k+1,[p,k]}\}$ is a basis for the space of piecewise polynomial functions of degree p and smoothness C^k, that is,

$$\mathscr{V}_{n,[p,k]} = \big\{v \in C^k([0,1]) : \ v|_{\left[\frac{i}{n}, \frac{i+1}{n}\right]} \in \mathbb{P}_p \text{ for all } i = 0, \dots, n-1\big\};$$

and $\{B_{2,[p,k]}, \dots, B_{n(p-k)+k,[p,k]}\}$ is a basis for the space

$$\mathscr{W}_{n,[p,k]} = \{w \in \mathscr{V}_{n,[p,k]} : \ w(0) = w(1) = 0\},$$

which has already been introduced in (3.5).

- Non-negativity and partition of unity:

$$B_{i,[p,k]} \geq 0 \text{ over } \mathbb{R}, \quad i = 1, \dots, n(p-k)+k+1, \tag{3.18}$$

$$\sum_{i=1}^{n(p-k)+k+1} B_{i,[p,k]} = 1 \text{ over } [0,1]. \tag{3.19}$$

- Bounds for derivatives:

$$\sum_{i=1}^{n(p-k)+k+1} |B'_{i,[p,k]}| \leq c_p n \text{ over } [0,1], \tag{3.20}$$

for some constant c_p depending only on p. Note that the derivatives $B'_{i,[p,k]}$ may not be defined at some of the grid points $0, \frac{1}{n}, \dots, \frac{n-1}{n}, 1$ in the case of C^0 smoothness ($k = 0$). In (3.20) it is assumed that the undefined values are excluded from the summation.

- All the B-splines $B_{i,[p,k]}$, except for the first $k+1$ and the last $k+1$, are uniformly shifted-scaled versions of $p-k$ fixed reference functions $\hat{\varphi}_{1,[p,k]}, \dots, \hat{\varphi}_{p-k,[p,k]}$, namely the first $p-k$ B-splines defined on the reference knot sequence

$$\underbrace{0, \dots, 0}_{p-k}, \underbrace{1, \dots, 1}_{p-k}, \dots, \underbrace{\eta, \dots, \eta}_{p-k}, \quad \eta = \left\lceil \frac{p+1}{p-k} \right\rceil. \tag{3.21}$$

In formulas, setting

$$\nu = \left\lceil \frac{k+1}{p-k} \right\rceil, \tag{3.22}$$

for the B-splines $B_{k+2,[p,k]}, \dots, B_{k+1+(n-\nu)(p-k),[p,k]}$ we have

$$B_{k+1+(p-k)(r-1)+s,[p,k]}(x) = \hat{\varphi}_{s,[p,k]}(nx - r + 1), \quad \begin{aligned} r &= 1, \dots, n-\nu, \\ s &= 1, \dots, p-k. \end{aligned} \tag{3.23}$$

We point out that the supports of the reference B-splines $\hat{\varphi}_{s,[p,k]}$ satisfy

$$\operatorname{supp}(\hat{\varphi}_{1,[p,k]}) \subseteq \operatorname{supp}(\hat{\varphi}_{2,[p,k]}) \subseteq \dots \subseteq \operatorname{supp}(\hat{\varphi}_{p-k,[p,k]}) = [0, \eta]. \tag{3.24}$$

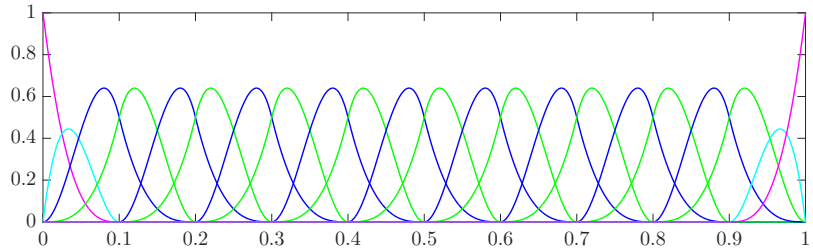

FIG. 3.1. *B-splines* $B_{1,[p,k]}, \dots, B_{n(p-k)+k+1,[p,k]}$ *for* $p = 3$ *and* $k = 1$, *with* $n = 10$.

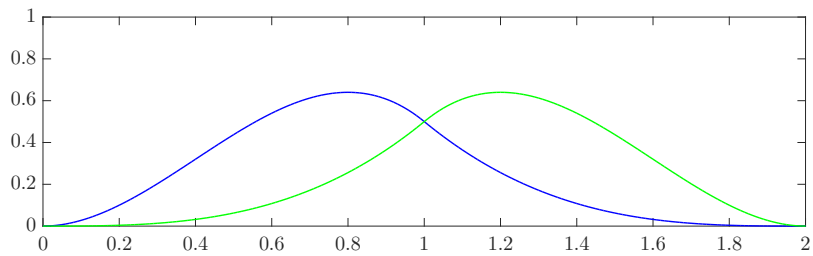

FIG. 3.2. *Reference B-splines* $\hat{\varphi}_{1,[p,k]}, \hat{\varphi}_{2,[p,k]}$ *for* $p = 3$ *and* $k = 1$.

Figs. 3.1–3.2 show the graphs of the B-splines $B_{1,[p,k]}, \dots, B_{n(p-k)+k+1,[p,k]}$ for the degree $p = 3$ and the smoothness $k = 1$, and the graphs of the associated reference B-splines $\hat{\varphi}_{1,[p,k]}, \hat{\varphi}_{2,[p,k]}$. For the formal definition of the B-splines, as well as for the proof of the properties mentioned above, we refer the reader to [3, 27].

The basis $\{\varphi_1, \dots, \varphi_{\overline{n}}\} = \{\varphi_{\mathbf{1}}, \dots, \varphi_{\boldsymbol{n}(\boldsymbol{p}-\boldsymbol{k})+\boldsymbol{k}-\mathbf{1}}\}$ for $\mathscr{W}_{\boldsymbol{n},[\boldsymbol{p},\boldsymbol{k}]}$ is defined as follows:

$$\begin{aligned} \varphi_{\boldsymbol{s}} &= B_{\boldsymbol{s}+\mathbf{1},[\boldsymbol{p},\boldsymbol{k}]} \\ &= B_{s_1+1,[p_1,k_1]} \otimes \cdots \otimes B_{s_d+1,[p_d,k_d]}, \quad \boldsymbol{s} = \mathbf{1}, \dots, \boldsymbol{n}(\boldsymbol{p}-\boldsymbol{k}) + \boldsymbol{k} - \mathbf{1}. \end{aligned} \tag{3.25}$$

3.4. Space-time matrix assembly. For every $\boldsymbol{i}, \boldsymbol{j} = \mathbf{1}, \dots, \boldsymbol{N}$, the $(\boldsymbol{i}, \boldsymbol{j})$ entry of the matrix A_m appearing in (3.12) is given by

$$\begin{aligned} (A_m)_{\boldsymbol{ij}} &= \hat{a}_m(\psi_{\boldsymbol{j}}, \psi_{\boldsymbol{i}}) \\ &= -\int_{\mathcal{E}^m} \psi_{\boldsymbol{j}}(t,\mathbf{x}) \partial_t \psi_{\boldsymbol{i}}(t,\mathbf{x}) \mathrm{d}t\mathrm{d}\mathbf{x} + \int_{\mathcal{E}^m} \nabla\psi_{\boldsymbol{j}}(t,\mathbf{x}) \cdot \nabla\psi_{\boldsymbol{i}}(t,\mathbf{x}) \mathrm{d}t\mathrm{d}\mathbf{x} \\ &\quad + \int_{[0,1]^d} \big[\psi_{\boldsymbol{j}}(t_{m+1}^-,\mathbf{x})\psi_{\boldsymbol{i}}(t_{m+1}^-,\mathbf{x}) - \psi_{\boldsymbol{j}}(t_m^-,\mathbf{x})\psi_{\boldsymbol{i}}(t_m^+,\mathbf{x})\big] \mathrm{d}\mathbf{x} \\ &= -\int_{t_m}^{t_{m+1}} \phi_{j_1}(t)\phi_{i_1}'(t)\mathrm{d}t \int_{[0,1]^d} \varphi_{j_2}(\mathbf{x})\varphi_{i_2}(\mathbf{x})\mathrm{d}\mathbf{x} \\ &\quad + \int_{t_m}^{t_{m+1}} \phi_{j_1}(t)\phi_{i_1}(t)\mathrm{d}t \int_{[0,1]^d} \nabla\varphi_{j_2}(\mathbf{x}) \cdot \nabla\varphi_{i_2}(\mathbf{x})\mathrm{d}\mathbf{x} \\ &\quad + \big[\phi_{j_1}(t_{m+1}^-)\phi_{i_1}(t_{m+1}^-) - \phi_{j_1}(t_m^-)\phi_{i_1}(t_m^+)\big] \int_{[0,1]^d} \varphi_{j_2}(\mathbf{x})\varphi_{i_2}(\mathbf{x})\mathrm{d}\mathbf{x} \end{aligned} \tag{3.26}$$

for $m = 1, \dots, N-1$, and similarly

$$
\begin{aligned}
(A_0)_{\boldsymbol{ij}} = &- \int_{t_0}^{t_1} \phi_{j_1}(t)\phi'_{i_1}(t)\mathrm{d}t \int_{[0,1]^d} \varphi_{j_2}(\mathbf{x})\varphi_{i_2}(\mathbf{x})\mathrm{d}\mathbf{x} \\
&+ \int_{t_0}^{t_1} \phi_{j_1}(t)\phi_{i_1}(t)\mathrm{d}t \int_{[0,1]^d} \nabla\varphi_{j_2}(\mathbf{x}) \cdot \nabla\varphi_{i_2}(\mathbf{x})\mathrm{d}\mathbf{x} \\
&+ \phi_{j_1}(t_1^-)\phi_{i_1}(t_1^-) \int_{[0,1]^d} \nabla\varphi_{j_2}(\mathbf{x}) \cdot \nabla\varphi_{i_2}(\mathbf{x})\mathrm{d}\mathbf{x}
\end{aligned} \tag{3.27}
$$

Since the values $\phi_j(t_0^-)$ are not defined, we set for convenience $\phi_j(t_0^-) = 0$ for all $j = 1, \dots, \overline{N}$, so that formula (3.26) is true also for $m = 0$. For all $m = 0, \dots, N-1$, define the matrices

$$
M_{N,[q]}^{[m]} = \left[\int_{t_m}^{t_{m+1}} \phi_j(t)\phi_i(t)\mathrm{d}t \right]_{i,j=1}^{\overline{N}}, \tag{3.28}
$$

$$
H_{N,[q]}^{[m]} = \left[-\int_{t_m}^{t_{m+1}} \phi_j(t)\phi'_i(t)\mathrm{d}t \right]_{i,j=1}^{\overline{N}}, \tag{3.29}
$$

$$
C_{N,[q]}^{[m]} = \left[\phi_j(t_{m+1}^-)\phi_i(t_{m+1}^-) \right]_{i,j=1}^{\overline{N}}, \tag{3.30}
$$

$$
J_{N,[q]}^{[m]} = \left[\phi_j(t_m^-)\phi_i(t_m^+) \right]_{i,j=1}^{\overline{N}}, \tag{3.31}
$$

$$
\begin{aligned}
M_{\boldsymbol{n},[\boldsymbol{p},\boldsymbol{k}]} &= \left[\int_{[0,1]^d} \varphi_j(\mathbf{x})\varphi_i(\mathbf{x})\mathrm{d}\mathbf{x} \right]_{i,j=1}^{\overline{n}} \\
&= \left[\int_{[0,1]^d} B_{\boldsymbol{j}+\mathbf{1},[\boldsymbol{p},\boldsymbol{k}]}(\mathbf{x}) B_{\boldsymbol{i}+\mathbf{1},[\boldsymbol{p},\boldsymbol{k}]}(\mathbf{x})\mathrm{d}\mathbf{x} \right]_{\boldsymbol{i},\boldsymbol{j}=\mathbf{1}}^{\boldsymbol{n}(\boldsymbol{p}-\boldsymbol{k})+\boldsymbol{k}-\mathbf{1}},
\end{aligned} \tag{3.32}
$$

$$
\begin{aligned}
K_{\boldsymbol{n},[\boldsymbol{p},\boldsymbol{k}]} &= \left[\int_{[0,1]^d} \nabla\varphi_j(\mathbf{x}) \cdot \nabla\varphi_i(\mathbf{x})\mathrm{d}\mathbf{x} \right]_{i,j=1}^{\overline{n}} \\
&= \left[\int_{[0,1]^d} \nabla B_{\boldsymbol{j}+\mathbf{1},[\boldsymbol{p},\boldsymbol{k}]}(\mathbf{x}) \cdot \nabla B_{\boldsymbol{i}+\mathbf{1},[\boldsymbol{p},\boldsymbol{k}]}(\mathbf{x})\mathrm{d}\mathbf{x} \right]_{\boldsymbol{i},\boldsymbol{j}=\mathbf{1}}^{\boldsymbol{n}(\boldsymbol{p}-\boldsymbol{k})+\boldsymbol{k}-\mathbf{1}}.
\end{aligned} \tag{3.33}
$$

From (3.26) we obtain

$$
\begin{aligned}
(A_m)_{\boldsymbol{ij}} = (H_{N,[q]}^{[m]})_{i_1j_1}(M_{\boldsymbol{n},[\boldsymbol{p},\boldsymbol{k}]})_{i_2j_2} + (M_{N,[q]}^{[m]})_{i_1j_1}(K_{\boldsymbol{n},[\boldsymbol{p},\boldsymbol{k}]})_{i_2j_2} \\
+ \left[(C_{N,[q]}^{[m]})_{i_1j_1} - (J_{N,[q]}^{[m]})_{i_1j_1} \right] (M_{\boldsymbol{n},[\boldsymbol{p},\boldsymbol{k}]})_{i_2j_2}.
\end{aligned}
$$

Hence, by (2.6),

$$
A_m = (H_{N,[q]}^{[m]} + C_{N,[q]}^{[m]} - J_{N,[q]}^{[m]}) \otimes M_{\boldsymbol{n},[\boldsymbol{p},\boldsymbol{k}]} + M_{N,[q]}^{[m]} \otimes K_{\boldsymbol{n},[\boldsymbol{p},\boldsymbol{k}]}. \tag{3.34}
$$

In view of the choice (3.14) for the basis in time, we can further simplify the expressions of the matrices (3.28)–(3.31). Concerning the matrices (3.28)–(3.30), the components $(M_{N,[q]}^{[m]})_{ij}$, $(H_{N,[q]}^{[m]})_{ij}$, $(C_{N,[q]}^{[m]})_{ij}$ are nonzero at most for the pairs (i,j) such that

$i = (q+1)m + s$ and $j = (q+1)m + s'$ with $1 \le s, s' \le q+1$. Moreover, for such pairs we have

$$\begin{aligned}
(M^{[m]}_{N,[q]})_{ij} &= \int_{t_m}^{t_{m+1}} \phi_{(q+1)m+s'}(t)\phi_{(q+1)m+s}(t)\mathrm{d}t \\
&= \int_{t_m}^{t_{m+1}} \hat{\phi}_{s',[q]}\left(\frac{2t-(2m+1)\Delta t}{\Delta t}\right)\hat{\phi}_{s,[q]}\left(\frac{2t-(2m+1)\Delta t}{\Delta t}\right)\mathrm{d}t \\
&= \frac{\Delta t}{2}\int_{-1}^{1} \ell_{s',[q]}(\tau)\ell_{s,[q]}(\tau)\mathrm{d}\tau, \\
(H^{[m]}_{N,[q]})_{ij} &= -\int_{t_m}^{t_{m+1}} \phi_{(q+1)m+s'}(t)\phi'_{(q+1)m+s}(t)\mathrm{d}t \\
&= -\int_{t_m}^{t_{m+1}} \hat{\phi}_{s',[q]}\left(\frac{2t-(2m+1)\Delta t}{\Delta t}\right)\frac{2}{\Delta t}\hat{\phi}'_{s,[q]}\left(\frac{2t-(2m+1)\Delta t}{\Delta t}\right)\mathrm{d}t \\
&= -\int_{-1}^{1} \ell_{s',[q]}(\tau)\ell'_{s,[q]}(\tau)\mathrm{d}\tau, \\
(C^{[m]}_{N,[q]})_{ij} &= \phi_{(q+1)m+s'}(t^-_{m+1})\phi_{(q+1)m+s}(t^-_{m+1}) = \ell_{s',[q]}(1)\ell_{s,[q]}(1).
\end{aligned}$$

