How to extend the application scope of GLT-sequences

Stanislav Morozov[a], Stefano Serra-Capizzano[b,c], Eugene Tyrtyshnikov[a,d,e]

[a] *Lomonosov Moscow State University, Leninskie Gory 1, Moscow 119991, Russia*
[b] *Department of Science and high Technology, University of Insubria, Via Valleggio 11, Como 22100, Italy*
[c] *Department of Information Technology, division of Scientific Computing, Uppsala University, Box 337 SE-751 05, Uppsala, Sweden*
[d] *Marchuk Institute of Numerical Mathematics, Russian Academy of Sciences, Gubkin Street, 8, Moscow 119333, Russia*
[e] *Siedlce University, Faculty of Sciences, 08-110 Siedlce, Poland*

Abstract

In this paper we address the problem of finding the distribution of eigenvalues and singular values for matrix sequences. The main focus of this paper is the spectral distribution for matrix sequences arising in discretization of PDE. In the last two decades the theory of GLT-sequences aimed at this problem has been developed. We investigate the possibility of application of GLT-theory to discretization of PDE on non-rectangular domains and show that in many cases the present GLT-theory is insufficient. We also propose a generalization of GLT-sequences that enables one to cope with a wide range of PDE discretization problems defined on polygonal domains.

Keywords: locally Toeplitz, GLT sequences, PDE discretization, eigenvalue distribution, singular value distribution, preconditioning
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1. Introduction

In last decades there was an active research of theory of matrix sequences arising during the discretization of PDE and allowing one to describe spectral distribution of such sequences. Recently it was developed into the machinery of generalized locally Toeplitz sequences which solves this problem for PDE defined on rectangular domains. There was as well a discussion of possibilities to extend the GLT-theory to the non-rectangular case. In this paper we propose an example with a very simple problem on non-rectangular domain for which the sequence of discretization matrices is not GLT. However, we have still discovered that this sequences can be transformed to a GLT-sequence by some similarity. The latter observations led us to a concept of generalization of GLT-sequences, which allow one to handle the sequences of discretization matrices defined on non-rectangular domains.

Email addresses: stanis-morozov@yandex.ru (Stanislav Morozov), stefano.serrac@uninsubria.it (Stefano Serra-Capizzano), eugene.tyrtyshnikov@gmail.com (Eugene Tyrtyshnikov)

The analysis is carried out in the context of finite elements on triangular and polygonal domains, with a uniform triangulations. Other approaches can be found in [1][pp. 395–399] in the case of equispaced finite differences on a general Peano-Jordan measurable domains and in [2] for finite elements on a L-shaped domain with graded meshes. The present generalization lies in the same direction of the notion of reduced GLT sequences considered in [3][Section 3.1.4], where the idea is sketched in great generality, but no technical details are given. In the current contribution we give a concrete analysis, which can be adapted for treating polygonal domains and in principle Peano-Jordan measurable sets, as done for equispaced finite differences in [1].

The rest of the paper is organized as follows: in Section 2 we give some preliminary notions and results about the spectral distributions. In Section 3 we provide a brief introduction to GLT-theory for self-containedness of the paper. In Section 4 we present our example, we prove that the matrix sequence does not belong to GLT class and then show that it is still possible for it to find the distribution of eigenvalues and singular values. After that in Section 5 we propose a generalization of GLT-sequences and prove some of its properties. In Section 6 we show that our generalization can be applied to the discretization of PDE defined on a broader class of polygonal domains. Next in Section 7 we provide an application of our theory to preconditioning and in Section 8 we put some concluding remarks.

2. Preliminaries

The very notion of the distribution of eigenvalues for a matrix sequence $\{A_n\}_n$, where A_n is of size n, is defined as follows [4].

Definition 1. *Let f be a measurable function defined on a domain D of nonzero, finite measure ($\mu(D) \neq 0, \infty$). A matrix sequence $\{A_n\}_n$ is said to have the distribution of eigenvalues with symbol f on the domain D if*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} F(\lambda_j(A_n)) = \frac{1}{\mu(D)} \int\limits_D F(f(x))dx \tag{1}$$

for any continuous function F with a compact support ($F \in C_C(\mathbb{C})$), where $\lambda_j(A_n)$ are the eigenvalues of matrix A_n. We denote this fact by $\{A_n\}_n \sim_\lambda f$.

The similar definition can be given for the distribution of singular values of a matrix sequence.

Definition 2. *Let f be a measurable function defined on a domain D of nonzero, finite measure ($\mu(D) \neq 0, \infty$). A matrix sequence $\{A_n\}_n$ is said to have the distribution of singular values with symbol f on the domain D if*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} F(\sigma_j(A_n)) = \frac{1}{\mu(D)} \int\limits_D F(|f(x)|)dx \tag{2}$$

for any continuous function F with a compact support ($F \in C_C(\mathbb{C})$), where $\sigma_j(A_n)$ are the singular values of matrix A_n. We denote this fact by $\{A_n\}_n \sim_\sigma f$.

First results in this area were obtained by G. Szegö [5]. Let f be a function from $L_\infty(-\pi, \pi)$. One can expand this function in the Fourier series

$$f(x) = \sum_{j=-\infty}^{\infty} f_j e^{\mathbb{i}jx}, \quad \mathbb{i}^2 = -1 \tag{3}$$

and define elements of a Toeplitz matrix T_n as $(T_n)_{i,j} = f_{i-j}$. G. Szegö proved that if f is real-valued, then T_n are all Hermitian in this case and the eigenvalues of T_n are distributed with symbol f on $[-\pi, \pi]$. Later F. Avram [6] and S. Parter [7] generalized this result to the case of distribution of singular values for non-Hermitian matrices corresponding to complex-valued f. Both results were generalized even more by E. Tyrtyshnikov and N. Zamarashkin [8, 4] to the case of arbitrary functions $f \in L_1(-\pi, \pi)$ and even to the Radom measures [9].

The techniques developed for Toeplitz matrices turned out to be very useful also for the study of spectral distributions of some matrices coming from the discretization of PDEs. As a simple example, we consider the following PDE problem with constant coefficients:

$$\begin{cases} -u''(x) = b(x), & x \in (0,1), \\ u(0) = u(1) = 0. \end{cases} \tag{4}$$

If the finite element method (FEM) is used on a uniform grid, then we naturally obtain a Toeplitz matrix equal up to a scaling factor to

$$T_n = \begin{bmatrix} 2 & -1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 2 \end{bmatrix}. \tag{5}$$

By refining the grid we come to a sequence of Toeplitz matrices whose eigenvalue and singular value distributions are already given by the above mentioned results. Now consider the case of variable coefficients

$$\begin{cases} -(a(x)u'(x))' = b(x), & x \in (0,1), \\ u(0) = u(1) = 0, \end{cases} \tag{6}$$

where $a(x) \in C^1(0,1)$. The discretization matrices now look like

$$A_n = \begin{bmatrix} a_{\frac{1}{2}} + a_{\frac{3}{2}} & -a_{\frac{3}{2}} & & \\ -a_{\frac{3}{2}} & a_{\frac{3}{2}} + a_{\frac{5}{2}} & -a_{\frac{5}{2}} & \\ & \ddots & \ddots & \ddots \\ & & -a_{n-\frac{1}{2}} & a_{n-\frac{1}{2}} + a_{n+\frac{1}{2}} \end{bmatrix}, \tag{7}$$

where $a_j = a(\frac{j}{n} + 1)$, $j = \frac{1}{2}, \frac{3}{2}, \ldots, n + \frac{1}{2}$.

The first theorems for such sequences were proved by P. Tilly [10] and described the distributions by the weighted symbol in the form $a(x)f(\theta)$ on the domain $[0,1]\times[-\pi,\pi]$, where $a(x)$ is called the weight function, and $f(\theta)$ is called the Toeplitz-case symbol. Precisely, the sequence of matrices $\{A_n\}_n$ has the distribution of eigenvalues with symbol $a(x)f(\theta)$ on the domain $[0,1]\times[-\pi,\pi]$, that is

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}F(\lambda_j(A_n))=\frac{1}{2\pi}\int_0^1\int_{-\pi}^{\pi}F(a(x)f(\theta))d\theta dx \tag{8}$$

for any function $F\in C_C(\mathbb{C})$. Sequences of matrices arising while solving such problems were called by P. Tilly *locally Toeplitz*. But the notion of locality means a bit more general thing. It is very natural to refer a locally Toeplitz matrix as a matrix that is approximately Toeplitz in every small subregion of the matrix, up to relatively low-rank corrections which may hyde the algebraic structure.

Definition 3. *Let f be an integrable function and $\{T_n(f)\}_n$ be a sequence of Toeplitz matrices generated by Fourier coefficients of f. Let also $a(x,y)$ be a continuous function defined on $[0,1]^2$. Let $D_n=\left[a(\frac{i}{n-1},\frac{j}{n-1})\right]_{i,j=0}^{n-1}$. A matrix sequence $\{A_n\}_n$ is said to be locally Toeplitz if it can be represented as $A_n=T_n(f)\circ D_n$ for some functions f and a, where $\circ$ denotes the Hadamard product.*

It can be proved (see for example [11]) that if $f\in L_2$ in the last defintion, then the corresponding locally Toeplitz sequence has a distribution of singular values with symbol $a(x,x)f(\theta)$. Further generalizations of P. Tilly's results inspired S. Serra-Capizzano and his colleagues to create the theory of *generalized locally Toeplitz sequences.*

3. Elements of the theory of generalized locally Toeplitz sequences

For the first time the notion of generalized locally Toeplitz sequences (GLT) appeared in the paper by S. Serra-Capizzano [1]. A systematic presentation of this theory is given, for example, in [11]. Here we focus only on a few key points of this theory, namely, we give the algebraic-topological definition of GLT-sequences and some of their basic algebraic and topological properties.

3.1. Approximating classes of sequences

A fundamental concept in all constructions is the notion of *approximating classes of sequences (a.c.s.).* It was explicitly formulated by S. Serra-Capizzano [12], although it was already used with a somewhat implicit notation in the papers by E. Tyrtyshnikov and N. Zamarashkin [8, 4, 9, 13] and then by P. Tilly [10].

Definition 4. *Let $\{A_n\}_n$ be a sequence of matrices. An approximating class of sequences for $\{A_n\}_n$ is a sequence of matrix sequences $\{\{B_{n,m}\}_n\}_m$ with the following property: for each m, there exists n_m such that, for $n\geq n_m$ a splitting exists*

$$A_n=B_{n,m}+R_{n,m}+N_{n,m} \tag{9}$$

$$\mathrm{rank}(R_{n,m}) \le c(m)n, \|N_{n,m}\| \le \omega(m), \tag{10}$$

where the quantities n_m, $c(m)$, $\omega(m)$ depend only on m, and

$$\lim_{m\to\infty} c(m) = \lim_{m\to\infty} \omega(m) = 0. \tag{11}$$

The latter definition can be interpreted as convergence in the space of matrix sequences, which in the world of the preconditioning is known as matrix approximation in norm and/or rank. That is, let $\mathcal{E}$ be the set of all matrix sequences. We define the topological space $(\mathcal{E}, \tau_{a.c.s.})$ in the following way [14]. A sequence $\{\{B_{n,m}\}_n\}_m \subset \mathcal{E}$ converges to $\{A_n\}_n \in \mathcal{E}$ if and only if $\{\{B_{n,m}\}_n\}_m$ is an approximating class of sequences for $\{A_n\}_n$.

The topological space $(\mathcal{E}, \tau_{a.c.s.})$ can be pseudo-metrized, that is endowed by a function $d_{a.c.s.}$ which satisfies all the properties of the metric function except that $d_{a.c.s.}(x, y)$ can be zero for $x \ne y$. To begin with, we define a function $p_n(A)$ on the set of matrices of size n as follows:

$$p_n(A) = \min_{i=0,\dots,n} \left(\frac{i}{n} + \sigma_{i+1}(A)\right), A \in \mathbb{C}^{n\times n} \tag{12}$$

$\sigma_{n+1}(A) = 0$ by the definition. Now we define the function

$$p_{a.c.s.}(\{A_n\}_n) = \limsup_{n\to\infty} p_n(A_n) = \limsup_{n\to\infty} \min_{i=0,\dots,n} \left(\frac{i}{n} + \sigma_{i+1}(A_n)\right) \tag{13}$$

and then the function

$$d_{a.c.s.}(\{A_n\}_n, \{B_n\}_n) = p_{a.c.s.}(\{A_n - B_n\}_n). \tag{14}$$

It can be proved that $d_{a.c.s.}$ is a pseudometric on the set $\mathcal{E}$ and a sequence $\{\{B_{n,m}\}_n\}_m$ is an approximating class of sequences for $\{A_n\}_n$ if and only if

$$\lim_{m\to\infty} d_{a.c.s.}(\{A_n\}_n, \{\{B_{n,m}\}_n\}_m) = 0. \tag{15}$$

The key role of a.c.s. concept is explained by the following theorems [11].

Theorem 1. *Let $\{A_n\}_n$ be a matrix sequence. Assume that:*

1. *$\{\{B_{n,m}\}_n\}_m$ is an approximating class of sequences for $\{A_n\}_n$;*
2. *for every m $\{B_{n,m}\}_n \sim_\sigma f_m$ for some measurable function $f_m : D \subset \mathbb{R}^k \to \mathbb{C}$;*
3. *$f_m \to f$ in measure over D, where $f : D \to \mathbb{C}$ is another measurable function.*

Then $\{A_n\}_n \sim_\sigma f$.

Theorem 2. *Let $\{A_n\}_n$ be a sequence of Hermitian matrices. Assume that:*

1. *$\{\{B_{n,m}\}_n\}_m$ is an approximating class of sequences for $\{A_n\}_n$ with every $\{\{B_{n,m}\}_n\}_m$ Hermitian;*
2. *for every m $\{B_{n,m}\}_n \sim_\lambda f_m$ for some measurable function $f_m : D \subset \mathbb{R}^k \to \mathbb{C}$;*
3. *$f_m \to f$ in measure over D, where $f : D \to \mathbb{C}$ is another measurable function.*

Then $\{A_n\}_n \sim_\lambda f$.

3.2. Algebraic-topological definition of GLT sequences

Let $\mathcal{E}$ be a set of all matrix sequences. We can define elementwise algebraic operations between two sequences, where we mean elementwise is sence of sequence elements but not in sence of matrices. For example, if we multiply sequences $\{A_1, A_2, A_3, \dots\}$ and $\{B_1, B_2, B_3, \dots\}$ then we obtain a sequence $\{A_1B_1, A_2B_2, A_3B_3, \dots\}$. Obviously, $\mathcal{E}$ is an algebra with such elementwise operations. Moreover, for every matrix sequence in $\mathcal{E}$ the sequence of conjugate matrices also belongs to $\mathcal{E}$, that is, $\mathcal{E}$ is a *-algebra. In addition, $\mathcal{E}$ can be endowed a topology with convergence in the sense of approximating classes of sequences.

Consider a set $\mathcal{M}$ of all measurable functions

$$\zeta : [0,1]^d \times [-\pi, \pi]^d \to \mathbb{C} \tag{16}$$

with the topology τ_μ corresponding to convergence in measure. This topology can be generated by the pseudometric

$$d_\mu(\xi, \zeta) = q(\xi - \zeta), \tag{17}$$

$$q(\psi) = \inf\{\mu(|\psi \geq \alpha|) + \alpha : \alpha > 0\}. \tag{18}$$

Obviously, the set $\mathcal{M}$ is also a *-algebra with natural operations between functions.