Thus, if E_{ij} is the $N \times N$ matrix having 1 in position (i,j) and 0 elsewhere, the matrices $M^{[m]}_{N,[q]}$, $H^{[m]}_{N,[q]}$, $C^{[m]}_{N,[q]}$ can be written as

$$M^{[m]}_{N,[q]} = E_{m+1,m+1} \otimes \frac{\Delta t}{2} M_{[q]}, \quad M_{[q]} = \left[\int_{-1}^{1} \ell_{s',[q]}(\tau)\ell_{s,[q]}(\tau)\mathrm{d}\tau\right]_{s,s'=1}^{q+1}, \tag{3.35}$$

$$H^{[m]}_{N,[q]} = E_{m+1,m+1} \otimes H_{[q]}, \qquad H_{[q]} = \left[-\int_{-1}^{1} \ell_{s',[q]}(\tau)\ell'_{s,[q]}(\tau)\mathrm{d}\tau\right]_{s,s'=1}^{q+1}, \tag{3.36}$$

$$C^{[m]}_{N,[q]} = E_{m+1,m+1} \otimes C_{[q]}, \qquad C_{[q]} = \left[\ell_{s',[q]}(1)\ell_{s,[q]}(1)\right]_{s,s'=1}^{q+1}, \tag{3.37}$$

for all $m = 0, \dots, N-1$. Concerning the matrix (3.31), the component $(J^{[m]}_{N,[q]})_{ij}$ is nonzero at most for the pairs (i,j) such that $i = (q+1)m+s$ and $j = (q+1)(m-1)+s'$ with $1 \le s, s' \le q+1$. Moreover, for such pairs we have

$$(J^{[m]}_{N,[q]})_{ij} = \phi_{(q+1)(m-1)+s'}(t^-_m)\phi_{(q+1)m+s}(t^+_m) = \ell_{s',[q]}(1)\ell_{s,[q]}(-1).$$

Thus,

$$J^{[m]}_{N,[q]} = E_{m+1,m} \otimes J_{[q]}, \qquad J_{[q]} = \left[\ell_{s',[q]}(1)\ell_{s,[q]}(-1)\right]_{s,s'=1}^{q+1}, \tag{3.38}$$

for all $m = 0, \dots, N-1$; note that $E_{1,0}$ is the $N \times N$ zero matrix. In conclusion, the matrix (3.34) can be expressed as

$$A_m = E_{m+1,m} \otimes B^{[q,\boldsymbol{p},\boldsymbol{k}]}_{\boldsymbol{n}} + E_{m+1,m+1} \otimes A^{[q,\boldsymbol{p},\boldsymbol{k}]}_{\boldsymbol{n}} \tag{3.39}$$

for all $m = 0, \dots, N-1$, where

$$A^{[q,\boldsymbol{p},\boldsymbol{k}]}_{\boldsymbol{n}} = (H_{[q]} + C_{[q]}) \otimes M_{\boldsymbol{n},[\boldsymbol{p},\boldsymbol{k}]} + \frac{\Delta t}{2} M_{[q]} \otimes K_{\boldsymbol{n},[\boldsymbol{p},\boldsymbol{k}]}, \tag{3.40}$$

$$B^{[q,\boldsymbol{p},\boldsymbol{k}]}_{\boldsymbol{n}} = -J_{[q]} \otimes M_{\boldsymbol{n},[\boldsymbol{p},\boldsymbol{k}]}. \tag{3.41}$$

Now we recall that the computation of the numerical solution is equivalent to finding the vector $\mathbf{u}$ satisfying the linear system (3.10) for all $m = 0, \dots, N-1$. In view of the definition (3.11) of $\mathbf{F}_m$ and the choice (3.14) for the basis in time, the component $(\mathbf{F}_m)_{\boldsymbol{i}}$ is nonzero at most for the pairs $\boldsymbol{i} = (i_1, i_2)$ such that $i_1 = (q+1)m+s$ with $1 \le s \le q+1$. Partition the vectors $\mathbf{u}$ and $\mathbf{F}_m$ as follows:

$$\mathbf{u} = \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \vdots \\ \mathbf{u}_N \end{bmatrix}, \quad \mathbf{F}_m = \begin{bmatrix} (\mathbf{F}_m)_1 \\ (\mathbf{F}_m)_2 \\ \vdots \\ (\mathbf{F}_m)_N \end{bmatrix},$$

where each $\mathbf{u}_j$ and $(\mathbf{F}_m)_j$ is a vector of length $\overline{n}(q+1)$. Then, $(\mathbf{F}_m)_j = \mathbf{0}$ for all $j \neq m+1$, and in view of (3.39) the requirement that $\mathbf{u}$ satisfies the linear system (3.10) for all $m = 0, \dots, N-1$ is equivalent to the requirement that $\mathbf{u}$ satisfies

$$\begin{bmatrix} A_{\boldsymbol{n}}^{[q,\boldsymbol{p},\boldsymbol{k}]} & & & \\ B_{\boldsymbol{n}}^{[q,\boldsymbol{p},\boldsymbol{k}]} & A_{\boldsymbol{n}}^{[q,\boldsymbol{p},\boldsymbol{k}]} & & \\ & \ddots & \ddots & \\ & & B_{\boldsymbol{n}}^{[q,\boldsymbol{p},\boldsymbol{k}]} & A_{\boldsymbol{n}}^{[q,\boldsymbol{p},\boldsymbol{k}]} \end{bmatrix} \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \vdots \\ \mathbf{u}_N \end{bmatrix} = \begin{bmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \\ \vdots \\ \mathbf{f}_N \end{bmatrix} \iff C_{N,\boldsymbol{n}}^{[q,\boldsymbol{p},\boldsymbol{k}]}\mathbf{u} = \mathbf{f},$$

where $\mathbf{f} = [\mathbf{f}_1, \mathbf{f}_2, \dots, \mathbf{f}_N]^T$, $\mathbf{f}_{m+1} = (\mathbf{F}_m)_{m+1}$ for all $m = 0, \dots, N-1$, and

$$C_{N,\boldsymbol{n}}^{[q,\boldsymbol{p},\boldsymbol{k}]} = \begin{bmatrix} A_{\boldsymbol{n}}^{[q,\boldsymbol{p},\boldsymbol{k}]} & & & \\ B_{\boldsymbol{n}}^{[q,\boldsymbol{p},\boldsymbol{k}]} & A_{\boldsymbol{n}}^{[q,\boldsymbol{p},\boldsymbol{k}]} & & \\ & \ddots & \ddots & \\ & & B_{\boldsymbol{n}}^{[q,\boldsymbol{p},\boldsymbol{k}]} & A_{\boldsymbol{n}}^{[q,\boldsymbol{p},\boldsymbol{k}]} \end{bmatrix}. \tag{3.42}$$

4. Properties of the space matrices. We investigate here some properties of the space matrices $M_{\boldsymbol{n},[\boldsymbol{p},\boldsymbol{k}]}$ and $K_{\boldsymbol{n},[\boldsymbol{p},\boldsymbol{k}]}$. Using the tensor-product structure of the B-spline basis functions $B_{\boldsymbol{2},[\boldsymbol{p},\boldsymbol{k}]}, \dots, B_{\boldsymbol{n}(\boldsymbol{p}-\boldsymbol{k})+\boldsymbol{k}-\boldsymbol{1},[\boldsymbol{p},\boldsymbol{k}]}$ and the rectangularity of the domain $(0,1)^d$, in the next lemma we show that $M_{\boldsymbol{n},[\boldsymbol{p},\boldsymbol{k}]}$ and $K_{\boldsymbol{n},[\boldsymbol{p},\boldsymbol{k}]}$ possess a tensor-product structure.

LEMMA 4.1. *Let $\boldsymbol{p}, \boldsymbol{n} \in \mathbb{N}^d$ and $\mathbf{0} \le \boldsymbol{k} \le \boldsymbol{p}-\mathbf{1}$. Then,*

$$K_{\boldsymbol{n},[\boldsymbol{p},\boldsymbol{k}]} = \sum_{s=1}^{d} \left(\bigotimes_{r=1}^{s-1} M_{n_r,[p_r,k_r]} \right) \otimes K_{n_s,[p_s,k_s]} \otimes \left(\bigotimes_{r=s+1}^{d} M_{n_r,[p_r,k_r]} \right), \tag{4.1}$$

$$M_{\boldsymbol{n},[\boldsymbol{p},\boldsymbol{k}]} = \bigotimes_{r=1}^{d} M_{n_r,[p_r,k_r]}, \tag{4.2}$$

where, for $p, n \in \mathbb{N}$ and $0 \le k \le p-1$, the matrices $M_{n,[p,k]}$ and $K_{n,[p,k]}$ are defined by

$$K_{n,[p,k]} = \left[\int_0^1 B'_{j+1,[p,k]}(x) B'_{i+1,[p,k]}(x) \mathrm{d}x \right]_{i,j=1}^{n(p-k)+k-1}, \tag{4.3}$$

$$M_{n,[p,k]} = \left[\int_0^1 B_{j+1,[p,k]}(x) B_{i+1,[p,k]}(x) \mathrm{d}x \right]_{i,j=1}^{n(p-k)+k-1}. \tag{4.4}$$

Proof. We only prove (4.1) as (4.2) is proved in the same way. For every $\boldsymbol{i},\boldsymbol{j} = \mathbf{1},\dots,\boldsymbol{n}(\boldsymbol{p}-\boldsymbol{k})+\boldsymbol{k}-\mathbf{1}$, we have

$$\begin{aligned}
(K_{\boldsymbol{n},[\boldsymbol{p},\boldsymbol{k}]})_{\boldsymbol{ij}} &= \int_{[0,1]^d} \nabla B_{\boldsymbol{j}+\mathbf{1},[\boldsymbol{p},\boldsymbol{k}]}(\mathbf{x})\cdot\nabla B_{\boldsymbol{i}+\mathbf{1},[\boldsymbol{p},\boldsymbol{k}]}(\mathbf{x})\mathrm{d}\mathbf{x}\\
&= \sum_{s=1}^{d}\int_0^1 B'_{j_s+1,[p_s,k_s]}(x_s)B'_{i_s+1,[p_s,k_s]}(x_s)\mathrm{d}x_s \times\\
&\qquad \times \prod_{\substack{r=1\\ r\neq s}}^{d}\int_0^1 B_{j_r+1,[p_r,k_r]}(x_r)B_{i_r+1,[p_r,k_r]}(x_r)\mathrm{d}x_r\\
&= \sum_{s=1}^{d}\left(\left(\bigotimes_{r=1}^{s-1} M_{n_r,[p_r,k_r]}\right)\otimes K_{n_s,[p_s,k_s]}\otimes\left(\bigotimes_{r=s+1}^{d} M_{n_r,[p_r,k_r]}\right)\right)_{\boldsymbol{ij}}\\
&= \left(\sum_{s=1}^{d}\left(\bigotimes_{r=1}^{s-1} M_{n_r,[p_r,k_r]}\right)\otimes K_{n_s,[p_s,k_s]}\otimes\left(\bigotimes_{r=s+1}^{d} M_{n_r,[p_r,k_r]}\right)\right)_{\boldsymbol{ij}},
\end{aligned}$$

where the third equality holds by (2.6) and by definition of $K_{n,[p,k]}$ and $M_{n,[p,k]}$. ∎

For $p,n\in\mathbb{N}$ and $0\le k\le p-1$, let

$$K^{[\ell]}_{[p,k]} = \left[\int_{\mathbb{R}} \hat{\varphi}'_{s',[p,k]}(x)\hat{\varphi}'_{s,[p,k]}(x-\ell)\mathrm{d}x\right]_{s,s'=1}^{p-k}, \quad \ell\in\mathbb{Z}, \tag{4.5}$$

$$M^{[\ell]}_{[p,k]} = \left[\int_{\mathbb{R}} \hat{\varphi}_{s',[p,k]}(x)\hat{\varphi}_{s,[p,k]}(x-\ell)\mathrm{d}x\right]_{s,s'=1}^{p-k}. \quad \ell\in\mathbb{Z}, \tag{4.6}$$

Due to (3.24), the integrals over $\mathbb{R}$ appearing in (4.5)–(4.6) actually reduce to integrals over $[0,\eta]$. For the same reason, the blocks (4.5)–(4.6) corresponding to indices $\ell\notin\{-\eta+1,\dots,\eta-1\}$ reduce to the zero block:

$$K^{[\ell]}_{[p,k]} = M^{[\ell]}_{[p,k]} = O_{p-k}, \quad |\ell|\ge\eta.$$

By direct computation one can show that $M^{[-\ell]}_{[p,k]} = (M^{[\ell]}_{[p,k]})^T$ and $K^{[-\ell]}_{[p,k]} = (K^{[\ell]}_{[p,k]})^T$ for all $\ell\in\mathbb{Z}$. Define the following $(p-k)\times(p-k)$ Hermitian matrix-valued functions:

$$\mathbf{f}_{[p,k]} : [-\pi,\pi]\to\mathbb{C}^{(p-k)\times(p-k)},$$

$$\mathbf{f}_{[p,k]}(\theta) = \sum_{\ell\in\mathbb{Z}} K^{[\ell]}_{[p,k]}\mathrm{e}^{\mathrm{i}\ell\theta} = K^{[0]}_{[p,k]} + \sum_{\ell=1}^{\eta-1}\Big(K^{[\ell]}_{[p,k]}\mathrm{e}^{\mathrm{i}\ell\theta} + (K^{[\ell]}_{[p,k]})^T\mathrm{e}^{-\mathrm{i}\ell\theta}\Big), \tag{4.7}$$

$$\mathbf{h}_{[p,k]} : [-\pi,\pi]\to\mathbb{C}^{(p-k)\times(p-k)},$$

$$\mathbf{h}_{[p,k]}(\theta) = \sum_{\ell\in\mathbb{Z}} M^{[\ell]}_{[p,k]}\mathrm{e}^{\mathrm{i}\ell\theta} = M^{[0]}_{[p,k]} + \sum_{\ell=1}^{\eta-1}\Big(M^{[\ell]}_{[p,k]}\mathrm{e}^{\mathrm{i}\ell\theta} + (M^{[\ell]}_{[p,k]})^T\mathrm{e}^{-\mathrm{i}\ell\theta}\Big). \tag{4.8}$$

Lemma 4.2. *Let $p,n\in\mathbb{N}$ and $0\le k\le p-1$. Let $\tilde{M}_{n,[p,k]}$ (resp., $\tilde{K}_{n,[p,k]}$) be the principal submatrix of $M_{n,[p,k]}$ (resp., $K_{n,[p,k]}$) of size $(n-\nu)(p-k)$ corresponding to the indices $k+1,\dots,k+(n-\nu)(p-k)$, where ν is defined in (3.22). Then,*

$$\tilde{M}_{n,[p,k]} = n^{-1}T_{n-\nu}(\mathbf{h}_{[p,k]}), \qquad \tilde{K}_{n,[p,k]} = n\,T_{n-\nu}(\mathbf{f}_{[p,k]}). \tag{4.9}$$

Proof. We only prove the left equation in (4.9) as the proof of the right equation is completely analogous. For convenience, we index the B-splines (3.23), as well as the entries of the matrices $\tilde{M}_{n,[p,k]}$ and $T_{n-\nu}(\mathbf{h}_{[p,k]})$, by a bi-index (r,s) such that $1 \le r \le n-\nu$ and $1 \le s \le p-k$. Of course, it is understood that (r,s) varies in the bi-index range $(1,1),\dots,(n-\nu,p-k)$ according to the standard lexicographic ordering (2.1). In this notation, we can rewrite (3.23) as follows:

$$\begin{aligned} B_{(r,s),[p,k]}(x) &= B_{k+1+(p-k)(r-1)+s,[p,k]}(x) \\ &= \hat{\varphi}_{s,[p,k]}(nx-r+1), \quad (r,s)=(1,1),\dots,(n-\nu,p-k). \end{aligned} \tag{4.10}$$