Now consider the set $\mathcal{E} \times \mathcal{M}$ endowed by the topology $\tau_{a.c.s.} \times \tau_\mu$. This is the set of pairs of matrix sequences and measurable functions that is also a *-algebra as a product of two *-algebras. Consider the set $\mathcal{T} \subset \mathcal{E} \times \mathcal{M}$, where

$$\mathcal{T} = \Big\{ \big(\{T_{\boldsymbol{n}}(e^{i(\boldsymbol{j},\boldsymbol{\theta})})\}_n, e^{i(\boldsymbol{j},\boldsymbol{\theta})}\big) : \boldsymbol{j} \in \mathbb{Z}^d \Big\}, \tag{19}$$

where $T_{\boldsymbol{n}}(f(\boldsymbol{\theta}))$ is a d-level Toeplitz matrix generated by multidimensional Fourier series for the function f. Here and further vectors are marked in bold in order to avoid misunderstandings. Also, consider the set $\mathcal{D} \subset \mathcal{E} \times \mathcal{M}$:

$$\mathcal{D} = \Big\{ \big(\{D_{\boldsymbol{n}}(a)\}_n, a(\boldsymbol{x})\big) : a \in C^\infty([0,1]^d) \Big\}, \tag{20}$$

where $D_{\boldsymbol{n}}(a)$ is a diagonal matrix the with values of the function a on the diagonal, sampled from the uniform grid over $[0,1]^d$.

Definition 5. *Let $\{A_{\boldsymbol{n}}\}_n$ be a matrix sequence and $\zeta : [0, 1^d] \times [-\pi, \pi]^d \to \mathbb{C}$ be a measurable function. Let us say that $\{A_{\boldsymbol{n}}\}_n$ is a GLT-sequence with symbol $\zeta(\boldsymbol{x}, \boldsymbol{\theta})$, and write $\{A_{\boldsymbol{n}}\}_n \sim_{GLT} \zeta(\boldsymbol{x}, \boldsymbol{\theta})$, if $(\{A_{\boldsymbol{n}}\}_n, \zeta(\boldsymbol{x}, \boldsymbol{\theta}))$ belongs to the smallest closed subalgebra of $\mathcal{E} \times \mathcal{M}$ containing $\mathcal{T} \cup \mathcal{D}$.*

3.3. Basic properties of GLT-sequences

The following is an incomplete list of the properties of GLT-sequences, the proofs can be found in [15, 3, 1, 16].

- If $\{A_{\boldsymbol{n}}\}_n \sim_{GLT} \kappa$, then $\{A_{\boldsymbol{n}}\}_n \sim_\sigma \kappa$. If moreover the matrices $\{A_{\boldsymbol{n}}\}_n$ are Hermitian, then $\{A_{\boldsymbol{n}}\}_n \sim_\lambda \kappa$;

- If $\{A_{\boldsymbol{n}}\}_n \sim_{GLT} \kappa$ and $\{A_{\boldsymbol{n}}\}_n \sim_{GLT} \xi$, then $\kappa = \xi$ almost everywhere in $[0,1]^d \times [-\pi,\pi]^d$;
- $\{T_{\boldsymbol{n}}(f)\}_n \sim_{GLT} f(\boldsymbol{\theta})$ for every $f \in L_1([-\pi,\pi]^d)$;
- $\{D_{\boldsymbol{n}}(a)\}_n \sim_{GLT} a(\boldsymbol{x})$ for every $a : [0,1]^d \to \mathbb{C}$ which is continuous almost everywhere;
- $\{Z_{\boldsymbol{n}}\}_n \sim_{GLT} 0$ if and only if $\{Z_{\boldsymbol{n}}\}_n \sim_\sigma 0$;
- If $\{A_{\boldsymbol{n}}\}_n \sim_{GLT} \kappa$, then $\{A^*_{\boldsymbol{n}}\}_n \sim_{GLT} \overline{\kappa}$;
- If $A_{\boldsymbol{n}} = \sum\limits_{i=1}^{r} \alpha_i \prod\limits_{j=1}^{q_i} A_{\boldsymbol{n}}^{(i,j)}$, where $r, q_1, \ldots, q_r \in \mathbb{N}$, $\alpha_1, \ldots, \alpha_r \in \mathbb{C}$ and $\{A_{\boldsymbol{n}}^{(i,j)}\}_n \sim_{GLT} \kappa_{ij}$, then $\{A_{\boldsymbol{n}}\}_n \sim_{GLT} \kappa = \sum\limits_{i=1}^{r} \alpha_i \prod\limits_{j=1}^{q_i} \kappa_{ij}$;
- If $\{A_{\boldsymbol{n}}\}_n \sim_{GLT} \kappa$ and $\kappa \neq 0$ almost everywhere, then $\{A^\dagger_{\boldsymbol{n}}\}_n \sim_{GLT} \kappa^{-1}$;
- If $\{A_{\boldsymbol{n}}\}_n \sim_{GLT} \kappa$ and each $A_{\boldsymbol{n}}$ is Hermitian, then $\{f(A_{\boldsymbol{n}})\}_n \sim_{GLT} f(\kappa)$ for all continuous functions $f : \mathbb{R} \to \mathbb{C}$;
- $\{A_{\boldsymbol{n}}\}_n \sim_{GLT} \kappa$ if and only if there exist GLT-sequences $\{\{B_{\boldsymbol{n},m}\}_n\}_m$ for which we have $\{B_{\boldsymbol{n},m}\}_n \sim_{GLT} \kappa_m$, $\{\{B_{\boldsymbol{n},m}\}_n\}_m$ is an approximating class of sequences for $\{A_{\boldsymbol{n}}\}_n$, and $\kappa_m \to \kappa$ in measure;
- The GLT class is isometrically equivalent to the algebra of measurable functions and it is a maximal algebra.

4. Spectral distribution of a sequence of discrete operators defined on a triangular domain

4.1. Problem statement

Consider the following problem. Given a domain Ω and an elliptic differential operator $\mathcal{L}$, we need to solve

$$\begin{cases} \mathcal{L}u = f, & x \in \Omega, \\ + \text{ boundary conditions}. \end{cases} \tag{21}$$

One of the common numerical methods for solving such problems is FEM. The described techniques can be applied not only to elliptic differential operators and not only for FEM, but we consider this particular case for clarity of presentation.

First of all, FEM requires the construction of some grid on the domain Ω. Each grid generates a system of linear equations to be solved. By refining the grid, we obtain a sequence of matrices with growing sizes. The theory of GLT-sequences enables to find the distribution of eigenvalues and singular values when Ω in (21) is a d-dimensional parallelepiped and the grid is uniform. For more general domains GLT-theory suggests [3, 1], the following technique: Ω is described by some map $G : [0,1]^d \to \overline{\Omega}$ that defines the geometry of the domain. The function G has to be continuously invertible, its image has to be equal to $\overline{\Omega}$, and it has to map boundary to the boundary: $G(\partial([0,1]^d)) = \partial\overline{\Omega}$. Then the grid on Ω is obtained by mapping a uniform grid from $[0,1]^d$, and the basis functions ϕ_i for FEM are obtained as $\phi_i(\boldsymbol{x}) = \hat{\phi}_i(G^{-1}(\boldsymbol{x}))$, where $\hat{\phi}_i$ are basis functions on $[0,1]^d$ with the uniform grid. Then by the change of variables, we get the problem defined on $[0,1]^d$. For example, using the above approach in [2] it was derived a direct formula for the asymptotic eigenvalue distribution of stiffness matrices obtained by applying finite elements to the semielliptic PDE of second order.

However, this technique of grid constructing, which is natural in world of the isogeometric analysis (see [11, 17] and references therein), has significant drawbacks when considering more classical approximation techniques. The first of them is that the construction of effective maps G is extremely challenging [18, 19]. In addition to the above requirements for the function G, it should also induce a conformal grid on Ω, and also enables to control the grid coarseness in different regions of Ω, for example, to induce a uniform grid on Ω. Another drawback is the calculation of integrals for FEM. Complex maps G may not enable to compute integrals analytically which is necessary for some problems.

Another way to deal with non-rectangular domains is reduced GLT theory. This theory proposes the following procedure for the constructing of the grid:

1. To choose the affine transformation F which maps the domain Ω to d-dimensional cube $[0,1]^d$ with uniform square grid and maximizes the measure of the new set;
2. To continue all coefficient functions to the whole cube by zeros;
3. We choose the part of the uniform square grid which lies inside $F(\Omega)$ as the grid for $F(\Omega)$. It can be easily proved that if we denote $\{A_n\}_n$ the sequence of discretization matrices on the domain $F(\Omega)$ and $\{B_n\}_n$ the sequence of discretization matrices on $[0,1]^d$ then $A_n = \Pi_n B_n \Pi_n$, where Π_n is a matrix consisting of several columns of permutation matrix. It remains note that the spectral distribution of $\{B_n\}_n$ can be obtained from the classical GLT theory.