For all $(r,s),(r',s')=(1,1),\dots,(n-\nu,p-k)$, we have

$$\begin{aligned} (\tilde{M}_{n,[p,k]})_{(r,s),(r',s')} &= \int_0^1 B_{(r',s'),[p,k]}(x)B_{(r,s),[p,k]}(x)\mathrm{d}x \\ &= \int_0^1 \hat{\varphi}_{s',[p,k]}(nx-r'+1)\hat{\varphi}_{s,[p,k]}(nx-r+1)\mathrm{d}x \\ &= n^{-1}\int_{-r'+1}^{n-r'+1} \hat{\varphi}_{s',[p,k]}(y)\hat{\varphi}_{s,[p,k]}(y-r+r')\mathrm{d}y \end{aligned}$$

and

$$(T_{n-\nu}(\mathbf{h}_{[p,k]}))_{(r,s),(r',s')} = (M_{[p,k]}^{[r-r']})_{s,s'} = \int_{\mathbb{R}} \hat{\varphi}_{s',[p,k]}(y)\hat{\varphi}_{s,[p,k]}(y-r+r')\mathrm{d}y.$$

Since, by (3.24), $\operatorname{supp}(\hat{\varphi}_{s',[p,k]}) \subseteq [0,\eta] \subseteq [-r'+1, n-r'+1]$ for all $r'=1,\dots,n-\nu$ and $s'=1,\dots,p-k$, we have

$$\int_{-r'+1}^{n-r'+1} \hat{\varphi}_{s',[p,k]}(y)\hat{\varphi}_{s,[p,k]}(y-r+r')\mathrm{d}y = \int_{\mathbb{R}} \hat{\varphi}_{s',[p,k]}(y)\hat{\varphi}_{s,[p,k]}(y-r+r')\mathrm{d}y$$

for all $(r,s),(r',s')=(1,1),\dots,(n-\nu,p-k)$. Hence,

$$(\tilde{M}_{n,[p,k]})_{(r,s),(r',s')} = (n^{-1}T_{n-\nu}(\mathbf{h}_{[p,k]}))_{(r,s),(r',s')}$$

for all $(r,s),(r',s')=(1,1),\dots,(n-\nu,p-k)$, i.e., $\tilde{M}_{n,[p,k]} = n^{-1}T_{n-\nu}(\mathbf{h}_{[p,k]})$. ∎

LEMMA 4.3. *For all $p,n \in \mathbb{N}$ and $0 \le k \le p-1$ we have*

$$\|nM_{n,[p,k]}\|_\infty,\ \|n^{-1}K_{n,[p,k]}\|_\infty \le C_p$$

for some constant C_p depending only on p.

Proof. By the local support property, non-negativity and partition of unity property of B-splines (see (3.16), (3.18) and (3.19)), we have

$$\begin{aligned} \|nM_{n,[p,k]}\|_\infty &= \max_{i=1,\dots,n(p-k)+k-1} \sum_{j=1}^{n(p-k)+k-1} |(nM_{n,[p,k]})_{ij}| \\ &= \max_{i=1,\dots,n(p-k)+k-1} \sum_{j=1}^{n(p-k)+k-1} n\int_0^1 B_{j+1,[p,k]}(x)B_{i+1,[p,k]}(x)\mathrm{d}x \\ &\le n \max_{i=1,\dots,n(p-k)+k-1} \int_0^1 B_{i+1,[p,k]}(x)\mathrm{d}x \end{aligned}$$

$$
\begin{aligned}
&= n \max_{i=1,\dots,n(p-k)+k-1} \int_{[\xi_{i+1},\xi_{i+p+2}]} B_{i+1,[p,k]}(x)\mathrm{d}x \\
&\le n \max_{i=1,\dots,n(p-k)+k-1} (\xi_{i+p+2}-\xi_{i+1}) \le p+1.
\end{aligned}
$$

By the local support property and the bounds for the derivatives of B-splines (see (3.16) and (3.20)), we have

$$
\begin{aligned}
\|n^{-1}K_{n,[p,k]}\|_\infty &= \max_{i=1,\dots,n(p-k)+k-1} \sum_{j=1}^{n(p-k)+k-1} |(n^{-1}K_{n,[p,k]})_{ij}| \\
&\le \max_{i=1,\dots,n(p-k)+k-1} \sum_{j=1}^{n(p-k)+k-1} n^{-1}\int_0^1 |B'_{j+1,[p,k]}(x)|\,|B'_{i+1,[p,k]}(x)|\mathrm{d}x \\
&\le c_p \max_{i=1,\dots,n(p-k)+k-1} \int_0^1 |B'_{i+1,[p,k]}(x)|\mathrm{d}x \\
&= c_p \max_{i=1,\dots,n(p-k)+k-1} \int_{[\xi_{i+1},\xi_{i+p+2}]} |B'_{i+1,[p,k]}(x)|\mathrm{d}x \\
&\le c_p^2 n \max_{i=1,\dots,n(p-k)+k-1} (\xi_{i+p+2}-\xi_{i+1}) \le c_p^2(p+1). \qquad \square
\end{aligned}
$$

5. Spectral symbol for the sequence of normalized space-time matrices. This section is devoted to the proof of the first main result of this paper, which gives the spectral distribution and the spectral symbol of the normalized space-time matrices $C_{N,\boldsymbol{n}}^{[q,\boldsymbol{p},\boldsymbol{k}]}$ under suitable assumptions on the discretization parameters N and $\boldsymbol{n}$. Drawing inspiration from the multi-index notation, for any vector $\boldsymbol{\alpha}=(\alpha_1,\dots,\alpha_d)$ we set $P(\boldsymbol{\alpha})=\prod_{i=1}^d \alpha_i$.

THEOREM 5.1. *Let $q\in\mathbb{N}$, $\boldsymbol{p}\in\mathbb{N}^d$ and $\boldsymbol{0}\le\boldsymbol{k}\le\boldsymbol{p}-\boldsymbol{1}$. Suppose that the following conditions are met:*

- *$\boldsymbol{n}=\boldsymbol{\alpha}n$, where $\boldsymbol{\alpha}=(\alpha_1,\dots,\alpha_d)$ is a vector with positive components in $\mathbb{Q}^d$ and n varies in some infinite subset of $\mathbb{N}$ such that $\boldsymbol{n}=\boldsymbol{\alpha}n\in\mathbb{N}^d$;*
- *$N=N(n)$ is such that $N\to\infty$ and $N/n^2\to 0$ as $n\to\infty$.*

Then, for the sequence of normalized space-time matrices $\{2Nn^{d-2}C_{N,\boldsymbol{n}}^{[q,\boldsymbol{p},\boldsymbol{k}]}\}_n$ we have

$$
\{2Nn^{d-2}C_{N,\boldsymbol{n}}^{[q,\boldsymbol{p},\boldsymbol{k}]}\}_n \sim_\lambda \mathbf{f}_{[q,\boldsymbol{p},\boldsymbol{k}]}^{[\boldsymbol{\alpha}]},
$$

where:

- *$\mathbf{f}_{[q,\boldsymbol{p},\boldsymbol{k}]}^{[\boldsymbol{\alpha}]}$ is defined as*

$$
\begin{aligned}
&\mathbf{f}_{[q,\boldsymbol{p},\boldsymbol{k}]}^{[\boldsymbol{\alpha}]} : [-\pi,\pi]^d \to \mathbb{C}^{(q+1)P(\boldsymbol{p}-\boldsymbol{k})\times(q+1)P(\boldsymbol{p}-\boldsymbol{k})}, \\
&\mathbf{f}_{[q,\boldsymbol{p},\boldsymbol{k}]}^{[\boldsymbol{\alpha}]}(\boldsymbol{\theta}) = \mathbf{f}_{[\boldsymbol{p},\boldsymbol{k}]}^{[\boldsymbol{\alpha}]}(\boldsymbol{\theta}) \otimes \mathrm{T}M_{[q]};
\end{aligned} \tag{5.1}
$$

- *$\mathbf{f}_{[\boldsymbol{p},\boldsymbol{k}]}^{[\boldsymbol{\alpha}]}$ is defined as*

$$
\begin{aligned}
&\mathbf{f}_{[\boldsymbol{p},\boldsymbol{k}]}^{[\boldsymbol{\alpha}]} : [-\pi,\pi]^d \to \mathbb{C}^{P(\boldsymbol{p}-\boldsymbol{k})\times P(\boldsymbol{p}-\boldsymbol{k})}, \\
&\mathbf{f}_{[\boldsymbol{p},\boldsymbol{k}]}^{[\boldsymbol{\alpha}]}(\boldsymbol{\theta}) = \frac{1}{P(\boldsymbol{\alpha})}\sum_{s=1}^d \alpha_s^2 \left(\bigotimes_{r=1}^{s-1} \mathbf{h}_{[p_r,k_r]}(\theta_r)\right) \otimes \mathbf{f}_{[p_s,k_s]}(\theta_s) \otimes \left(\bigotimes_{r=s+1}^{d} \mathbf{h}_{[p_r,k_r]}(\theta_r)\right);
\end{aligned} \tag{5.2}
$$

- $\mathbf{f}_{[p,k]}$ *and* $\mathbf{h}_{[p,k]}$ *are given by* (4.7) *and* (4.8) *for all* $p \in \mathbb{N}$ *and* $0 \le k \le p-1$*;*
- T *is the final time in* (3.1) *and* $M_{[q]}$ *is given in* (3.35)*.*

Proof. The proof consists of two steps. The letter C will be used to denote a generic constant independent of n.

Step 1. By (3.40)–(3.42),

$$\begin{aligned} C_{N,\boldsymbol{n}}^{[q,\boldsymbol{p},\boldsymbol{k}]} &= T_N(\mathrm{e}^{\mathrm{i}\theta}) \otimes B_{\boldsymbol{n}}^{[q,\boldsymbol{p},\boldsymbol{k}]} + I_N \otimes A_{\boldsymbol{n}}^{[q,\boldsymbol{p},\boldsymbol{k}]} \\ &= -T_N(\mathrm{e}^{\mathrm{i}\theta}) \otimes J_{[q]} \otimes M_{\boldsymbol{n},[\boldsymbol{p},\boldsymbol{k}]} + I_N \otimes (H_{[q]} + C_{[q]}) \otimes M_{\boldsymbol{n},[\boldsymbol{p},\boldsymbol{k}]} \\ &\quad + \frac{\Delta t}{2} I_N \otimes M_{[q]} \otimes K_{\boldsymbol{n},[\boldsymbol{p},\boldsymbol{k}]}. \end{aligned}$$

Consider the decomposition of $2Nn^{d-2}C_{N,\boldsymbol{n}}^{[q,\boldsymbol{p},\boldsymbol{k}]}$ given by

$$2Nn^{d-2}C_{N,\boldsymbol{n}}^{[q,\boldsymbol{p},\boldsymbol{k}]} = X_{N,\boldsymbol{n}}^{[q,\boldsymbol{p},\boldsymbol{k}]} + Y_{N,\boldsymbol{n}}^{[q,\boldsymbol{p},\boldsymbol{k}]},$$

where

$$\begin{aligned} X_{N,\boldsymbol{n}}^{[q,\boldsymbol{p},\boldsymbol{k}]} &= 2Nn^{d-2}\frac{\Delta t}{2} I_N \otimes M_{[q]} \otimes K_{\boldsymbol{n},[\boldsymbol{p},\boldsymbol{k}]} = \mathrm{T}n^{d-2} I_N \otimes M_{[q]} \otimes K_{\boldsymbol{n},[\boldsymbol{p},\boldsymbol{k}]}, \\ Y_{N,\boldsymbol{n}}^{[q,\boldsymbol{p},\boldsymbol{k}]} &= -2Nn^{d-2} T_N(\mathrm{e}^{\mathrm{i}\theta}) \otimes J_{[q]} \otimes M_{\boldsymbol{n},[\boldsymbol{p},\boldsymbol{k}]} + 2Nn^{d-2} I_N \otimes (H_{[q]} + C_{[q]}) \otimes M_{\boldsymbol{n},[\boldsymbol{p},\boldsymbol{k}]}. \end{aligned}$$

By Lemma 4.1 and the equation $\boldsymbol{n} = \boldsymbol{\alpha} n$, we have

$$\begin{aligned} n^{d-2} K_{\boldsymbol{n},[\boldsymbol{p},\boldsymbol{k}]} = \frac{1}{P(\boldsymbol{\alpha})} \sum_{s=1}^{d} \left(\bigotimes_{r=1}^{s-1} n_r M_{n_r,[p_r,k_r]} \right) \otimes \alpha_s^2 n_s^{-1} K_{n_s,[p_s,k_s]} \\ \otimes \left(\bigotimes_{r=s+1}^{d} n_r M_{n_r,[p_r,k_r]} \right), \end{aligned} \tag{5.3}$$

$$n^{d-2} M_{\boldsymbol{n},[\boldsymbol{p},\boldsymbol{k}]} = \frac{n^{-2}}{P(\boldsymbol{\alpha})} \bigotimes_{r=1}^{d} n_r M_{n_r,[p_r,k_r]}. \tag{5.4}$$

By property (2.5), Lemma 4.3, and the equation $\|T_N(\mathrm{e}^{\mathrm{i}\theta})\| = 1$, we have

$$\|X_{N,\boldsymbol{n}}^{[q,\boldsymbol{p},\boldsymbol{k}]}\| \le C, \quad \|Y_{N,\boldsymbol{n}}^{[q,\boldsymbol{p},\boldsymbol{k}]}\| \le CN/n^2.$$

Since $N/n^2 \to 0$ by assumption, we have $\|Y_{N,\boldsymbol{n}}^{[q,\boldsymbol{p},\boldsymbol{k}]}\| \to 0$ as $n \to \infty$. Hence, by (2.3),

$$\|Y_{N,\boldsymbol{n}}^{[q,\boldsymbol{p},\boldsymbol{k}]}\|_1 \le \|Y_{N,\boldsymbol{n}}^{[q,\boldsymbol{p},\boldsymbol{k}]}\| \overline{n}\overline{N} = o(\overline{n}\overline{N}).$$

Since $X_{N,\boldsymbol{n}}^{[q,\boldsymbol{p},\boldsymbol{k}]}$ is symmetric, by Theorem 2.4 the thesis is proved if we show that

$$\{X_{N,\boldsymbol{n}}^{[q,\boldsymbol{p},\boldsymbol{k}]}\}_n \sim_\lambda \mathbf{f}_{[q,\boldsymbol{p},\boldsymbol{k}]}^{[\boldsymbol{\alpha}]}. \tag{5.5}$$

By (2.7), there exists a permutation matrix $\Pi_{\overline{n},\overline{N}}$, depending only on $\overline{n}$ and $\overline{N}$, such that

$$\begin{aligned} X_{N,\boldsymbol{n}}^{[q,\boldsymbol{p},\boldsymbol{k}]} &= \mathrm{T}n^{d-2} I_N \otimes M_{[q]} \otimes K_{\boldsymbol{n},[\boldsymbol{p},\boldsymbol{k}]} = T_N(\mathrm{T}M_{[q]}) \otimes n^{d-2} K_{\boldsymbol{n},[\boldsymbol{p},\boldsymbol{k}]} \\ &= \Pi_{\overline{n},\overline{N}} \big(n^{d-2} K_{\boldsymbol{n},[\boldsymbol{p},\boldsymbol{k}]} \otimes T_N(\mathrm{T}M_{[q]}) \big) \Pi_{\overline{n},\overline{N}}^T. \end{aligned}$$

Proving (5.5) is then equivalent to proving that

$$\{n^{d-2}K_{\boldsymbol{n},[\boldsymbol{p},\boldsymbol{k}]} \otimes T_N(\mathrm{T}M_{[q]})\}_n \sim_\lambda \mathbf{f}^{[\boldsymbol{\alpha}]}_{[q,\boldsymbol{p},\boldsymbol{k}]} = \mathbf{f}^{[\boldsymbol{\alpha}]}_{[\boldsymbol{p},\boldsymbol{k}]} \otimes \mathrm{T}M_{[q]}. \tag{5.6}$$

By Lemma 2.6 and Theorem 2.7, the spectral distribution (5.6) is established as soon as we have proved that

$$\{n^{d-2}K_{\boldsymbol{n},[\boldsymbol{p},\boldsymbol{k}]}\}_n \sim_\lambda \mathbf{f}^{[\boldsymbol{\alpha}]}_{[\boldsymbol{p},\boldsymbol{k}]}. \tag{5.7}$$

The next step is devoted to the proof of (5.7).

Step 2. For $p, n \in \mathbb{N}$ and $0 \le k \le p-1$, let ν be defined as in (3.22) and let $P_{n,[p,k]} \in \mathbb{C}^{(n(p-k)+k-1)\times(n-\nu)(p-k)}$ be the matrix having $I_{(n-\nu)(p-k)}$ as the submatrix corresponding to the row and column indices $i, j = k+1, \dots, k+(n-\nu)(p-k)$ and zeros elsewhere. Let

$$P_{\boldsymbol{n},[\boldsymbol{p},\boldsymbol{k}]} = P_{n_1,[p_1,k_1]} \otimes \cdots \otimes P_{n_d,[p_d,k_d]}.$$

Noting that $P^T_{n,[p,k]}P_{n,[p,k]} = I_{(n-\nu)(p-k)}$, by (2.4) we have

$$P^T_{\boldsymbol{n},[\boldsymbol{p},\boldsymbol{k}]}P_{\boldsymbol{n},[\boldsymbol{p},\boldsymbol{k}]} = I_{(n_1-\nu_1)(p_1-k_1)} \otimes \cdots \otimes I_{(n_d-\nu_d)(p_d-k_d)} = I_{P((\boldsymbol{n}-\boldsymbol{\nu})(\boldsymbol{p}-\boldsymbol{k}))}.$$

By (5.3), Lemma 4.2, and (2.4),

$$\begin{aligned}
&P^T_{\boldsymbol{n},[\boldsymbol{p},\boldsymbol{k}]}(n^{d-2}K_{\boldsymbol{n},[\boldsymbol{p},\boldsymbol{k}]})P_{\boldsymbol{n},[\boldsymbol{p},\boldsymbol{k}]} \\
&= \frac{1}{P(\boldsymbol{\alpha})}\sum_{s=1}^{d}\alpha_s^2\left(\bigotimes_{r=1}^{s-1}T_{n_r-\nu_r}(\mathbf{h}_{[p_r,k_r]})\right)\otimes T_{n_s-\nu_s}(\mathbf{f}_{[p_s,k_s]})\otimes\left(\bigotimes_{r=s+1}^{d}T_{n_r-\nu_r}(\mathbf{h}_{[p_r,k_r]})\right).
\end{aligned}$$

Hence, by Lemma 2.8,

$$P^T_{\boldsymbol{n},[\boldsymbol{p},\boldsymbol{k}]}(n^{d-2}K_{\boldsymbol{n},[\boldsymbol{p},\boldsymbol{k}]})P_{\boldsymbol{n},[\boldsymbol{p},\boldsymbol{k}]} = \Gamma_{\boldsymbol{n}-\boldsymbol{\nu},\boldsymbol{p}-\boldsymbol{k}}\, T_{\boldsymbol{n}-\boldsymbol{\nu}}(\mathbf{f}^{[\boldsymbol{\alpha}]}_{[\boldsymbol{p},\boldsymbol{k}]})\,\Gamma^T_{\boldsymbol{n}-\boldsymbol{\nu},\boldsymbol{p}-\boldsymbol{k}}.$$

Thus, Theorems 2.5 and 2.7 yield $\{n^{d-2}K_{\boldsymbol{n},[\boldsymbol{p},\boldsymbol{k}]}\}_n \sim_\lambda \mathbf{f}^{[\boldsymbol{\alpha}]}_{[\boldsymbol{p},\boldsymbol{k}]}$. ∎

5.1. Heat equation with variable diffusion coefficients. A more general version of the heat equation (3.1) is given by

$$\begin{cases}
\partial_t u(t,\mathbf{x}) - \nabla\cdot K(\mathbf{x})\nabla u(t,\mathbf{x}) = f(t,\mathbf{x}), & (t,\mathbf{x}) \in (0,\mathrm{T})\times(0,1)^d,\\
u(t,\mathbf{x}) = 0, & (t,\mathbf{x}) \in (0,\mathrm{T})\times\partial((0,1)^d),\\
u(t,\mathbf{x}) = 0, & (t,\mathbf{x}) \in \{0\}\times(0,1)^d,
\end{cases} \tag{5.8}$$

where $K(\mathbf{x})$ is a symmetric matrix in $\mathbb{R}^{d\times d}$, which is now supposed to vary with $\mathbf{x}$. We only assume that the symmetric matrix-valued function $K : [0,1]^d \to \mathbb{R}^{d\times d}$ belongs to $L^\infty([0,1]^d)$, i.e., its components K_{ij} belong to $L^\infty([0,1]^d)$. Suppose we adopt for (5.8) the same discretization as described in section 3, and denote by $C^{[q,\boldsymbol{p},\boldsymbol{k}]}_{N,\boldsymbol{n}}(K)$ the resulting space-time discretization matrix. Following the derivation of section 3, we see that $C^{[q,\boldsymbol{p},\boldsymbol{k}]}_{N,\boldsymbol{n}}(K)$ is given again by (3.42), with $A^{[q,\boldsymbol{p},\boldsymbol{k}]}_{\boldsymbol{n}}$ and $B^{[q,\boldsymbol{p},\boldsymbol{k}]}_{\boldsymbol{n}}$ defined as in (3.40)–(3.41). The only difference is that the matrix $K_{\boldsymbol{n},[\boldsymbol{p},\boldsymbol{k}]}$ appearing in (3.40) is now replaced by the matrix

$$\begin{aligned}
K_{\boldsymbol{n},[\boldsymbol{p},\boldsymbol{k}]}(K) &= \left[\int_{[0,1]^d}\big[K(\mathbf{x})\nabla\varphi_j(\mathbf{x})\big]\cdot\nabla\varphi_i(\mathbf{x})\mathrm{d}\mathbf{x}\right]_{i,j=1}^{\overline{n}}\\
&= \left[\int_{[0,1]^d}\big[K(\mathbf{x})\nabla B_{\boldsymbol{j}+\boldsymbol{1},[\boldsymbol{p},\boldsymbol{k}]}(\mathbf{x})\big]\cdot\nabla B_{\boldsymbol{i}+\boldsymbol{1},[\boldsymbol{p},\boldsymbol{k}]}(\mathbf{x})\mathrm{d}\mathbf{x}\right]_{\boldsymbol{i},\boldsymbol{j}=\boldsymbol{1}}^{\boldsymbol{n}(\boldsymbol{p}-\boldsymbol{k})+\boldsymbol{k}-\boldsymbol{1}}.
\end{aligned} \tag{5.9}$$

THEOREM 5.2. *Let $q \in \mathbb{N}$, $\boldsymbol{p} \in \mathbb{N}^d$ and $\mathbf{0} \le \boldsymbol{k} \le \boldsymbol{p}-\mathbf{1}$. Suppose that $K : (0,1)^d \to \mathbb{R}^{d\times d}$ is a symmetric matrix-valued function in $L^\infty([0,1]^d)$ and that the following conditions are met:*

- *$\boldsymbol{n} = \boldsymbol{\alpha} n$, where $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_d)$ is a vector with positive components in $\mathbb{Q}^d$ and n varies in some infinite subset of $\mathbb{N}$ such that $\boldsymbol{n} = \boldsymbol{\alpha} n \in \mathbb{N}^d$;*
- *$N = N(n)$ is such that $N \to \infty$ and $N/n^2 \to 0$ as $n \to \infty$.*

Then, for the sequence of normalized space-time matrices $\{2Nn^{d-2}C_{N,\boldsymbol{n}}^{[q,\boldsymbol{p},\boldsymbol{k}]}(K)\}_n$ we have

$$\{2Nn^{d-2}C_{N,\boldsymbol{n}}^{[q,\boldsymbol{p},\boldsymbol{k}]}(K)\}_n \sim_\lambda \mathbf{f}_{[q,\boldsymbol{p},\boldsymbol{k}]}^{[\boldsymbol{\alpha},K]},$$

where:

- *$\mathbf{f}_{[q,\boldsymbol{p},\boldsymbol{k}]}^{[\boldsymbol{\alpha},K]}$ is defined as*

$$\begin{aligned} &\mathbf{f}_{[q,\boldsymbol{p},\boldsymbol{k}]}^{[\boldsymbol{\alpha},K]} : [0,1]^d \times [-\pi,\pi]^d \to \mathbb{C}^{(q+1)P(\boldsymbol{p}-\boldsymbol{k})\times(q+1)P(\boldsymbol{p}-\boldsymbol{k})}, \\ &\mathbf{f}_{[q,\boldsymbol{p},\boldsymbol{k}]}^{[\boldsymbol{\alpha},K]}(\mathbf{x},\boldsymbol{\theta}) = \mathbf{f}_{[\boldsymbol{p},\boldsymbol{k}]}^{[\boldsymbol{\alpha},K]}(\mathbf{x},\boldsymbol{\theta}) \otimes \mathrm{T}M_{[q]}; \end{aligned} \tag{5.10}$$

- *$\mathbf{f}_{[\boldsymbol{p},\boldsymbol{k}]}^{[\boldsymbol{\alpha},K]}$ is defined as*

$$\begin{aligned} &\mathbf{f}_{[\boldsymbol{p},\boldsymbol{k}]}^{[\boldsymbol{\alpha},K]} : [0,1]^d \times [-\pi,\pi]^d \to \mathbb{C}^{P(\boldsymbol{p}-\boldsymbol{k})\times P(\boldsymbol{p}-\boldsymbol{k})}, \\ &\mathbf{f}_{[\boldsymbol{p},\boldsymbol{k}]}^{[\boldsymbol{\alpha},K]}(\mathbf{x},\boldsymbol{\theta}) = \frac{1}{\alpha_1\cdots\alpha_d}\sum_{i,j=1}^d \alpha_i\alpha_j K_{ij}(\mathbf{x})(H_{[\boldsymbol{p},\boldsymbol{k}]})_{ij}(\boldsymbol{\theta}), \end{aligned} \tag{5.11}$$

- *$H_{[\boldsymbol{p},\boldsymbol{k}]}$ is the $d\times d$ block matrix whose (i,j) entry is the $P(\boldsymbol{p}-\boldsymbol{k})\times P(\boldsymbol{p}-\boldsymbol{k})$ block defined as follows:*

$$(H_{[\boldsymbol{p},\boldsymbol{k}]})_{ij} = \begin{cases} \big(\bigotimes_{r=1}^{i-1}\mathbf{h}_{[p_r,k_r]}\big)\otimes\mathbf{f}_{[p_i,k_i]}\otimes\big(\bigotimes_{r=i+1}^{d}\mathbf{h}_{[p_r,k_r]}\big), & i=j, \\ \big(\bigotimes_{r=1}^{i-1}\mathbf{h}_{[p_r,k_r]}\big)\otimes\mathbf{g}_{[p_i,k_i]}\otimes\big(\bigotimes_{r=i+1}^{j-1}\mathbf{h}_{[p_r,k_r]}\big) & \\ \qquad\otimes\mathbf{g}_{[p_j,k_j]}\otimes\big(\bigotimes_{r=j+1}^{d}\mathbf{h}_{[p_r,k_r]}\big), & i<j, \\ \big(\bigotimes_{r=1}^{j-1}\mathbf{h}_{[p_r,k_r]}\big)\otimes\mathbf{g}_{[p_j,k_j]}\otimes\big(\bigotimes_{r=j+1}^{i-1}\mathbf{h}_{[p_r,k_r]}\big) & \\ \qquad\otimes\mathbf{g}_{[p_i,k_i]}\otimes\big(\bigotimes_{r=i+1}^{d}\mathbf{h}_{[p_r,k_r]}\big), & i>j; \end{cases} \tag{5.12}$$

- *$\mathbf{f}_{[p,k]}, \mathbf{h}_{[p,k]}$ are given by (4.7)–(4.8) for all $p \in \mathbb{N}$ and $0 \le k \le p-1$, while $\mathbf{g}_{[p,k]}$ is defined similarly to $\mathbf{f}_{[p,k]}, \mathbf{h}_{[p,k]}$ as follows:*

$$\begin{aligned} &\mathbf{g}_{[p,k]} : [-\pi,\pi] \to \mathbb{C}^{(p-k)\times(p-k)}, \\ &\mathbf{g}_{[p,k]}(\theta) = \sum_{\ell\in\mathbb{Z}} H_{[p,k]}^{[\ell]}\mathrm{e}^{\mathrm{i}\ell\theta} = H_{[p,k]}^{[0]} + \sum_{\ell=1}^{\eta-1}\Big(H_{[p,k]}^{[\ell]}\mathrm{e}^{\mathrm{i}\ell\theta} - (H_{[p,k]}^{[\ell]})^T\mathrm{e}^{-\mathrm{i}\ell\theta}\Big), \end{aligned} \tag{5.13}$$

with

$$H_{[p,k]}^{[\ell]} = \left[\int_{\mathbb{R}} \hat{\varphi}'_{s',[p,k]}(x)\hat{\varphi}_{s,[p,k]}(x-\ell)\mathrm{d}x\right]_{s,s'=1}^{p-k}, \quad \ell \in \mathbb{Z}; \tag{5.14}$$

- *T is the final time in (5.8) and $M_{[q]}$ is given in (3.35).*

REMARK 5.3. *Theorem 5.2 is clearly an extension of Theorem 5.1. Indeed, in the case where $K(\mathbf{x}) = I_d$, the differential problem (5.8) reduces to the standard heat equation (3.1), the space-time matrix $C_{N,\boldsymbol{n}}^{[q,\boldsymbol{p},\boldsymbol{k}]}(K)$ reduces to $C_{N,\boldsymbol{n}}^{[q,\boldsymbol{p},\boldsymbol{k}]}(1) = C_{N,\boldsymbol{n}}^{[q,\boldsymbol{p},\boldsymbol{k}]}$, and the spectral symbol $\mathbf{f}_{[q,\boldsymbol{p},\boldsymbol{k}]}^{[\boldsymbol{\alpha},K]}$ in (5.10) reduces to $\mathbf{f}_{[q,\boldsymbol{p},\boldsymbol{k}]}^{[\boldsymbol{\alpha},1]} = \mathbf{f}_{[q,\boldsymbol{p},\boldsymbol{k}]}^{[\boldsymbol{\alpha}]}$ in (5.1).*

REMARK 5.4. *As the reader can easily figure out, the reason why this paper has been devoted to the heat equation* (3.1)*, rather than its more general version* (5.8)*, is essentially of technical type. Working with matrices coming from the discretization of variable-coefficient PDEs has a degree of mathematical complexity that has been successfully tackled through the theory of generalized locally Toeplitz (GLT) sequences* [14, 15, 29, 30]*, but only in the case of scalar-valued symbols. On the other hand, the use of a DG/FE discretization as in this paper necessarily leads to block-structured matrices with a matrix-valued symbol; see also* [17]*. This adds further complications that were only partially addressed in* [30, *section* 3.3]*, whereas a systematic treatment is now being developed in* [18] *along with the theory of block GLT sequences.*

Despite the technical difficulties highlighted by Remark 5.4, for the sake of completeness we report a proof of Theorem 5.2 in the appendix.

6. Numerical experiments. In this section we validate Theorem 5.2 through numerical examples that illustrate the effectiveness of the spectral symbol $\mathbf{f}_{[q,\boldsymbol{p},\boldsymbol{k}]}^{[\boldsymbol{\alpha},K]}$ in describing the asymptotic spectral distribution of the space-time discretization matrix $C_{N,\boldsymbol{n}}^{[q,\boldsymbol{p},\boldsymbol{k}]}(K)$. In all the examples we consider the differential problem (5.8) with $d=1$ and $\mathrm{T}=1$, and we choose as a reference basis $\{\ell_{1,[q]},\dots,\ell_{q+1,[q]}\}$ for $\mathbb{P}_q$ the Lagrange polynomials associated with the right Gauss–Radau nodes in $[-1,1]$. The diffusion coefficient $K(x)$ and the parameters N,n,q,p,k are specified in each example. Since $d=1$ and $\mathrm{T}=1$, the spectral symbol of the sequence $\{2Nn^{-1}C_{N,n}^{[q,p,k]}(K)\}_n$ under the assumptions of Theorem 5.2 is given by

$$\begin{aligned}&\mathbf{f}_{[q,p,k]}^{[1,K]}:[0,1]\times[-\pi,\pi]\to\mathbb{C}^{(q+1)(p-k)\times(q+1)(p-k)},\\ &\mathbf{f}_{[q,p,k]}^{[1,K]}(x,\theta)=K(x)\mathbf{f}_{[p,k]}(\theta)\otimes M_{[q]},\end{aligned}\tag{6.1}$$

where $\mathbf{f}_{[p,k]}(\theta)$ is defined in (4.7) and $M_{[q]}$ is defined in (3.35).