However this approach allows to use only uniform grid and the construction of the affine map which maximazes the image measure can be difficult problem for complicated domains.

A more natural and efficient way is to try to find a distribution of eigenvalues and singular values for a sequence of matrices given by a particular sequence of refining grids and basis functions.

Now we consider one simple but extremely important particular case when Ω is a triangle. We assume that the triangle has a uniform grid, and Courant functions (that is functions which are equal to 1 exactly in one node of the grid, equal to 0 in all other nodes and piecewise linear on every triangle of the triangulation) are taken as basis functions. The last condition is not critical for the analysis and is taken only for simplicity. In fact, basis functions are only required to have compact support. Then for a given sequence of grids and a set of basis functions we try to find a distribution of eigenvalues and singular values for the obtained sequence of discretization matrices.

4.2. Laplace operator case

Consider the following problem:

$$\begin{cases} -\Delta u(\boldsymbol{x}) = f(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega, \\ u(\boldsymbol{x})\Big|_{\partial\Omega} = 0, \end{cases} \tag{22}$$

where Ω is a triangle with generator vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ (Figure 1).

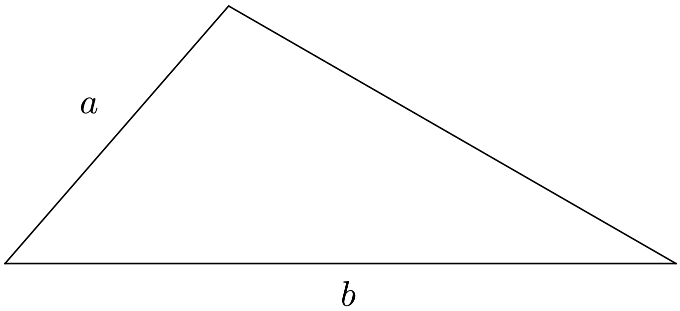

Figure 1: A triangle with generators $\boldsymbol{a}$ and $\boldsymbol{b}$.

We introduce a uniform grid containing n nodes on a triangle, namely, the grid generated by splitting the original triangle into small equal triangles that are similar to the original one. We use a natural node numbering consistent with the generating vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ (Figure 2) and solve the problem by FEM, taking Courant functions as basis ones.

The weak formulation of (22):

$$\int\limits_{\Omega} (\nabla u, \nabla \phi) d\Omega = \int\limits_{\Omega} (f, \phi) d\Omega, \ \forall \phi \in \mathring{W}_2^1, \tag{23}$$

where $\mathring{W}_2^1$ is a Sobolev space with zero boundaries.

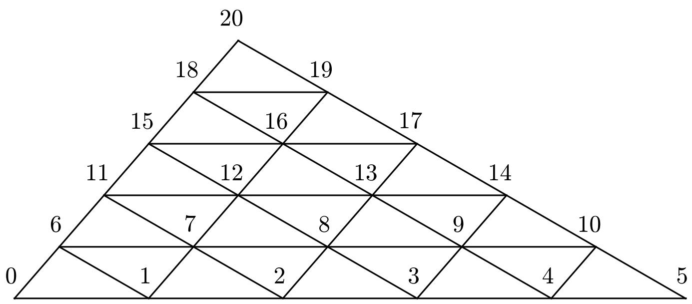

Figure 2: Uniform grid on a triangle and the numbering induced by generators.

The basis functions are

$$\phi_i(\boldsymbol{x}_j) = \begin{cases} 0, & \boldsymbol{x}_i \neq \boldsymbol{x}_j \\ 1, & \boldsymbol{x}_i = \boldsymbol{x}_j \end{cases}, \ i, j = 0, \dots, n-1, \tag{24}$$

and $\phi_i(\boldsymbol{x}_j)$ are linear on every triangle of the triangulation, that is it is nonzero on all small triangles which are incident to the node i and zero on all other triangles.

Write an explicit expression for ϕ_i. On all the triangles that are not adjacent to $\boldsymbol{x}_i$ the function ϕ_i is equal to zero. Consider a triangle adjacent to the node $\boldsymbol{x}_i$ with the generators $\boldsymbol{c}$ and $\boldsymbol{d}$. Then, by writing the general form of a linear function (we denote coordinates of vectors with superscript to avoid the confusion with indices denoted with subscript):

$$\phi(\boldsymbol{x}) = a_0\boldsymbol{x}^{(1)} + a_1\boldsymbol{x}^{(2)} + a_2 \tag{25}$$

and the system of constraints

$$\begin{cases} \phi_i(\boldsymbol{x}_i) = 1 \\ \phi_i(\boldsymbol{x}_i + \boldsymbol{c}) = 0 \\ \phi_i(\boldsymbol{x}_i + \boldsymbol{d}) = 0 \end{cases}, \tag{26}$$

we obtain the expression for the coefficients of function ϕ_i on this triangle

$$a_0 = \frac{\boldsymbol{d}^{(2)} - \boldsymbol{c}^{(2)}}{\boldsymbol{c}^{(2)}\boldsymbol{d}^{(1)} - \boldsymbol{c}^{(1)}\boldsymbol{d}^{(2)}}, \tag{27}$$

$$a_1 = \frac{\boldsymbol{c}^{(1)} - \boldsymbol{d}^{(1)}}{\boldsymbol{c}^{(2)}\boldsymbol{d}^{(1)} - \boldsymbol{c}^{(1)}\boldsymbol{d}^{(2)}}, \tag{28}$$

$$a_2 = 1 + \frac{\boldsymbol{x}_i^{(1)}(\boldsymbol{c}^{(2)} - \boldsymbol{d}^{(2)}) + \boldsymbol{x}_i^{(2)}(\boldsymbol{c}^{(1)} - \boldsymbol{d}^{(1)})}{\boldsymbol{c}^{(2)}\boldsymbol{d}^{(1)} - \boldsymbol{c}^{(1)}\boldsymbol{d}^{(2)}}. \tag{29}$$

Correspondingly

$$\frac{\partial \phi_i}{\partial \boldsymbol{x}^{(1)}} = \frac{\boldsymbol{d}^{(2)} - \boldsymbol{c}^{(2)}}{\boldsymbol{c}^{(2)}\boldsymbol{d}^{(1)} - \boldsymbol{c}^{(1)}\boldsymbol{d}^{(2)}}, \tag{30}$$

$$\frac{\partial \phi_i}{\partial \boldsymbol{x}^{(2)}} = \frac{\boldsymbol{c}^{(1)} - \boldsymbol{d}^{(1)}}{\boldsymbol{c}^{(2)}\boldsymbol{d}^{(1)} - \boldsymbol{c}^{(1)}\boldsymbol{d}^{(2)}}. \tag{31}$$

Calculating the considered integrals, we can write an equation for each node of the grid, and then the discretization matrix itself. Let us introduce the notation:

$$t_0 = 2\frac{(\boldsymbol{a}^{(1)})^2 + (\boldsymbol{a}^{(2)})^2 + (\boldsymbol{b}^{(1)})^2 + (\boldsymbol{b}^{(2)})^2 - \boldsymbol{a}^{(1)}\boldsymbol{b}^{(1)} - \boldsymbol{a}^{(2)}\boldsymbol{b}^{(2)}}{|\boldsymbol{a}^{(2)}\boldsymbol{b}^{(1)} - \boldsymbol{a}^{(1)}\boldsymbol{b}^{(2)}|} \tag{32}$$

$$t_1 = \frac{\boldsymbol{a}^{(1)}\boldsymbol{b}^{(1)} + \boldsymbol{a}^{(2)}\boldsymbol{b}^{(2)} - (\boldsymbol{b}^{(1)})^2 - (\boldsymbol{b}^{(2)})^2}{|\boldsymbol{a}^{(2)}\boldsymbol{b}^{(1)} - \boldsymbol{a}^{(1)}\boldsymbol{b}^{(2)}|}, \tag{33}$$

$$t_2 = \frac{-(\boldsymbol{a}^{(1)}\boldsymbol{b}^{(1)} + \boldsymbol{a}^{(2)}\boldsymbol{b}^{(2)})}{|\boldsymbol{a}^{(2)}\boldsymbol{b}^{(1)} - \boldsymbol{a}^{(1)}\boldsymbol{b}^{(2)}|}, \tag{34}$$

$$t_3 = \frac{\boldsymbol{a}^{(1)}\boldsymbol{b}^{(1)} + \boldsymbol{a}^{(2)}\boldsymbol{b}^{(2)} - (\boldsymbol{a}^{(1)})^2 - (\boldsymbol{a}^{(2)})^2}{|\boldsymbol{a}^{(2)}\boldsymbol{b}^{(1)} - \boldsymbol{a}^{(1)}\boldsymbol{b}^{(2)}|}. \tag{35}$$