REMARK 6.1. *Let $\lambda_1(\mathbf{f}_{[p,k]}(\theta))\le\ldots\le\lambda_{p-k}(\mathbf{f}_{[p,k]}(\theta))$ be the eigenvalues of $\mathbf{f}_{[p,k]}(\theta)$ sorted in non-decreasing order. Since $\mathbf{f}_{[p,k]}(-\theta)=(\mathbf{f}_{[p,k]}(\theta))^T$ has the same eigenvalues as $\mathbf{f}_{[p,k]}(\theta)$, we infer that each eigenvalue function $\lambda_i(\mathbf{f}_{[p,k]}(\theta))$ is symmetric in θ, i.e., it is an even function. Therefore, also the eigenvalue functions of the symbol $\mathbf{f}_{[q,p,k]}^{[1,K]}(x,\theta)$ in* (6.1)*, namely $K(x)\lambda_i(\mathbf{f}_{[p,k]}(\theta))\lambda_j(M_{[q]})$, $i=1,\dots,p-k$, $j=1,\dots,q+1$, are symmetric in θ. Thus, the restriction*

$$\begin{aligned}&\mathbf{f}_{[q,p,k]}^{[1,K]}:[0,1]\times[0,\pi]\to\mathbb{C}^{(q+1)(p-k)\times(q+1)(p-k)},\\ &\mathbf{f}_{[q,p,k]}^{[1,K]}(x,\theta)=K(x)\mathbf{f}_{[p,k]}(\theta)\otimes M_{[q]},\end{aligned}\tag{6.2}$$

is again a spectral symbol for $\{2Nn^{-1}C_{N,n}^{[q,p,k]}(K)\}_n$ according to Definition 2.1. Furthermore, if $K(x)=1$ then $\mathbf{f}_{[q,p,k]}^{[1,K]}(x,\theta)=\mathbf{f}_{[q,p,k]}^{[1,1]}(x,\theta)$ does not depend on the variable x and the restriction

$$\begin{aligned}&\mathbf{f}_{[q,p,k]}^{[1,1]}:[0,\pi]\to\mathbb{C}^{(q+1)(p-k)\times(q+1)(p-k)},\\ &\mathbf{f}_{[q,p,k]}^{[1,1]}(\theta)=\mathbf{f}_{[p,k]}(\theta)\otimes M_{[q]},\end{aligned}\tag{6.3}$$

is again a spectral symbol for $\{2Nn^{-1}C_{N,n}^{[q,p,k]}(K)\}_n=\{2Nn^{-1}C_{N,n}^{[q,p,k]}(1)\}_n$ according to Definition 2.1. In all the examples of this section, when referring to the spectral symbol of the sequence $\{2Nn^{-1}C_{N,n}^{[q,p,k]}(K)\}_n$, we mean either (6.2) *or* (6.3)*, depending on whether $K(x)$ is non-constant or $K(x)=1$.*

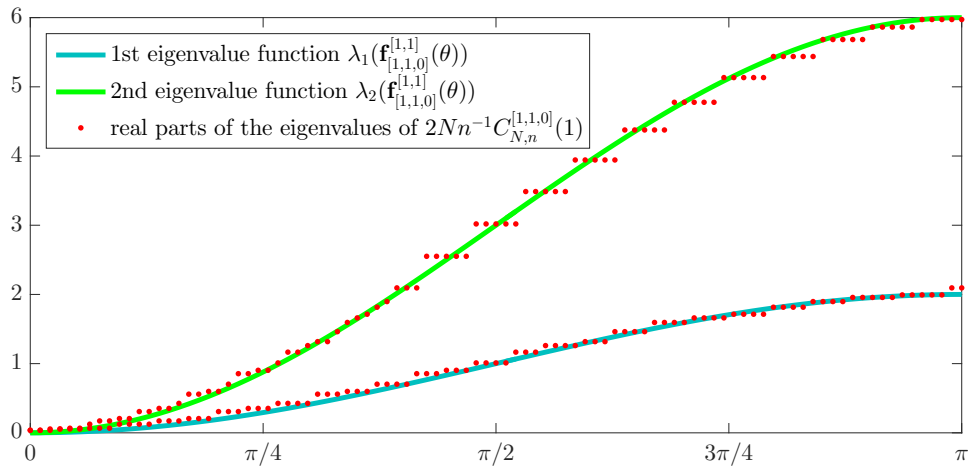

FIG. 6.1. *Example 6.2: Comparison between the spectrum of* $2Nn^{-1}C^{[1,1,0]}_{N,n}(1)$ *and the symbol* $\mathbf{f}^{[1,1]}_{[1,1,0]}(\theta)$ *for* $N = 5$ *and* $n = 20$.

EXAMPLE 6.2. *Let* $K(x) = 1$ *and take* $q = 1$, $p = 1$, $k = 0$. *In this basic case, a direct computation shows that the symbol* (6.3) *simplifies to*

$$\mathbf{f}^{[1,1]}_{[1,1,0]}(\theta) = (2 - 2\cos\theta)\begin{bmatrix} 3/2 & 0 \\ 0 & 1/2 \end{bmatrix}.$$

Its eigenvalue functions are then given by

$$\lambda_1(\mathbf{f}^{[1,1]}_{[1,1,0]}(\theta)) = 1 - 1\cos\theta,$$
$$\lambda_2(\mathbf{f}^{[1,1]}_{[1,1,0]}(\theta)) = 3 - 3\cos\theta.$$

Fig. 6.1 shows the graphs of the eigenvalue functions over $[0, \pi]$ *and the real parts* $\varrho_1, \ldots, \varrho_{2N(n-1)}$ *of the eigenvalues of* $2Nn^{-1}C^{[1,1,0]}_{N,n}(1)$ *for* $N = 5$ *and* $n = 20$. *The imaginary parts of the eigenvalues are not shown in the figure because they are negligible (their maximum modulus is about* $1.57 \cdot 10^{-4}$*). The real parts of the eigenvalues are sorted so as to match the graphs of the eigenvalue functions and are represented by the dots placed at the points*

$$(\theta_j, \varrho_j), \quad j = 1, \ldots, N(n-1),$$
$$(\theta_j, \varrho_{j+N(n-1)}), \quad j = 1, \ldots, N(n-1),$$

where

$$\theta_j = \frac{(j-1)\pi}{N(n-1)-1}, \quad j = 1, \ldots, N(n-1).$$

We clearly see from Fig. 6.1 that, although N *and* n *are not so large, the eigenvalues of* $2Nn^{-1}C^{[1,1,0]}_{N,n}(1)$ *can be subdivided into* 2 *subsets of the same cardinality* $N(n-1)$*; and the eigenvalues belonging to the* 1*st (resp.,* 2*nd) subset are approximately equal to the samples of the* 1*st (resp.,* 2*nd) eigenvalue function over the uniform grid* θ_j, $j = 1, \ldots, N(n-1)$. *This agrees with the informal meaning of the spectral distribution* $\{2Nn^{-1}C^{[1,1,0]}_{N,n}(1)\}_n \sim_\lambda \mathbf{f}^{[1,1]}_{[1,1,0]}$ *given in Remark 2.2. In particular, we observe no outliers in this case.*

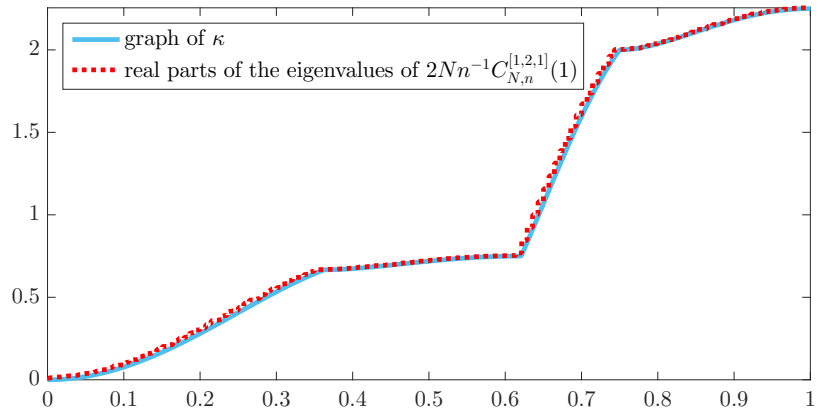

FIG. 6.2. *Comparison between the spectrum of* $2Nn^{-1}C_{N,n}^{[1,2,1]}(1)$ *and the canonical rearranged version* κ *of the symbol* $\mathbf{f}_{[1,2,1]}^{[1,1]}(\theta)$ *for* $N = 35$ *and* $n = 70$.

$2N = n$	20	30	40	50	60	70
$\|\kappa(\mathbf{y}) - \boldsymbol{\varrho}\|_\infty$	0.2909	0.2009	0.1533	0.1238	0.1039	0.0894
$\|\boldsymbol{\iota}\|_\infty$	0.0227	0.0196	0.0170	0.0146	0.0131	0.0117

TABLE 6.1

Computation of the error $\|\kappa(\mathbf{y}) - \boldsymbol{\varrho}\|_\infty$ *and the maximum modulus* $\|\boldsymbol{\iota}\|_\infty$ *of the imaginary parts of the eigenvalues of* $2Nn^{-1}C_{N,n}^{[1,2,1]}(1)$ *for increasing values of* $2N = n$.

EXAMPLE 6.3. *Let* $K(x) = 1$ *and take* $q = 1$, $p = 2$, $k = 1$. *In this case, the symbol* (6.3) *becomes*

$$\mathbf{f}_{[1,2,1]}^{[1,1]}(\theta) = \left(1 - \frac{2}{3}\cos\theta - \frac{1}{3}\cos(2\theta)\right)\begin{bmatrix} 3/2 & 0 \\ 0 & 1/2 \end{bmatrix}.$$

Let κ *be the canonical rearranged version of* $\mathbf{f}_{[1,2,1]}^{[1,1]}(\theta)$ *obtained as the limit of the piecewise linear functions* κ_r, *according to the construction of Remark* 2.3. *Then, by Remark* 2.3, κ *is again a spectral symbol for the sequence* $\{2Nn^{-1}C_{N,n}^{[1,2,1]}(1)\}_n$. *Fig.* 6.2 *shows the graph of* κ *and the real parts* $\varrho_1, \ldots, \varrho_{2Nn}$ *of the eigenvalues of* $2Nn^{-1}C_{N,n}^{[1,2,1]}(1)$ *for* $N = 35$ *and* $n = 70$. *The graph of* κ *has been obtained by plotting the graph of* κ_r *corresponding to a large value of* r ($r = 10000$). *The real parts of the eigenvalues have been sorted in non-decreasing order and placed at the points*

$$(y_j, \varrho_j), \qquad y_j = \frac{j}{2Nn}, \qquad j = 1, \ldots, 2Nn.$$

The imaginary parts of the eigenvalues are not shown in the figure because they are negligible (their maximum modulus is about 0.01*). We clearly see from the figure an excellent agreement between* κ *and the eigenvalues of* $2Nn^{-1}C_{N,n}^{[1,2,1]}(1)$. *As illustrated in Fig.* 6.3 *and Table* 6.1, *the agreement becomes perfect in the limit of mesh refinement* $2N = n \to \infty$. *Both Fig.* 6.3 *and Table* 6.1 *show that* $\|\kappa(\mathbf{y}) - \boldsymbol{\varrho}\|_\infty \to 0$ *and* $\|\boldsymbol{\iota}\|_\infty \to 0$ *as* $2N = n \to \infty$, *where* $\kappa(\mathbf{y}) - \boldsymbol{\varrho} = (\kappa(y_1) - \varrho_1, \ldots, \kappa(y_{2Nn}) - \varrho_{2Nn})$ *is the vector of the errors and* $\boldsymbol{\iota} = (\iota_1, \ldots, \iota_{2Nn})$ *is the vector of the imaginary parts of the eigenvalues of*

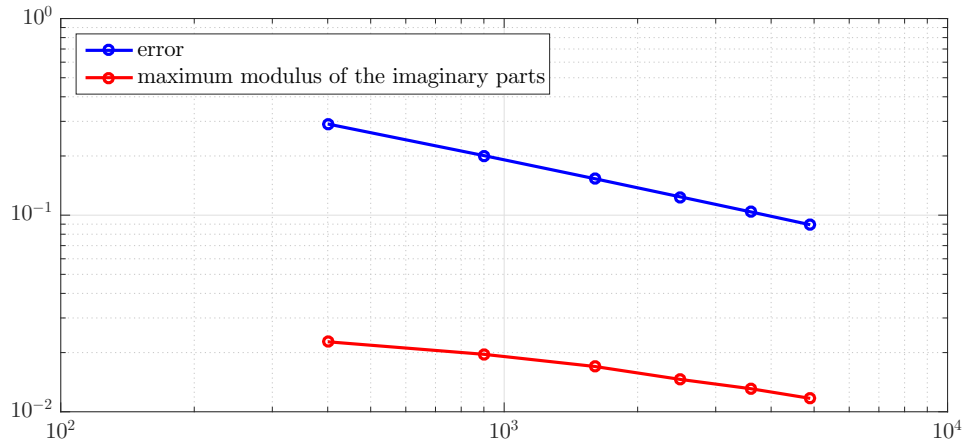

FIG. 6.3. *Graphs of the error* $\|\kappa(\mathbf{y}) - \boldsymbol{\varrho}\|_\infty$ *and the maximum modulus* $\|\boldsymbol{\iota}\|_\infty$ *of the imaginary parts of the eigenvalues of* $2Nn^{-1}C_{N,n}^{[1,2,1]}(1)$ *versus the matrix size* $2Nn$ *for increasing values of* $2N = n$.

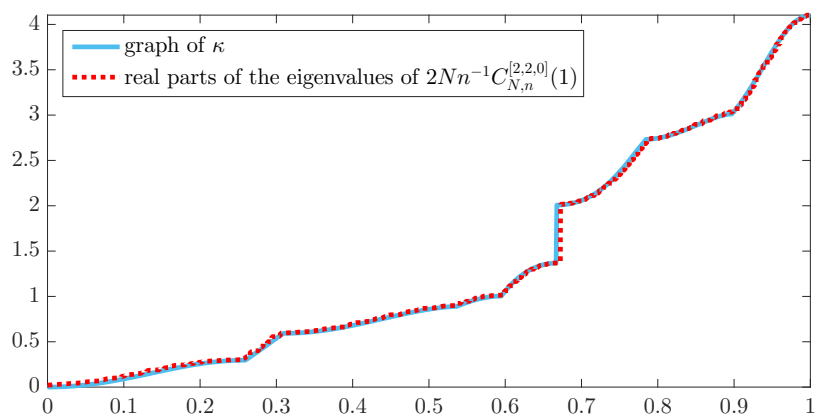

FIG. 6.4. *Comparison between the spectrum of* $2Nn^{-1}C_{N,n}^{[2,2,0]}(1)$ *and the canonical rearranged version* κ *of the symbol* $\mathbf{f}_{[2,2,0]}^{[1,1]}(\theta)$ *for* $N = n = 30$.