For the discretization we introduce the space of piecewise linear functions on Ω, which are linear on every triangle of the triangulation,

$$\mathring{W}^1_{2h} = \Big\{ \sum_{i:\text{internal vertices}} \alpha_i \phi_i(x) : \alpha_i \in \mathbb{R} \Big\}, \tag{36}$$

and $u^h \in \mathring{W}^1_{2h}$ is determined by the equations

$$\int_\Omega (\nabla u^h, \nabla \phi^h) d\Omega = \int_\Omega (f, \phi^h) d\Omega, \ \forall \phi^h \in \mathring{W}^1_{2h}, \tag{37}$$

so the problem reduces to linear algebraic equations. Refining the grid gives us a sequence of discretization matrices with growing size. In this notations it is easy to write down the discretization matrix. Firstly, let us imagine that we are solving the problem not in a triangle, but in a parallelogram spanned on the same generator vectors. Then the discretization matrix T_n has the following form:

$$T_4 = \left[\begin{array}{cccc|cccc|cccc|cccc}
t_0 & t_1 & & & t_3 & & & & & & & & & & & \\
t_1 & t_0 & t_1 & & t_2 & t_3 & & & & & & & & & & \\
 & t_1 & t_0 & t_1 & & t_2 & t_3 & & & & & & & & & \\
 & & t_1 & t_0 & & & t_2 & t_3 & & & & & & & & \\
\hline
t_3 & t_2 & & & t_0 & t_1 & & & t_3 & & & & & & & \\
 & t_3 & t_2 & & t_1 & t_0 & t_1 & & t_2 & t_3 & & & & & & \\
 & & t_3 & t_2 & & t_1 & t_0 & t_1 & & t_2 & t_3 & & & & & \\
 & & & t_3 & & & t_1 & t_0 & & & t_2 & t_3 & & & & \\
\hline
 & & & & t_3 & t_2 & & & t_0 & t_1 & & & t_3 & & & \\
 & & & & & t_3 & t_2 & & t_1 & t_0 & t_1 & & t_2 & t_3 & & \\
 & & & & & & t_3 & t_2 & & t_1 & t_0 & t_1 & & t_2 & t_3 & \\
 & & & & & & & t_3 & & & t_1 & t_0 & & & t_2 & t_3 \\
\hline
 & & & & & & & & t_3 & t_2 & & & t_0 & t_1 & & \\
 & & & & & & & & & t_3 & t_2 & & t_1 & t_0 & t_1 & \\
 & & & & & & & & & & t_3 & t_2 & & t_1 & t_0 & t_1 \\
 & & & & & & & & & & & t_3 & & & t_1 & t_0
\end{array}\right], \tag{38}$$

that is a two-level Toeplitz matrix. Secondly, the discretization matrix $\widehat{T'}_n$ for a trianglular domain can be obtained from T_n by removing the last column and row from the second block, the last two columns and rows from the third block and so on:

$$
\widehat{T'}_4 = \left[\begin{array}{cccc|ccc|cc|c}
t_0 & t_1 & & & t_3 & & & & & \\
t_1 & t_0 & t_1 & & t_2 & t_3 & & & & \\
 & t_1 & t_0 & t_1 & & t_2 & t_3 & & & \\
 & & t_1 & t_0 & & & t_2 & & & \\
\hline
t_3 & t_2 & & & t_0 & t_1 & & t_3 & & \\
 & t_3 & t_2 & & t_1 & t_0 & t_1 & t_2 & t_3 & \\
 & & t_3 & t_2 & & t_1 & t_0 & & t_2 & \\
\hline
 & & & & t_3 & t_2 & & t_0 & t_1 & t_3 \\
 & & & & & t_3 & t_2 & t_1 & t_0 & t_2 \\
\hline
 & & & & & & & t_3 & t_2 & t_0
\end{array}\right]. \tag{39}
$$

We refer such matrices as *truncated Toeplitz matrices*. We show that the sequence of matrices $\widehat{T'}_n$ is not a GLT-sequence.

Note that the sequence of matrices composed of only diagonal blocks of $\widehat{T'}_n$ generates a GLT-sequence. Indeed, consider a matrix composed of diagonal blocks of the matrix $\widehat{T'}_n$. This matrix can be transformed to a tridiagonal Toeplitz matrix by symmetric transformation of rank $o(n^2)$, where the matrix $\widehat{T'}_n$ is of size $\dfrac{n(n+1)}{2}$ (we need just "to glue" diagonal which contains t_1. For this we only need to add a matrix containing $2(n-1)$ values t_1). It is easy to see that a self-adjoint tridiagonal Toeplitz matrix is a GLT-sequence, therefore, due to the properties of GLT-sequences (Section 3.3), the diagonal blocks of matrices $\widehat{T'}_n$ can be put to zero and we can prove the statement for the sequence of matrices $\widehat{T}_n$ of the following form:

$$
\widehat{T}_4 = \left[\begin{array}{cccc|ccc|cc|c}
 & & & & t_3 & & & & & \\
 & & & & t_2 & t_3 & & & & \\
 & & & & & t_2 & t_3 & & & \\
 & & & & & & t_2 & & & \\
\hline
t_3 & t_2 & & & & & & t_3 & & \\
 & t_3 & t_2 & & & & & t_2 & t_3 & \\
 & & t_3 & t_2 & & & & & t_2 & \\
\hline
 & & & & t_3 & t_2 & & & & t_3 \\
 & & & & & t_3 & t_2 & & & t_2 \\
\hline
 & & & & & & & t_3 & t_2 &
\end{array}\right]. \tag{40}
$$

Assume that $\{\widehat{T}_n\}_n$ possesses a GLT-distribution on the domain $[0,1]^2 \times [-\pi,\pi]^2$, i.e. $\{\widehat{T}_n\}_n \sim_{GLT} f(x_1, x_2, \theta_1, \theta_2)$.

Note that matrix $\widehat{T}$ has the size $\dfrac{n(n+1)}{2}$. We introduce a $n \times n$ Jordan block J_n:

$$J_n = \begin{bmatrix} 0 & 1 & & & & & \\ 0 & 0 & 1 & & & & \\ & 0 & 0 & 1 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & & 0 & 0 & 1 \\ & & & & & 0 & 0 \end{bmatrix}. \tag{41}$$

Let $\widehat{J}_n = J_{[n/\sqrt{2}]} \otimes I_{[n/\sqrt{2}]}$. It is the two-level Toeplitz matrix and the sequence of such matrices belongs to the class of GLT-sequences $\{\widehat{J}_n\}_n \sim_{GLT} e^{-\mathrm{i}\theta_1}$. Note that

$$\left[\frac{n}{\sqrt{2}}\right]^2 - \frac{n(n+1)}{2} = O(n), \tag{42}$$

which means that we can extend the diagonals in matrix $\widehat{J}_n$ to have a matrix of order $\dfrac{n(n+1)}{2}$ by transformation of rank $o(n^2)$. We denote the obtained matrix by H_n. It has the same GLT-distribution as $\widehat{J}_n$ and the size $\dfrac{n(n+1)}{2}$.