$2Nn^{-1}C_{N,n}^{[1,2,1]}(1)$. *We point out that in Fig. 6.3 and Table 6.1 the (unknown) function* κ *has been approximated with* κ_r *for* $r = 100000$ *instead of* $r = 10000$, *so as to obtain more accurate results.*

EXAMPLE 6.4. *Let* $K(x) = 1$ *and* $q = 2$, $p = 2$, $k = 0$. *The symbol* (6.3) *becomes*

$$\mathbf{f}_{[2,2,0]}^{[1,1]}(\theta) = \frac{1}{3}\begin{bmatrix} 4 & -2 - 2\mathrm{e}^{\mathrm{i}\theta} \\ -2 - 2\mathrm{e}^{-\mathrm{i}\theta} & 8 - 4\cos\theta \end{bmatrix} \otimes \frac{1}{18}\begin{bmatrix} 16 + \sqrt{6} & 0 & 0 \\ 0 & 16 - \sqrt{6} & 0 \\ 0 & 0 & 4 \end{bmatrix}.$$

Let κ *be the canonical rearranged version of* $\mathbf{f}_{[2,2,0]}^{[1,1]}(\theta)$ *obtained as the limit of the functions* κ_r, *according to the construction of Remark 2.3. Fig. 6.4 shows the graph of* κ *and the real parts* $\varrho_1, \ldots, \varrho_{3N(2n-1)}$ *of the eigenvalues of* $2Nn^{-1}C_{N,n}^{[2,2,0]}(1)$ *for* $N = n = 30$. *The graph of* κ *has been obtained by plotting the graph of* κ_r *corresponding to*

$r = 10000$. *The real parts of the eigenvalues have been sorted in non-decreasing order and placed at the points*

$$(y_j, \varrho_j), \qquad y_j = \frac{j}{3N(2n-1)}, \qquad j = 1, \dots, 3N(2n-1).$$

The imaginary parts of the eigenvalues are not shown in the figure because they are negligible (their maximum modulus is about 0.04*). We clearly see from the figure an excellent agreement between* κ *and the eigenvalues of* $2Nn^{-1}C_{N,n}^{[2,2,0]}(1)$.

EXAMPLE 6.5. *Let* $K(x) = 3 - \cos(10x)$ *and* $q = 2$, $p = 3$, $k = 2$. *The symbol* (6.2) *becomes*

$$\mathbf{f}_{[2,3,2]}^{[1,K]}(x,\theta) = K(x)\Big(\frac{2}{3} - \frac{1}{4}\cos\theta - \frac{2}{5}\cos(2\theta) - \frac{1}{60}\cos(3\theta)\Big)\frac{1}{18}\begin{bmatrix} 16+\sqrt{6} & 0 & 0 \\ 0 & 16-\sqrt{6} & 0 \\ 0 & 0 & 4 \end{bmatrix}.$$

Let κ *be the canonical rearranged version of* $\mathbf{f}_{[2,3,2]}^{[1,K]}(x,\theta)$ *obtained as the limit of the functions* κ_r*, according to the construction of Remark* 2.3. *Fig.* 6.5 *shows the graph of* κ *and the real parts* $\varrho_1, \dots, \varrho_{3N(n+1)}$ *of the eigenvalues of* $2Nn^{-1}C_{N,n}^{[2,3,2]}(K)$ *for* $N = n = 30$. *The graph of* κ *has been obtained by plotting the graph of* κ_r *corresponding to* $r = 2500$. *The real parts of the eigenvalues have been sorted in non-decreasing order and placed at the points*

$$(y_j, \varrho_j), \qquad y_j = \frac{j}{3N(n+1)}, \qquad j = 1, \dots, 3N(n+1).$$

The imaginary parts of the eigenvalues are not shown in the figure because they are negligible (their maximum modulus is about 0.077*). We clearly see from the figure a good matching between* κ *and the eigenvalues of* $2Nn^{-1}C_{N,n}^{[2,3,2]}(K)$. *In this case, however, we also note the appearance of a few outliers at the right end of the spectrum. Nevertheless, the total number of outliers is* 59*, which is negligible with respect to the matrix size* $3N(n+1) = 2790$. *Moreover, when passing from* $N = n = 30$ *to* $N = n = 40$*, the matrix size passes from* 2790 *to* 4920 *but the number of outliers only passes from* 59 *to* 79. *Further numerical experiments reveal that the number of outliers should be equal to* $2n-1$ *for all* N, n *such that* $N = n$*, and this is in agreement with Remark* 2.2*, according to which the number of outliers divided by the matrix size goes to* 0 *as the matrix size diverges to* ∞.

7. Conclusions and perspectives. We have considered the discretization of the multidimensional heat equation (3.1) and of its more general version (5.8) involving variable diffusion coefficients by a DG method in time and a FE method of arbitrary regularity in space. We have shown that the resulting (normalized) space-time discretization matrices possess an asymptotic spectral distribution, and we have identified the corresponding spectral symbol in Theorems 5.1 and 5.2. A program for future research consists of the following two items.

- Study the properties of the spectral symbol.
- Exploit the symbol and its properties to design/analyze appropriate solvers (especially parallel solvers) for the linear systems arising from the DG/FE approximation of both the heat equation (3.1) and the heat equation with variable diffusion coefficients (5.8).

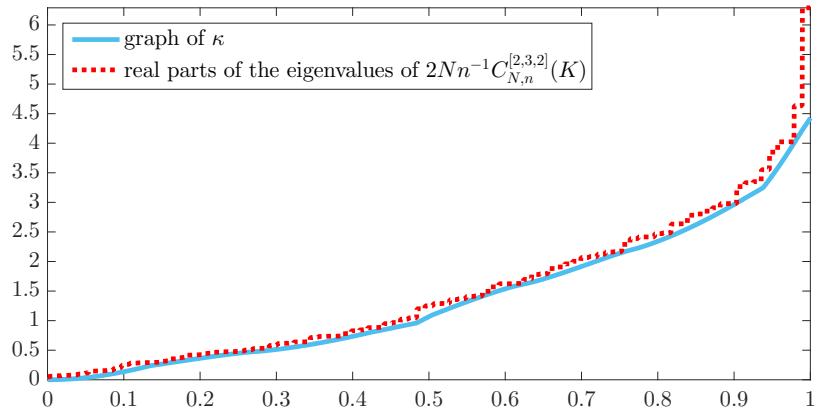

FIG. 6.5. *Comparison between the spectrum of* $2Nn^{-1}C_{N,n}^{[2,3,2]}(K)$ *and the canonical rearranged version* κ *of the symbol* $\mathbf{f}_{[2,3,2]}^{[1,K]}(x,\theta)$ *for* $N = n = 30$ *and* $K(x) = 3 - \cos(10x)$.

This program aims at repeating what has already been done for the IgA discretization of elliptic problems, where the symbol computed in [7, 10, 11, 12, 13] was studied and exploited in [5, 6, 8] to design/analyze ad-hoc fast solvers for IgA linear systems.

Appendix A. Proof of Theorem 5.2. This appendix is devoted to the proof of Theorem 5.2. As mentioned in Remark 5.4, such a proof is technically complicated. It is therefore convenient to first prove the theorem in the unidimensional case, where all the main ideas are already present but many technicalities can be avoided, and then illustrate the way to extend the proof to the multidimensional setting. We start with introducing the abstract notion of spectral distribution, which generalizes Definition 2.1, and the concept of approximating classes of sequences (a.c.s.es).

DEFINITION A.1. *Let* $\{X_n\}_n$ *be a sequence of matrices, with* X_n *of size* d_n *tending to infinity, and let* $\phi : C_c(\mathbb{C}) \to \mathbb{C}$. *We say that* $\{X_n\}_n$ *has an asymptotic spectral distribution described by* ϕ, *and we write* $\{X_n\}_n \sim_\lambda \phi$, *if*

$$\lim_{n\to\infty} \frac{1}{d_n} \sum_{j=1}^{d_n} F(\lambda_j(X_n)) = \phi(F), \quad \forall\, F \in C_c(\mathbb{C}).$$

DEFINITION A.2. *Let* $\{A_n\}_n$ *be a sequence of matrices and let* $\{\{B_{n,m}\}_n\}_m$ *be a sequence of sequences of matrices, with* A_n *and* $B_{n,m}$ *of size* d_n *tending to infinity. We say that* $\{\{B_{n,m}\}_n\}_m$ *is an approximating class of sequences (a.c.s.) for* $\{A_n\}_n$, *and we write* $\{B_{n,m}\}_n \xrightarrow{\text{a.c.s.}} \{A_n\}_n$, *if the following condition is met: for every* m *there exists* n_m *such that, for* $n \geq n_m$,

$$A_n = B_{n,m} + R_{n,m} + N_{n,m}, \qquad \operatorname{rank}(R_{n,m}) \leq c(m)d_n, \qquad \|N_{n,m}\| \leq \omega(m),$$

where n_m, $c(m)$, $\omega(m)$ *depend only on* m, *and* $\lim_{m\to\infty} c(m) = \lim_{m\to\infty} \omega(m) = 0$.

Roughly speaking, $\{\{B_{n,m}\}_n\}_m$ is an a.c.s. for $\{A_n\}_n$ if, for all sufficiently large m, the sequence $\{B_{n,m}\}_n$ approximates (asymptotically) the sequence $\{A_n\}_n$ in the sense that A_n is eventually equal to $B_{n,m}$ plus a small-rank matrix (with respect to the matrix size d_n) plus a small-norm matrix. We report below two fundamental results of the theory of a.c.s.es [14, chapter 5].

THEOREM A.3. *Let $\{A_n\}_n$ and $\{B_{n,m}\}_n$ be sequences of matrices, with A_n and $B_{n,m}$ of size d_n tending to infinity, and suppose that for every m there exists n_m, such that, for $n \geq n_m$,*

$$\|A_n - B_{n,m}\|_1 \leq \varepsilon(m) d_n$$

where $\varepsilon(m) \to 0$ as $m \to \infty$. Then $\{B_{n,m}\}_n \xrightarrow{\text{a.c.s.}} \{A_n\}_n$.

THEOREM A.4. *Let $\{A_n\}_n$, $\{B_{n,m}\}_n$ be sequences of Hermitian matrices, with A_n and $B_{n,m}$ of size d_n tending to infinity, and let ϕ, $\phi_m : C_c(\mathbb{C}) \to \mathbb{C}$. Suppose that*
(a) *$\{B_{n,m}\}_n \sim_\lambda \phi_m$ for every m,*
(b) *$\phi_m \to \phi$ pointwise over $C_c(\mathbb{C})$,*
(c) *$\{B_{n,m}\}_n \xrightarrow{\text{a.c.s.}} \{A_n\}_n$.*
Then $\{A_n\}_n \sim_\lambda \phi$.

In what follows, $D_n(a)$ will denote the diagonal sampling matrix of size n associated with the function $a : [0,1] \to \mathbb{C}$, i.e.,

$$D_n(a) = \operatorname*{diag}_{j=1,\dots,n} a\Big(\frac{j}{n}\Big).$$

Moreover, the symbol $X \oplus Y$ will denote the direct sum of the matrices X and Y, i.e.,

$$X \oplus Y = \begin{bmatrix} X & O \\ O & Y \end{bmatrix}.$$

The next lemma is essential for our purposes.

LEMMA A.5. *Let $\mathbf{f} : [-\pi, \pi] \to \mathbb{C}^{s\times s}$ be a matrix-valued trigonometric polynomial and let $a : [0,1] \to \mathbb{C}$ be a continuous function. Let*

$$LT_n^m(a, \mathbf{f}) = \big[D_m(a) \otimes T_{\lfloor n/m \rfloor}(\mathbf{f})\big] \oplus O_{(n \bmod m)s}.$$

Then, the following properties hold.
(a) *$\{LT_n^m(a,\mathbf{f})\}_n \sim_\lambda \phi_m$ for every m, where*

$$\phi_m(F) = \frac{1}{m}\sum_{j=1}^{m} \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\sum_{i=1}^{s} F(a(\frac{j}{m})\lambda_i(\mathbf{f}(\theta)))}{s} \mathrm{d}\theta, \quad F \in C_c(\mathbb{C}).$$

(b) *$\phi_m \to \phi$ pointwise over $C_c(\mathbb{C})$, where*

$$\phi(F) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\int_0^1 \frac{\sum_{i=1}^{s} F(a(x)\lambda_i(\mathbf{f}(\theta)))}{s} \mathrm{d}x\mathrm{d}\theta, \quad F \in C_c(\mathbb{C}).$$

(c) *$\{LT_n^m(a,\mathbf{f})\}_n \xrightarrow{\text{a.c.s.}} \{(D_n(a) \otimes I_s)T_n(\mathbf{f})\}_n$.*

Proof. (a) The eigenvalues of $LT_n^m(a,\mathbf{f})$ are given by

$$a\Big(\frac{j}{m}\Big)\lambda_i(T_{\lfloor n/m \rfloor}(\mathbf{f})), \quad j = 1,\dots,m, \quad i = 1,\dots,\lfloor n/m \rfloor s,$$

plus further $(n \bmod m)s$ eigenvalues which are equal to 0. Thus, by Theorem 2.7, for every $F \in C_c(\mathbb{C})$ we have

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{ns}\sum_{r=1}^{ns} F(\lambda_r(LT_n^m(a,\mathbf{f}))) &= \lim_{n\to\infty} \frac{1}{ns}\sum_{j=1}^{m}\sum_{i=1}^{\lfloor n/m \rfloor s} F\Big(a\Big(\frac{j}{m}\Big)\lambda_i(T_{\lfloor n/m \rfloor}(\mathbf{f}))\Big) \\ &= \phi_m(F).\end{aligned}$$

(b) For $F \in C_c(\mathbb{C})$ and $\theta \in [-\pi, \pi]$, the function

$$x \mapsto \frac{\sum_{i=1}^{s} F(a(x)\lambda_i(\mathbf{f}(\theta)))}{s}$$

is continuous (and hence Riemann-integrable) over $[0, 1]$. The quantity

$$\frac{1}{m}\sum_{j=1}^{m} \frac{\sum_{i=1}^{s} F(a(\frac{j}{m})\lambda_i(\mathbf{f}(\theta)))}{s}$$

is a Riemann sum of this function and so it converges to

$$\int_0^1 \frac{\sum_{i=1}^{s} F(a(x)\lambda_i(\mathbf{f}(\theta)))}{s} \mathrm{d}x.$$

As a consequence, $\phi_m(F) \to \phi(F)$ by the dominated convergence theorem.

(c) Let

$$\overline{LT}_n^m(a, \mathbf{f}) = [(D_m(a) \otimes I_{\lfloor n/m \rfloor s}) \oplus (a(1)I_{(n \bmod m)s})]T_n(\mathbf{f})$$

and set

$$\begin{aligned} N_{n,m} &= (D_n(a) \otimes I_s)T_n(\mathbf{f}) - \overline{LT}_n^m(a, \mathbf{f}), \\ R_{n,m} &= \overline{LT}_n^m(a, \mathbf{f}) - LT_n^m(a, \mathbf{f}). \end{aligned}$$

It is clear that

$$(D_n(a) \otimes I_s)T_n(\mathbf{f}) = LT_n^m(a, \mathbf{f}) + R_{n,m} + N_{n,m}. \tag{A.1}$$

If the degree of $\mathbf{f}$ is R, so that we can write $\mathbf{f}(\theta) = \sum_{j=-R}^{R} \mathbf{f}_j \mathrm{e}^{\mathrm{i}j\theta}$, then

$$\operatorname{rank}(R_{n,m}) \le (2R + 1)ms.$$

Assuming $n \ge m$ and using the inequality $\|T_n(\mathbf{f})\| \le \max_{\theta \in [-\pi,\pi]} \|\mathbf{f}(\theta)\|$ (see, e.g., [28, Corollary 3.5]) we have

$$\begin{aligned} \|N_{n,m}\| &\le \|D_n(a) \otimes I_s - [(D_m(a) \otimes I_{\lfloor n/m \rfloor s}) \oplus (a(1)I_{(n \bmod m)s})]\| \, \|T_n(\mathbf{f})\| \\ &\le \omega_a\Big(\frac{1}{m} + \frac{m}{n}\Big) \max_{\theta \in [-\pi,\pi]} \|\mathbf{f}(\theta)\|, \end{aligned}$$

where $\omega_a(\cdot)$ is the modulus of continuity of a. If $n \ge n_m = m^2$, then

$$\begin{aligned} \operatorname{rank}(R_{n,m}) &\le \frac{(2R+1)}{m} ns = c(m)ns, \\ \|N_{n,m}\| &\le \omega_a\Big(\frac{2}{m}\Big) \max_{\theta \in [-\pi,\pi]} \|\mathbf{f}(\theta)\| = \omega(m), \end{aligned}$$

and we conclude from (A.1) that $\{LT_n^m(a, \mathbf{f})\}_n \xrightarrow{\text{a.c.s.}} \{(D_n(a) \otimes I_s)T_n(\mathbf{f})\}_n$. ∎

We are now ready to prove Theorem 5.2 in the unidimensional case. For the reader's convenience, we also report the simplified statement of the theorem for $d = 1$; note that the parameter $\boldsymbol{\alpha}$ plays no role in this case, since we have a unique spatial direction.