Consider the sequence $\{\widehat{T}_n H_n - H_n \widehat{T}_n\}_n$. Due to the GLT-theory, the singular values of the resulting sequence have to be zero-distributed. Our immediate goal is to show that it is not possible.

Note that the multiplication $\widehat{T}_n H_n$ performs a shift of the elements of the matrix $\widehat{T}_n$ to the right. Similarly, $H_n \widehat{T}_n$ shifts the matrix elements upwards. The matrices $\widehat{T}_n$ have a block structure. We calculate the numbers of columns where the blocks of the matrix $\widehat{T}_n$ begin, as well as the numbers of columns where they move to after the multiplications $\widehat{T}_n H_n$ and $H_n \widehat{T}_n$. Initially, the column numbers where the blocks of the matrix $\widehat{T}_n$ begin are n, $n + (n-1)$, etc., the first column of the k-th block has the number $\sum_{j=1}^{k} n - j + 1 = k(n+1) - \dfrac{k(k+1)}{2}$. After the multiplication of $\widehat{T}_n$ by H_n from the right hand side the numbers of all columns increase by $\left[\dfrac{n}{\sqrt{2}}\right]$, while multiplication of $\widehat{T}_n$ by H_n from the left hand side does not change the column numbers. Consider an arbitrary superdiagonal block of $\widehat{T}_n$ that lies above the main diagonal and contains the quantities t_3. We assume that $t_3 \neq 0$, otherwise, the proof can be done similarly for $t_2 \neq 0$. t_2 and t_3 cannot be equal zero simultaneously, otherwise the triangle degenerates to a segment or a point. Note that any diagonal after the shift to the right or upwards by the same number of elements, moves to the one diagonal (in this case we mean by a diagonal the set of elements of the matrix $[\widehat{T}_n]_{ij}$ such that the difference $i - j$ is constant). The order of the k-th superdiagonal block is $n - k$. Then the diagonal of the considered block after the upwards shift has the column numbers from $k(n+1) - \dfrac{k(k+1)}{2}$

to $k(n+1) - \dfrac{k(k+1)}{2} + n - k$. At the same time, after the shift to the right it has the column numbers from $k(n+1) - \dfrac{k(k+1)}{2} + \left[\dfrac{n}{\sqrt{2}}\right]$ to $k(n+1) - \dfrac{k(k+1)}{2} + \left[\dfrac{n}{\sqrt{2}}\right] + n - k$. So, after the subtraction $\widehat{T}_n H_n - H_n \widehat{T}_n$ the elements equal to t_3 remain on the positions with column numbers from $k(n+1) - \dfrac{k(k+1)}{2} + n - k + 1$ to $k(n+1) - \dfrac{k(k+1)}{2} + \left[\dfrac{n}{\sqrt{2}}\right] + n - k$ (i.e. $\left[\dfrac{n}{\sqrt{2}}\right]$ elements) under the conditions

$$k(n+1) - \frac{k(k+1)}{2} + n - k \geq k(n+1) - \frac{k(k+1)}{2} + \left[\frac{n}{\sqrt{2}}\right], \tag{43}$$

$$n - \left[\frac{n}{\sqrt{2}}\right] \geq k. \tag{44}$$

The latter is true for all blocks whose numbers satisfy the inequality above and whose elements do not vanish from the matrix shifting upwards and to the right, i.e., such that the numbers k of these blocks satisfy the inequality

$$2 \leq k \leq n - \left[\frac{n}{\sqrt{2}}\right]. \tag{45}$$

Then in the matrix $\widehat{T}_n H_n - H_n \widehat{T}_n$ at least

$$\left(n - \left[\frac{n}{\sqrt{2}}\right] - 1\right)\left[\frac{n}{\sqrt{2}}\right] \geq C_0 \frac{n(n+1)}{2} \tag{46}$$

elements remain to be equal to t_3, where $C_0 > 0$. It is clear that in the case $t_3 = 0$, the proof is completely analogous to the elements equal to t_2.

Then $\|\widehat{T}_n\|_F \geq C\dfrac{n(n+1)}{2}$. In addition, it is obvious that $\|\widehat{T}_n\|_1 \leq 4(|t_2| + |t_3|)$ and $\|\widehat{T}_n\|_\infty \leq 4(|t_2| + |t_3|)$, hence $\|\widehat{T}_n\|_2 \leq 4(|t_2| + |t_3|)$. We have that $\sum\limits_k \sigma_k^2 \geq C\dfrac{n(n+1)}{2}$, but $\sigma_1^2 \leq 16(|t_2| + |t_3|)^2$, where σ_k are singular values of $\widehat{T}_n$ and σ_1 is the largest singular value. Assume that at least $\delta\dfrac{n(n+1)}{2}$ singular values are less than ε. $\sum\limits_1 \sigma_k^2 + \sum\limits_2 \sigma_k^2 \geq C\dfrac{n(n+1)}{2}$, where $\sum\limits_2$ denotes the sum by $\delta\dfrac{n(n+1)}{2}$ minimal elements and $\sum\limits_1$ by the others. Then $\sum\limits_1 \sigma_k^2 \geq (C - \delta\varepsilon^2)\dfrac{n(n+1)}{2}$ and

$$\sigma_1^2(1-\delta)\frac{n(n+1)}{2} \geq \sum_1 \sigma_k^2 \geq (C - \delta\varepsilon^2)\frac{n(n+1)}{2}, \tag{47}$$

$$16(|t_2| + |t_3|)^2(1-\delta) \geq C - \delta\varepsilon^2, \tag{48}$$

$$\delta \leq \frac{16(|t_2| + |t_3|)^2 - C}{16(|t_2| + |t_3|)^2 - \varepsilon^2}. \tag{49}$$

The formulas (34-35) show that t_2 and t_3 cannot be equal to zero at the same time. Choosing $\varepsilon = \frac{\sqrt{C}}{2}$, we obtain that there cannot be more than $\delta\frac{n(n+1)}{2}$ (where $\delta < 1$) elements that are less than ε, therefore, at least $(1-\delta)\frac{n(n+1)}{2}$ singular values are greater than ε and the sequence is not zero-distributed [11]. It means that our assumption about the GLT-distribution of $\{\widehat{T}_n\}_n$ is false. Thus, the following has been proved:

Theorem 3. *Let $\{\widehat{T}_n\}_n$ be a matrix sequence obtained by the discretization of (22) by FEM, where the domain Ω is a triangle, the grid is uniform and basis functions are Courant functions. Then $\{\widehat{T}_n\}_n$ is not a GLT-sequence on the domain $[0,1]^2 \times [-\pi,\pi]^2$.*

4.3. Finding the spectrum distribution

Although the considered sequences are not GLT-sequences, they still possess some distribution of eigenvalues and singular values. Moreover, it turns out that symbols of a distribution of eigenvalues and singular values are exactly the same as it is if we consider the problem with the same operator on a parallelogram, where sequences of discretization matrices are GLT-sequences.

We extend the triangle with generators $\boldsymbol{a}$ and $\boldsymbol{b}$ to the parallelogram with the same generators and take it as the domain Ω in the problem (22). Then, as it noted above, the discretization matrix looks like (38). This is a two-level Toeplitz matrix, which is a GLT-sequence with symbol $t_0 + 2t_3 \cos\theta_1 + 2t_1 \cos\theta_2 + 2t_2 \cos(\theta_1 - \theta_2)$, where $t_0, \ldots, t_3$ are defined in (32–35). Now we slightly change the problem statement by adding the artificial additional boundary condition which is the equality to zero on the diagonal of the parallelogram which divides the parallelogram into two triangles generated by $(\boldsymbol{a}, \boldsymbol{b})$ and $(-\boldsymbol{a}, -\boldsymbol{b})$. From the point of view of discretization we split the parallelogram into two independent equal triangles. Indeed, for our discretization the value in each node depends only on the values in neighboring nodes, which implies that the values in both triangles are independent due to they are partitioned by the zero diagonal. From the point of view of discretization matrices it is equivalent to removing columns and rows with the numbers corresponding to diagonal nodes. Since the diagonal contains about n nodes, while the matrix is of size n^2, we make only a low-rank transformation which does not change the GLT symbol. We denote the matrix after the removing corresponding rows and columns as $\widetilde{T}_n$. Now let us change the node numbering in the triangulation of the domain. Renumber nodes so that one triangle is numbered at first and then the another one is numbered and also such that the order of numbering is the same for both triangles (Figure 3).

Note that exchange of a pair of node numbers is equivalent in the matrix to the exchange of rows and columns with corresponding numbers, i.e. similarity transformation with permutation matrices. Then after such a renumbering, the sequence of discretization matrices

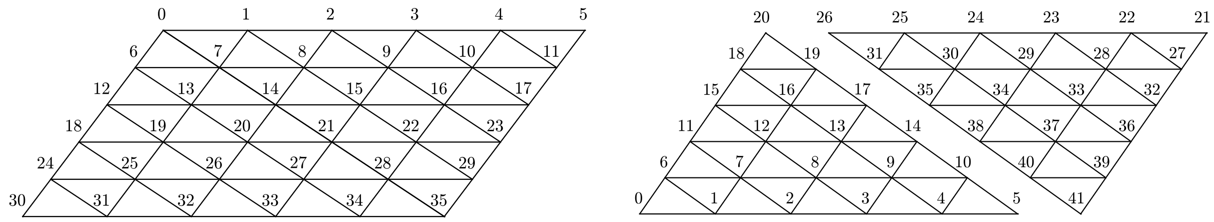

Figure 3: The grid on the parallelogram and division of the parallelogram into 2 triangles.

transforms to the form

$$P_n\widetilde{T}_nP_n^T = \begin{bmatrix} \widehat{T}_n & 0 \\ 0 & \widehat{T}_n \end{bmatrix}, \tag{50}$$

where $\widehat{T}_n$ is the discretization matrix for the triangle. Due to the last transformation is transformation of unitary similarity, it does not change eigenvalues and singular values. It follows that sequences of matrices $\{T_n\}_n$ and $\{\widehat{T}_n\}_n$ have the same distributions of eigenvalues and singular values.

Theorem 4. *Let $\{\widehat{T}_n\}_n$ be a matrix sequence obtained by a discretization of (22) by FEM, where the domain Ω is a triangle, the grid is uniform and basis functions are Courant functions. Then the sequence $\{\widehat{T}_n\}_n$ is not a GLT-sequence, however, it has a distribution of eigenvalues and singular values with symbol $t_0 + 2t_3\cos\theta_1 + 2t_1\cos\theta_2 + 2t_2\cos(\theta_1 - \theta_2)$ on the domain $[-\pi,\pi]^2$, where $t_0,\dots,t_3$ are defined in (32–35).*

Note 1. *It is easy to see that formulas (32–35) for t_0, t_1, t_2 and t_3 depend only on the dot-product of generators, their lengths and the square of the parallelogram that is spanned on the generators. Hence, the matrix sequences and the distributions of eigenvalues and singular values do not change after the isometry.*

4.4. Generalization in case of variable coefficients

The statements described and proved above have been stated for the Laplace operator, however, they are true for a much more general family of operators including operators depending on the values of some functions defined on the grid nodes. For certainty, consider the following problem:

$$\begin{cases} -\nabla\cdot(a(\boldsymbol{x})\nabla u(\boldsymbol{x})) = f(\boldsymbol{x}), & \boldsymbol{x}\in\Omega_T, \\ u(\boldsymbol{x})\Big|_{\partial\Omega_T} = 0, \end{cases} \tag{51}$$

where Ω_T is an isosceles rectangular triangle generated by $(0,1)^T$ and $(1,0)^T$. The described technique can be directly generalized to solve the problem in this case. We extend the triangle to the square with generators $(0,1)^T$ and $(1,0)^T$. The function $a(x,y)$ is defined on the triangle spanned by these generators. We extend the function $a(x,y)$ to the whole square, using the central symmetry relative to the center of the square, i.e., we define

$$a(x,y) = a(1-x,1-y),\ y \geq 1-x \tag{52}$$

We add the artificial zero boundary condition on the diagonal $x+y=1$, which corresponds to self-adjoint low-rank transformation, and then renumber the nodes such that the matrix has a block-diagonal structure with two equal blocks on the diagonal. As a result, we obtain that the distribution of eigenvalues in the discretization of the problem (51) coincides with the distribution of eigenvalues for the same problem on the square, where $a(x, y)$ is extended using central symmetry to the whole square. The presented technique enables to conclude the following:

Let us have the problem

$$\begin{cases} \mathcal{L}u(\boldsymbol{x}) = f(\boldsymbol{x}), & \boldsymbol{x} \in \Omega, \\ u(\boldsymbol{x})\Big|_{\partial\Omega} = 0, \end{cases} \tag{53}$$

where $\mathcal{L}$ is an elliptic differential operator that is possibly depends on some functions defined on the domain Ω, where Ω is the triangle given by its generator vectors. We construct the parallelogram spanned by this generators and extend all functions by central symmetry to the whole parallelogram, that is, we extend $\mathcal{L}$ to the whole parallelogram. Then we extend the uniform grid on Ω and map it by central symmetry to the parallelogram. Then if the discretization of the new problem has some distribution of eigenvalues or singular values, then the same discretization of the problem (53) has the same distribution of eigenvalues or singular values.

In particular, if the problem in a parallelogram leads to a sequence of matrices, which is a GLT-sequence with the symbol $\kappa(\boldsymbol{x}, \boldsymbol{\theta})$, then for the sequence of discretization matrices on a triangle we can map the same symbol $\kappa(\boldsymbol{x}, \boldsymbol{\theta})$.

Remark. It can be noted that the proposed technique shares similar ideas with reduced GLT theory [1, 3], but it has the advantage of being more direct, simpler and more intuitive.

5. Generalization of GLT-sequences

It turns out that the constructed symbols have all the same properties as classical GLT-symbols, despite the fact that the corresponding matrix sequences are not classical GLT-sequences. To justify this fact, we introduce the following definition.

Definition 6. *Let us say that a matrix sequence $\{A_n\}_n$ belongs to the class GLT_U^k with symbol $f(\boldsymbol{x}, \boldsymbol{\theta})$ on a domain D, if there exists a sequence of unitary matrices $\{U_n\}_n$ and a number k such that the matrix sequence $U_n^* B_n U_n$ is a GLT-sequence with symbol $f(\boldsymbol{x}, \boldsymbol{\theta})$ on the domain D, where B_n is a block-diagonal matrix, containing k blocks on the diagonal that are equal to A_n:*

$$B_n = \begin{bmatrix} A_n & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & A_n \end{bmatrix} \tag{54}$$

In particular, the set of classical GLT-sequences in this notation coincides with GLT_I^1.

Note that the sequence of matrices $\{A_n\}_n$ obtained by the discretization of the problem on a triangle can be transformed to a GLT-sequence by the following way. We build the sequence

$$B_n = \begin{bmatrix} A_n & 0 \\ 0 & A_n \end{bmatrix} \tag{55}$$

Then we apply to it self-adjoint low-rank transformation and after that similarity transformation with permutation matrices. It is clear that the last two transformations can be swapped (in the sense that we can firstly apply similarity transformation with permutation matrices, and then apply other self-adjoint low-rank transformation). But the class of GLT-sequences is closed in respect to low-rank transformations. It follows that the sequence of discretization matrices on a triangle belongs to GLT_P^2 for some sequence of permutation matrices $\{P_n\}_n$.

It turns out that matrix sequences of each class GLT_U^k have the properties similar to classical GLT-sequences. We prove some of these properties.