THEOREM A.6. *Let $q, p \in \mathbb{N}$ and $0 \le k \le p-1$. Suppose that $K : (0,1) \to \mathbb{R}$ belongs to $L^\infty([0,1])$ and that $N = N(n)$ satisfies $N \to \infty$ and $N/n^2 \to 0$ as $n \to \infty$. Then, for the sequence of normalized space-time matrices $\{2Nn^{-1}C_{N,n}^{[q,p,k]}(K)\}_n$ we have*

$$\{2Nn^{-1}C_{N,n}^{[q,p,k]}(K)\}_n \sim_\lambda \mathbf{f}_{[q,p,k]}^{[1,K]} = \mathbf{f}_{[q,p,k]}^{[K]},$$

where:

- $\mathbf{f}_{[q,p,k]}^{[K]}$ *is defined as*

$$\begin{aligned} &\mathbf{f}_{[q,p,k]}^{[K]} : [0,1] \times [-\pi,\pi] \to \mathbb{C}^{(q+1)(p-k)\times(q+1)(p-k)}, \\ &\mathbf{f}_{[q,p,k]}^{[K]}(x,\theta) = \mathbf{f}_{[p,k]}^{[K]}(x,\theta) \otimes \mathrm{T}M_{[q]}; \end{aligned} \tag{A.2}$$

- $\mathbf{f}_{[p,k]}^{[K]}$ *is defined as*

$$\begin{aligned} &\mathbf{f}_{[p,k]}^{[K]} : [0,1] \times [-\pi,\pi] \to \mathbb{C}^{(p-k)\times(p-k)}, \\ &\mathbf{f}_{[p,k]}^{[K]}(x,\theta) = K(x)\mathbf{f}_{[p,k]}(\theta); \end{aligned} \tag{A.3}$$

- $\mathbf{f}_{[p,k]}$ *is given by* (4.7)*;*
- T *is the final time and* $M_{[q]}$ *is given in* (3.35)*.*

Proof. The proof consists of the following steps. The letter C will be used to denote a generic constant independent of n. As noted in section 5.1, we have

$$\begin{aligned} C_{N,n}^{[q,p,k]}(K) = &-T_N(\mathrm{e}^{\mathrm{i}\theta}) \otimes J_{[q]} \otimes M_{n,[p,k]} + I_N \otimes (H_{[q]} + C_{[q]}) \otimes M_{n,[p,k]} \\ &+ \frac{\Delta t}{2} I_N \otimes M_{[q]} \otimes K_{n,[p,k]}(K), \end{aligned}$$

where

$$K_{n,[p,k]}(K) = \left[\int_0^1 K(x) B'_{j+1,[p,k]}(x) B'_{i+1,[p,k]}(x)\mathrm{d}x\right]_{i,j=1}^{n(p-k)+k-1}.$$

Step 1. By (2.2) we have

$$\begin{aligned} \|K_{n,[p,k]}(K)\| &\le \|K_{n,[p,k]}(K)\|_\infty \\ &\le \|K\|_{L^\infty} \left\| \left[\int_0^1 |B'_{j+1,[p,k]}(x)|\,|B'_{i+1,[p,k]}(x)|\mathrm{d}x\right]_{i,j=1}^{n(p-k)+k-1} \right\|_\infty \le C_p n, \end{aligned} \tag{A.4}$$

where C_p is a constant depending only on p and the last inequality in (A.4) is contained in the proof of Lemma 4.3. In view of (A.4), the same argument as in the first step of the proof of Theorem 5.1 shows that the theorem is established as soon as we have proved that

$$\{n^{-1}K_{n,[p,k]}(K)\}_n \sim_\lambda \mathbf{f}_{[p,k]}^{[K]}(x,\theta) = K(x)\mathbf{f}_{[p,k]}(\theta). \tag{A.5}$$

The next steps are devoted to the proof of (A.5).

Step 2. Define the linear operator

$$\begin{aligned} &K_{n,[p,k]}(\cdot) : L^1([0,1]) \to \mathbb{C}^{(n(p-k)+k-1)\times(n(p-k)+k-1)}, \\ &K_{n,[p,k]}(a) = \left[\int_0^1 a(x) B'_{j+1,[p,k]}(x) B'_{i+1,[p,k]}(x)\mathrm{d}x\right]_{i,j=1}^{n(p-k)+k-1}. \end{aligned}$$

In the next two steps we show that

$$\{n^{-1}K_{n,[p,k]}(a)\}_n \sim_\lambda a(x)\mathbf{f}_{[p,k]}(\theta), \quad \forall\, a \in L^1([0,1]). \tag{A.6}$$

Once this is done, the thesis (A.5) follows immediately by selecting $a = K$ in (A.6).

Step 3. We first prove (A.6) in the case where $a \in C([0,1])$. We will show that

$$\{n^{-1}\tilde{K}_{n,[p,k]}(a)\}_n \sim_\lambda a(x)\mathbf{f}_{[p,k]}(\theta), \tag{A.7}$$

where

$$\tilde{K}_{n,[p,k]}(a) = \left[\int_0^1 a(x)B'_{j+k+1,[p,k]}(x)B'_{i+k+1,[p,k]}(x)\mathrm{d}x\right]_{i,j=1}^{(n-\nu)(p-k)}$$

is the principal submatrix of $K_{n,[p,k]}(a)$ corresponding to the indices $k+1,\dots,k+(n-\nu)(p-k)$ and ν is defined in (3.22). By Theorem 2.5, proving (A.7) is equivalent to proving (A.6); note that $\tilde{K}_{n,[p,k]}(a) = P^T_{n,[p,k]}K_{n,[p,k]}(a)P_{n,[p,k]}$, with $P_{n,[p,k]}$ defined as in the second step of the proof of Theorem 5.1. To prove (A.7), we use Lemma A.5. We first show that

$$\|n^{-1}\tilde{K}_{n,[p,k]}(a) - (D_{n-\nu}(a)\otimes I_{p-k})T_{n-\nu}(\mathbf{f}_{[p,k]})\| \to 0 \quad \text{as} \quad n\to\infty. \tag{A.8}$$

For $\ell = 1,\dots,n-\nu$ and $(\ell-1)(p-k)+1 \le i \le \ell(p-k)$, the support of $B_{i+k+1,[p,k]}$ is located near the point $\frac{\ell}{n-\nu}$, in the sense that, for all $x \in \operatorname{supp}(B_{i+k+1,[p,k]}) = [\xi_{i+k+1}, \xi_{i+k+p+2}]$,

$$\begin{aligned}\left|x - \frac{\ell}{n-\nu}\right| &\le \max\left(\left|\xi_{i+k+1} - \frac{\ell}{n-\nu}\right|, \left|\xi_{i+k+p+2} - \frac{\ell}{n-\nu}\right|\right)\\ &\le \max\left(\left|\xi_{(\ell-1)(p-k)+k+2} - \frac{\ell}{n-\nu}\right|, \left|\xi_{\ell(p-k)+k+p+2} - \frac{\ell}{n-\nu}\right|\right) \le Cn^{-1}.\end{aligned}$$

Moreover, it is clear that the length of $\operatorname{supp}(B_{i+k+1,[p,k]})$ is bounded by $(p+1)/n$. Hence, for all $i,j = 1,\dots,(n-\nu)(p-k)$, if ℓ is the index in $\{1,\dots,n-\nu\}$ such that $(\ell-1)(p-k)+1 \le i \le \ell(p-k)$, by Lemma 4.2 and (3.20) we have

$$\begin{aligned}&|(\tilde{K}_{n,[p,k]}(a) - (D_{n-\nu}(a)\otimes I_{p-k})nT_{n-\nu}(\mathbf{f}_{[p,k]}))_{ij}|\\ &= |(\tilde{K}_{n,[p,k]}(a) - (D_{n-\nu}(a)\otimes I_{p-k})\tilde{K}_{n,[p,k]}(1))_{ij}|\\ &= \left|\int_0^1 \left[a(x) - a\Big(\frac{\ell}{n-\nu}\Big)\right]B'_{j+k+1,[p,k]}(x)B'_{i+k+1,[p,k]}(x)\mathrm{d}x\right|\\ &= \left|\int_{\operatorname{supp}(B_{i+k+1,[p,k]})} \left[a(x) - a\Big(\frac{\ell}{n-\nu}\Big)\right]B'_{j+k+1,[p,k]}(x)B'_{i+k+1,[p,k]}(x)\mathrm{d}x\right|\\ &\le \omega_a(Cn^{-1})(c_pn)^2\frac{p+1}{n} \le C\omega_a(n^{-1})n,\end{aligned} \tag{A.9}$$

where $\omega_a(\cdot)$ is the modulus of continuity of a. Considering that both $\tilde{K}_{n,[p,k]}(a)$ and $(D_{n-\nu}(a)\otimes I_{p-k})T_{n-\nu}(\mathbf{f}_{[p,k]})$ are banded with bandwidth bounded by C, due to the local support property (3.16), from (A.9) and (2.2) we infer that

$$\|n^{-1}\tilde{K}_{n,[p,k]}(a) - (D_{n-\nu}(a)\otimes I_{p-k})T_{n-\nu}(\mathbf{f}_{[p,k]})\| \le C\omega_a(n^{-1}),$$

which tends to 0 as $n\to\infty$. This completes the proof of (A.8). Now, let

$$\tilde{K}^{(m)}_{n,[p,k]}(a) = n\, LT^m_{n-\nu}(a, \mathbf{f}_{[p,k]}). \tag{A.10}$$

(a) By Lemma A.5, $\{n^{-1}\tilde{K}^{(m)}_{n,[p,k]}(a)\}_n \sim_\lambda \phi_m$ for every m, where

$$\phi_m(F) = \frac{1}{m}\sum_{j=1}^{m}\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\sum_{i=1}^{p-k}F(a(\frac{j}{m})\lambda_i(\mathbf{f}_{[p,k]}(\theta)))}{p-k}\mathrm{d}\theta, \quad F\in C_c(\mathbb{C}).$$

(b) By Lemma A.5, $\phi_m \to \phi$ pointwise over $C_c(\mathbb{C})$, where

$$\phi(F) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\int_0^1\frac{\sum_{i=1}^{p-k}F(a(x)\lambda_i(\mathbf{f}_{[p,k]}(\theta)))}{p-k}\mathrm{d}x\mathrm{d}\theta, \quad F\in C_c(\mathbb{C}).$$

(c) By Lemma A.5 and (A.8), $\{n^{-1}\tilde{K}^{(m)}_{n,[p,k]}(a)\}_n \xrightarrow{\text{a.c.s.}} \{n^{-1}\tilde{K}_{n,[p,k]}(a)\}_n$.
Thus, the result (A.7) follows from Theorem A.4.

Step 4. We now prove (A.6) in the general case where $a \in L^1([0,1])$. Let $\{a_m\}_m$ be a sequence of functions in $C([0,1])$ such that $a_m \to a$ in $L^1([0,1])$ and a.e. We show that:
(a) $\{n^{-1}K_{n,[p,k]}(a_m)\}_n \sim_\lambda \phi_m$ for every m, where

$$\phi_m(F) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\int_0^1\frac{\sum_{i=1}^{(p-k)}F(a_m(x)\lambda_i(\mathbf{f}_{[p,k]}(\theta)))}{p-k}\mathrm{d}x\mathrm{d}\theta;$$

(b) $\phi_m \to \phi$ pointwise over $C_c(\mathbb{C})$, where

$$\phi(F) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\int_0^1\frac{\sum_{i=1}^{p-k}F(a(x)\lambda_i(\mathbf{f}_{[p,k]}(\theta)))}{p-k}\mathrm{d}x\mathrm{d}\theta;$$

(c) $\{n^{-1}K_{n,[p,k]}(a_m)\}_n \xrightarrow{\text{a.c.s.}} \{n^{-1}K_{n,[p,k]}(a)\}_n$.
Once this is done, the result (A.6) follows from Theorem A.4.

Proof of (a). This was proved in Step 3, because $a_m \in C([0,1])$.

Proof of (b). This follows from the dominated convergence theorem, since $a_m \to a$ a.e.

Proof of (c). We prove that

$$\|n^{-1}K_{n,[p,k]}(a_m) - n^{-1}K_{n,[p,k]}(a)\|_1 \le \varepsilon(m)n \tag{A.11}$$

for some $\varepsilon(m)$ such that $\varepsilon(m) \to 0$ as $m \to \infty$. Once this is done, item (c) follows from Theorem A.3. To prove (A.11), we use the following trace-norm inequality, whose proof can be found, e.g., in [14, section 2.4.3]:

$$\|X\|_1 \le \sum_{i,j=1}^{\ell}|x_{ij}|, \quad \forall\, X\in\mathbb{C}^{\ell\times\ell}. \tag{A.12}$$

By (A.12) and (3.20),

$$\begin{aligned}
&\|K_{n,[p,k]}(a_m) - K_{n,[p,k]}(a)\|_1 = \|K_{n,[p,k]}(a_m - a)\|_1\\
&\le \sum_{i,j=1}^{n(p-k)+k-1}|(K_{n,[p,k]}(a_m-a))_{ij}|\\
&\le \sum_{i,j=1}^{n(p-k)+k-1}\int_0^1|a_m(x)-a(x)|\,|B'_{j+1,[p,k]}(x)|\,|B'_{i+1,[p,k]}(x)|\mathrm{d}x\\
&\le \int_0^1|a_m(x)-a(x)|\left(\sum_{i=1}^{n(p-k)+k-1}|B'_{i+1,[p,k]}(x)|\right)^2\mathrm{d}x \le (c_pn)^2\|a_m-a\|_{L^1}
\end{aligned}$$

and (A.11) is proved with $\varepsilon(m) = c_p^2 \|a_m - a\|_{L^1}$. ∎

We conclude by illustrating the idea to prove Theorem 5.2 in the general d-dimensional case. The proof consists of the following steps, which are essentially the same as in the proof of Theorem A.6 (but involve more technical difficulties).

Step 1. With the same argument as in the first step of the proof of Theorem A.6 one can show that Theorem 5.2 is established as soon as we have proved that

$$\{n^{d-2} K_{\boldsymbol{n},[\boldsymbol{p},\boldsymbol{k}]}(K)\}_n \sim_\lambda \mathbf{f}_{[\boldsymbol{p},\boldsymbol{k}]}^{[\boldsymbol{\alpha},K]}(\mathbf{x},\boldsymbol{\theta}). \tag{A.13}$$

The next steps deal with the proof of (A.13).

Step 2. Let $\mathbb{R}^{s\times s}_{\text{sym}}$ be the space of $s\times s$ real symmetric matrices, and let $L^1([0,1]^d, \mathbb{R}^{s\times s}_{\text{sym}})$ be the space consisting of all symmetric matrix-valued functions $A : [0,1]^d \to \mathbb{R}^{s\times s}_{\text{sym}}$ belonging to $L^1([0,1]^d)$. Define the linear operator

$$K_{\boldsymbol{n},[\boldsymbol{p},\boldsymbol{k}]}(\cdot) : L^1([0,1]^d, \mathbb{R}^{s\times s}_{\text{sym}}) \to \mathbb{C}^{P(\boldsymbol{n}(\boldsymbol{p}-\boldsymbol{k})+\boldsymbol{k}-\boldsymbol{1})\times P(\boldsymbol{n}(\boldsymbol{p}-\boldsymbol{k})+\boldsymbol{k}-\boldsymbol{1})},$$

$$K_{\boldsymbol{n},[\boldsymbol{p},\boldsymbol{k}]}(A) = \left[\int_{[0,1]^d} \left[A(\mathbf{x})\nabla B_{\boldsymbol{j}+\boldsymbol{1},[\boldsymbol{p},\boldsymbol{k}]}(\mathbf{x})\right]\cdot \nabla B_{\boldsymbol{i}+\boldsymbol{1},[\boldsymbol{p},\boldsymbol{k}]}(\mathbf{x})\mathrm{d}\mathbf{x}\right]_{\boldsymbol{i},\boldsymbol{j}=\boldsymbol{1}}^{\boldsymbol{n}(\boldsymbol{p}-\boldsymbol{k})+\boldsymbol{k}-\boldsymbol{1}}.$$

The next two steps are devoted to show that

$$\{n^{d-2} K_{\boldsymbol{n},[\boldsymbol{p},\boldsymbol{k}]}(A)\}_n \sim_\lambda \mathbf{f}_{[\boldsymbol{p},\boldsymbol{k}]}^{[\boldsymbol{\alpha},A]}(\mathbf{x},\boldsymbol{\theta}), \quad \forall\, A \in L^1([0,1]^d, \mathbb{R}^{s\times s}_{\text{sym}}). \tag{A.14}$$

Once this is done, the thesis (A.13) follows immediately by selecting $A = K$ in (A.14).

Step 3. We first prove (A.14) in the case where $A \in C([0,1]^d, \mathbb{R}^{s\times s}_{\text{sym}})$, that is, the components of A are continuous functions. By Theorem 2.5, proving (A.14) is equivalent to proving that

$$\{n^{d-2} \tilde{K}_{\boldsymbol{n},[\boldsymbol{p},\boldsymbol{k}]}(A)\}_n \sim_\lambda \mathbf{f}_{[\boldsymbol{p},\boldsymbol{k}]}^{[\boldsymbol{\alpha},A]}. \tag{A.15}$$

where

$$\tilde{K}_{\boldsymbol{n},[\boldsymbol{p},\boldsymbol{k}]}(A) = P^T_{\boldsymbol{n},[\boldsymbol{p},\boldsymbol{k}]} K_{\boldsymbol{n},[\boldsymbol{p},\boldsymbol{k}]}(A) P_{\boldsymbol{n},[\boldsymbol{p},\boldsymbol{k}]}$$

and $P_{\boldsymbol{n},[\boldsymbol{p},\boldsymbol{k}]}$ is defined in the second step of the proof of Theorem 5.1. To prove (A.15), the argument is the same as in the third step of the proof of Theorem A.6; it relies on the construction of a proper matrix $\tilde{K}^{(m)}_{\boldsymbol{n},[\boldsymbol{p},\boldsymbol{k}]}(A)$ for which the d-dimensional analogs of properties (a)–(c) in the third step of the proof of Theorem A.6 are satisfied. The matrix $\tilde{K}^{(m)}_{\boldsymbol{n},[\boldsymbol{p},\boldsymbol{k}]}(A)$ is the d-dimensional version of the matrix $\tilde{K}^{(m)}_{n,[p,k]}(a)$ in (A.10), and it is formally defined as

$$\tilde{K}^{(m)}_{\boldsymbol{n},[\boldsymbol{p},\boldsymbol{k}]}(A) = \sum_{i,j=1}^{d} LT^m_{\boldsymbol{n}-\boldsymbol{\nu}}(A_{ij}, (H_{[\boldsymbol{p},\boldsymbol{k}]})_{ij}).$$

The explicit construction of $LT^m_{\boldsymbol{n}-\boldsymbol{\nu}}(A_{ij}, (H_{[\boldsymbol{p},\boldsymbol{k}]})_{ij})$ and the proof of the d-dimensional versions of the properties (a)–(c) in the third step of the proof of Theorem A.6 are very complicated from a technical viewpoint and are therefore omitted here. For more insights about the so-called locally Toeplitz operator $LT^{\boldsymbol{m}}_{\boldsymbol{n}}(\cdot,\cdot)$ we refer the reader to [15, Section 6.1] or [18].

Step 4. We now prove (A.14) in the general case where $A \in L^1([0,1]^d, \mathbb{R}^{s\times s}_{\text{sym}})$. Let $\{A_m\}_m$ be a sequence in $C([0,1]^d, \mathbb{R}^{s\times s}_{\text{sym}})$ such that $(A_m)_{ij} \to A_{ij}$ in $L^1([0,1]^d)$ and a.e. We show that:

(a) $\{n^{d-2}K_{\boldsymbol{n},[\boldsymbol{p},\boldsymbol{k}]}(A_m)\}_n \sim_\lambda \phi_m$, where

$$\phi_m(F) = \frac{1}{(2\pi)^d}\int_{[-\pi,\pi]^d}\int_{[0,1]^d} \frac{\sum_{i=1}^{P(\boldsymbol{p}-\boldsymbol{k})} F(\lambda_i(\mathbf{f}_{[\boldsymbol{p},\boldsymbol{k}]}^{[\boldsymbol{\alpha},A_m]}(\mathbf{x},\boldsymbol{\theta})))}{P(\boldsymbol{p}-\boldsymbol{k})}\mathrm{d}\mathbf{x}\mathrm{d}\boldsymbol{\theta};$$

(b) $\phi_m \to \phi$ pointwise over $C_c(\mathbb{C})$, where

$$\phi(F) = \frac{1}{(2\pi)^d}\int_{[-\pi,\pi]}\int_{[0,1]^d} \frac{\sum_{i=1}^{P(\boldsymbol{p}-\boldsymbol{k})} F(\lambda_i(\mathbf{f}_{[\boldsymbol{p},\boldsymbol{k}]}^{[\boldsymbol{\alpha},A]}(\mathbf{x},\boldsymbol{\theta})))}{P(\boldsymbol{p}-\boldsymbol{k})}\mathrm{d}\mathbf{x}\mathrm{d}\boldsymbol{\theta};$$

(c) $\{n^{d-1}K_{\boldsymbol{n},[\boldsymbol{p},\boldsymbol{k}]}(A_m)\}_n \xrightarrow{\text{a.c.s.}} \{n^{d-2}K_{\boldsymbol{n},[\boldsymbol{p},\boldsymbol{k}]}(A)\}_n$.
Once this is done, the result (A.14) follows from Theorem A.4.

Proof of (a). This was proved in Step 3, because $A_m \in C([0,1],\mathbb{R}^{s\times s}_{\text{sym}})$.

Proof of (b). This follows from the dominated convergence theorem, since $(A_m)_{ij} \to A_{ij}$ a.e. for all $i,j = 1,\dots,d$.

Proof of (c). We prove that

$$\|n^{d-2}K_{\boldsymbol{n},[\boldsymbol{p},\boldsymbol{k}]}(A_m) - n^{d-2}K_{\boldsymbol{n},[\boldsymbol{p},\boldsymbol{k}]}(A)\|_1 \le \varepsilon(m)n^d \tag{A.16}$$

for some $\varepsilon(m)$ such that $\varepsilon(m) \to 0$ as $m \to \infty$. Once this is done, item (c) follows from Theorem A.3. By (A.12) and (3.18)–(3.20),

$$\begin{aligned}
&\|K_{\boldsymbol{n},[\boldsymbol{p},\boldsymbol{k}]}(A_m) - K_{\boldsymbol{n},[\boldsymbol{p},\boldsymbol{k}]}(A)\|_1 = \|K_{\boldsymbol{n},[\boldsymbol{p},\boldsymbol{k}]}(A_m - A)\|_1\\
&\le \sum_{\boldsymbol{i},\boldsymbol{j}=\boldsymbol{1}}^{\boldsymbol{n}(\boldsymbol{p}-\boldsymbol{k})+\boldsymbol{k}-\boldsymbol{1}} |(K_{\boldsymbol{n},[\boldsymbol{p},\boldsymbol{k}]}(A_m - A))_{\boldsymbol{ij}}|\\
&\le \sum_{\boldsymbol{i},\boldsymbol{j}=\boldsymbol{1}}^{\boldsymbol{n}(\boldsymbol{p}-\boldsymbol{k})+\boldsymbol{k}-\boldsymbol{1}} \int_{[0,1]^d} |(A_m(\mathbf{x}) - A(\mathbf{x}))\nabla B_{\boldsymbol{j}+\boldsymbol{1},[\boldsymbol{p},\boldsymbol{k}]}(\mathbf{x})\cdot\nabla B_{\boldsymbol{i}+\boldsymbol{1},[\boldsymbol{p},\boldsymbol{k}]}(\mathbf{x})|\mathrm{d}\mathbf{x}\\
&\le \sum_{\boldsymbol{i},\boldsymbol{j}=\boldsymbol{1}}^{\boldsymbol{n}(\boldsymbol{p}-\boldsymbol{k})+\boldsymbol{k}-\boldsymbol{1}} \int_{[0,1]^d} \sum_{r,s=1}^{d} |(A_m)_{rs}(\mathbf{x}) - A_{rs}(\mathbf{x})| \left|\frac{\partial B_{\boldsymbol{j}+\boldsymbol{1},[\boldsymbol{p},\boldsymbol{k}]}}{\partial x_r}(\mathbf{x})\right| \left|\frac{\partial B_{\boldsymbol{i}+\boldsymbol{1},[\boldsymbol{p},\boldsymbol{k}]}}{\partial x_s}(\mathbf{x})\right| \mathrm{d}\mathbf{x}\\
&= \sum_{r,s=1}^{d} \int_{[0,1]^d} |(A_m)_{rs}(\mathbf{x}) - A_{rs}(\mathbf{x})| \left(\sum_{\boldsymbol{i}=\boldsymbol{1}}^{\boldsymbol{n}(\boldsymbol{p}-\boldsymbol{k})+\boldsymbol{k}-\boldsymbol{1}} \left|\frac{\partial B_{\boldsymbol{i}+\boldsymbol{1},[\boldsymbol{p},\boldsymbol{k}]}}{\partial x_s}(\mathbf{x})\right|\right)^2 \mathrm{d}\mathbf{x}\\
&\le C_{\boldsymbol{p}} n^2 \sum_{r,s=1}^{d} \|(A_m)_{rs} - A_{rs}\|_{L^1},
\end{aligned}$$

where $C_{\boldsymbol{p}}$ is a constant depending only on $\boldsymbol{p}$. Thus, (A.16) is proved with $\varepsilon(m) = C_{\boldsymbol{p}}\sum_{r,s=1}^{d}\|(A_m)_{rs} - A_{rs}\|_{L^1}$.

Acknowledgements. The authors wish to thank Hendrik Speleers for useful discussions and for his invaluable help in the numerical experiments.

REFERENCES

[1] B. Beckermann and A. B. J. Kuijlaars, *Superlinear convergence of conjugate gradients*, SIAM J. Numer. Anal., 39 (2001), pp. 300–329.
[2] R. Bhatia, *Matrix Analysis*, Springer, New York, 1997.

[3] C. De Boor, *A Practical Guide to Splines*, Revised ed., Springer, New York, 2001.
[4] V. Dolejsi and M. Feistauer, *Discontinuous Galerkin Method: Analysis and Applications to Compressible Flow*, Springer, Cham, 2015.
[5] M. Donatelli, C. Garoni, C. Manni, S. Serra-Capizzano, and H. Speleers, *Robust and optimal multi-iterative techniques for IgA Galerkin linear systems*, Comput. Methods Appl. Mech. Engrg., 284 (2015), pp. 230–264.
[6] M. Donatelli, C. Garoni, C. Manni, S. Serra-Capizzano, and H. Speleers, *Robust and optimal multi-iterative techniques for IgA collocation linear systems*, Comput. Methods Appl. Mech. Engrg., 284 (2015), pp. 1120–1146.
[7] M. Donatelli, C. Garoni, C. Manni, S. Serra-Capizzano, and H. Speleers, *Spectral analysis and spectral symbol of matrices in isogeometric collocation methods*, Math. Comp., 85 (2016), pp. 1639–1680.
[8] M. Donatelli, C. Garoni, C. Manni, S. Serra-Capizzano, and H. Speleers, *Symbol-based multigrid methods for Galerkin B-spline isogeometric analysis*, SIAM J. Numer. Anal., 55 (2017), pp. 31–62.
[9] K. Eriksson, D. Estep, P. Hansbo, and C. Johnson, *Introduction to adaptive methods for differential equations*, Acta Numerica, 4 (1995), pp. 105–158.
[10] C. Garoni, *Spectral distribution of PDE discretization matrices from isogeometric analysis: the case of L^1 coefficients and non-regular geometry*, J. Spectral Theory (in press).
[11] C. Garoni, C. Manni, F. Pelosi, S. Serra-Capizzano, and H. Speleers *On the spectrum of stiffness matrices arising from isogeometric analysis*, Numer. Math., 127 (2014), pp. 751–799.
[12] C. Garoni, C. Manni, S. Serra-Capizzano, D. Sesana, and H. Speleers, *Spectral analysis and spectral symbol of matrices in isogeometric Galerkin methods*, Math. Comp., 86 (2017), pp. 1343–1373.
[13] C. Garoni, C. Manni, S. Serra-Capizzano, D. Sesana, and H. Speleers, *Lusin theorem, GLT sequences and matrix computations: an application to the spectral analysis of PDE discretization matrices*, J. Math. Anal. Appl., 446 (2017), pp. 365–382.
[14] C. Garoni and S. Serra-Capizzano, *Generalized Locally Toeplitz Sequences: Theory and Applications*, vol. I, Springer, Cham, 2017.
[15] C. Garoni and S. Serra-Capizzano, *Generalized Locally Toeplitz Sequences: Theory and Applications*, Technical report 2017-002, Department of Information Technology, Uppsala University, Uppsala, 2017. Preliminary version of: C. Garoni and S. Serra-Capizzano, *Generalized Locally Toeplitz Sequences: Theory and Applications*, vol. II, in preparation.
[16] C. Garoni, S. Serra-Capizzano, and D. Sesana, *Tools for determining the asymptotic spectral distribution of non-Hermitian perturbations of Hermitian matrix-sequences and applications*, Integr. Equ. Oper. Theory, 81 (2015), pp. 213–225.
[17] C. Garoni, S. Serra-Capizzano, and D. Sesana, *Spectral analysis and spectral symbol of d-variate $\mathbb{Q}_{\boldsymbol{p}}$ Lagrangian FEM stiffness matrices*, SIAM J. Matrix Anal. Appl., 36 (2015), pp. 1100–1128.
[18] C. Garoni, S. Serra-Capizzano, and D. Sesana, *Block Generalized Locally Toeplitz Sequences: Theory and Applications*, in preparation.
[19] L. Golinskii and S. Serra-Capizzano, *The asymptotic properties of the spectrum of nonsymmetrically perturbed Jacobi matrix sequences*, J. Approx. Theory, 144 (2007), pp. 84–102.
[20] G. H. Golub and C. F. Van Loan, *Matrix Computations*, 4th ed., The Johns Hopkins University Press, Baltimore, 2013.
[21] J. S. Hesthaven and T. Warburton, *Nodal Discontinuous Galerkin Methods: Algorithms, Analysis, and Applications*, Springer, New York, 2008.
[22] P. Jamet, *Galerkin-type approximations which are discontinuous in time for parabolic equations in a variable domain*, SIAM J. Numer. Anal., 15 (1978), pp. 912–928.
[23] P. Lesaint and P. A. Raviart, *On a finite element method for solving the neutron transport equation*, Chapter 4 (pp. 89–123) of the book "Mathematical Aspects of Finite Elements in Partial Differential Equations (Proc. Sympos., Math. Res. Center, Univ. Wisconsin–Madison, 1974)" edited by C. de Boor, Academic Press, 1974.
[24] M. Mazza, A. Ratnani, and S. Serra-Capizzano, *Spectral analysis and spectral symbol for the 2D curl-curl (stabilized) operator with applications to the related iterative solutions*, submitted.
[25] A. Quarteroni, *Numerical Models for Differential Problems*, Springer, Milan, 2009.
[26] B. Rivière, *Discontinuous Galerkin Methods for Solving Elliptic and Parabolic Equations: Theory and Implementation*, SIAM, Philadelphia, 2008.
[27] L. L. Schumaker, *Spline Functions: Basic Theory*, 3rd ed., Cambridge Mathematical Library, 2007.

[28] S. Serra-Capizzano, *More inequalities and asymptotics for matrix valued linear positive operators: the noncommutative case*, Oper. Theory Adv. Appl., 135 (2002), pp. 293–315.

[29] S. Serra-Capizzano, *Generalized locally Toeplitz sequences: spectral analysis and applications to discretized partial differential equations*, Linear Algebra Appl., 366 (2003), pp. 371–402.

[30] S. Serra-Capizzano, *The GLT class as a generalized Fourier analysis and applications*, Linear Algebra Appl., 419 (2006), pp. 180–233.

[31] P. Tilli, *A note on the spectral distribution of Toeplitz matrices*, Linear Multilinear Algebra, 45 (1998), pp. 147–159.