Theorem 5. *Let a matrix sequence $\{A_n\}_n$ belongs to the class GLT_U^k with symbol $f(\boldsymbol{x}, \boldsymbol{\theta})$ on the domain D. Then $\{A_n\}_n \sim_\sigma f(\boldsymbol{x}, \boldsymbol{\theta})$ on the domain D. If every matrix A_n is Hermitian, then $\{A_n\}_n \sim_\lambda f(\boldsymbol{x}, \boldsymbol{\theta})$ on the domain D.*

Proof. Obviously from the definition. □

Theorem 6. *Let a matrix sequence $\{A_n\}_n$ belong to the class GLT_U^k with symbols $f(\boldsymbol{x}, \boldsymbol{\theta})$ and $g(\boldsymbol{x}, \boldsymbol{\theta})$ on the domain D. Then $f(\boldsymbol{x}, \boldsymbol{\theta}) = g(\boldsymbol{x}, \boldsymbol{\theta})$ almost everywhere on the domain D.*

Proof. Immediately follows from the definition of the class GLT_U^k and a similar property for classical GLT-sequences. □

Theorem 7. *Let $\{A_n\}_n$ and $\{B_n\}_n$ belong to the class GLT_U^k for some $\{U_n\}_n$ and k with symbols $f(\boldsymbol{x}, \boldsymbol{\theta})$ and $g(\boldsymbol{x}, \boldsymbol{\theta})$ on a domain D. Then the matrix sequences $\{A_n + B_n\}_n$, $\{A_nB_n\}_n$, $\{\alpha A_n\}_n$ and $\{A_n^*\}_n$ belong to the class GLT_U^k with symbols $f(\boldsymbol{x}, \boldsymbol{\theta}) + g(\boldsymbol{x}, \boldsymbol{\theta})$, $f(\boldsymbol{x}, \boldsymbol{\theta})g(\boldsymbol{x}, \boldsymbol{\theta})$, $\alpha f(\boldsymbol{x}, \boldsymbol{\theta})$ and $\overline{f}(\boldsymbol{x}, \boldsymbol{\theta})$ respectively on the domain D.*

Proof. We prove this statement, for example, for the case of the product of matrix sequences. The remaining can be proved similarly. Since $\{A_n\}_n$ and $\{B_n\}_n$ belong to GLT_U^k, then the sequences

$$\widehat{A}_n = U_n \begin{bmatrix} A_n & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & A_n \end{bmatrix} U_n^* \text{ and } \widehat{B}_n = U_n \begin{bmatrix} B_n & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & B_n \end{bmatrix} U_n^* \tag{56}$$

are GLT-sequences with symbols $f(\boldsymbol{x}, \boldsymbol{\theta})$ and $g(\boldsymbol{x}, \boldsymbol{\theta})$. Then, on the one hand, the sequence $\{\widehat{A}_n\widehat{B}_n\}_n$ has GLT-distribution $f(\boldsymbol{x}, \boldsymbol{\theta})g(\boldsymbol{x}, \boldsymbol{\theta})$ and, on the other hand, is equal to

$$U_n \begin{bmatrix} A_nB_n & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & A_nB_n \end{bmatrix} U_n^*. \tag{57}$$

Hence $\{A_nB_n\}_n$ belongs to GLT_U^k with symbol $f(\boldsymbol{x}, \boldsymbol{\theta})g(\boldsymbol{x}, \boldsymbol{\theta})$. □

Theorem 8. *Let a matrix sequence $\{A_n\}_n$ belong to the class GLT_U^k with symbol $f(\boldsymbol{x}, \boldsymbol{\theta}) \neq 0$ almost everywhere. Then the matrix sequence $\{A_n^\dagger\}_n$ belongs to the class GLT_U^k with symbol $f^{-1}(\boldsymbol{x}, \boldsymbol{\theta})$.*

Proof. The sequence $\widehat{A}_n = U_n \begin{bmatrix} A_n & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & A_n \end{bmatrix} U_n^*$ is a GLT-sequence with symbol $f(\boldsymbol{x}, \boldsymbol{\theta}) \neq 0$ almost everywhere, therefore, $U_n \begin{bmatrix} A_n^\dagger & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & A_n^\dagger \end{bmatrix} U_n^* = \widehat{A}_n^\dagger$ has the GLT-distribution $f^{-1}(\boldsymbol{x}, \boldsymbol{\theta})$. □

Theorem 9. *Let a matrix sequence $\{A_n\}_n$ belong to the class GLT_U^k with symbol $f(\boldsymbol{x}, \boldsymbol{\theta})$ and every matrix A_n is Hermitian. Then the matrix sequence $\{g(A_n)\}_n$ belongs to the class GLT_U^k with symbol $g(f(\boldsymbol{x}, \boldsymbol{\theta}))$ for every continuous function $g : \mathbb{R} \to \mathbb{C}$.*

Proof. The sequence $\widehat{A}_n = U_n \begin{bmatrix} A_n & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & A_n \end{bmatrix} U_n^*$ is a GLT-sequence with symbol $f(\boldsymbol{x}, \boldsymbol{\theta})$, therefore, $g(\widehat{A})_n = U_n \begin{bmatrix} g(A_n) & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & g(A_n) \end{bmatrix} U_n^*$ has the GLT-distribution $g(f(\boldsymbol{x}, \boldsymbol{\theta}))$. □

Theorem 10. *A matrix sequence $\{A_n\}_n$ belongs to the class GLT_U^k with symbol $f(\boldsymbol{x}, \boldsymbol{\theta})$ on the domain D if and only if there is a sequence of matrix sequences $\{\{B_{n,m}\}_m\}_n$ such that $\{B_{n,m}\}_n$ belongs to the class GLT_U^k with symbol $f_m(\boldsymbol{x}, \boldsymbol{\theta})$ on the domain D, $\{\{B_{n,m}\}_m\}_n$ is an approximating class of sequences for $\{A_n\}_n$ and $f_m \to f$ in measure over D.*

Proof. If $\{A_n\}_n$ belongs to GLT_U^k, then we can take $\{A_n\}_n$ as $\{B_{n,m}\}_n$ for every m and the statement is obvious. Let there be a sequence of matrix sequences $\{\{B_{n,m}\}_m\}_n$, being an approximating class of sequences for $\{A_n\}_n$, and every sequence in $\{B_{n,m}\}_n$ belongs to the class GLT_U^k with symbol $f_m(\boldsymbol{x}, \boldsymbol{\theta})$ and f_m converges to a measurable function f in measure. The fact that $\{\{B_{n,m}\}_m\}_n$ is an approximating class of sequences for $\{A_n\}_n$ in terms of the pseudonorm (Section 3.1) writes like:

$$p_{a.c.s.}(\{A_n - B_{n,m}\}_n) \to 0, \; m \to \infty. \tag{58}$$

Due to the fact that every sequence $\{B_{n,m}\}_n$ belongs to the class GLT_U^k, the matrices

$$\widehat{B}_{n,m} = U_n \begin{bmatrix} B_{n,m} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & B_{n,m} \end{bmatrix} U_n^* \tag{59}$$

produce GLT-sequences with symbols f_m. Consider the matrix

$$\widehat{A}_n = U_n \begin{bmatrix} A_n & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & A_n \end{bmatrix} U_n^*. \tag{60}$$

Calculate the singular values of the matrix $\widehat{A}_n - \widehat{B}_{n,m}$. They coincide with the singular values of the matrix $A_n - B_{n,m}$, where every singular value is repeated k times. From here it is easy to see that $p_{kn}(\{\widehat{A}_n - \widehat{B}_{n,m}\}) \leq kp_n(A_n - B_{n,m})$, hence

$$p_{a.c.s.}(\{\widehat{A}_n - \widehat{B}_{n,m}\}) \to 0, \ m \to \infty, \tag{61}$$

therefore, $\{\{\widehat{B}_{n,m}\}_m\}_n$ is an approximating class of sequences for $\{\widehat{A}_n\}_n$. We obtain that $\{A_n\}_n$ is a GLT-sequence with symbol f, therefore, by the definition $\{A_n\}_n$ belongs to the class GLT_U^k with symbol f. □

Coming back to problems of discretization of differential equations on a triangle, it is worth noting that the permutation matrices for the implementation of similarity transformation do not depend on the choice of the triangle generators, nor on the operator, nor on the triangle itself, but are determined only by the uniformity of the grid. It follows that sequences of discretization matrices for all problems given on triangles with uniform grids belong to the same class GLT_P^2 for some fixed sequence of permutation matrices $\{P_n\}_n$.

6. Spectral distribution in case of polygonal domains

Consider a more general problem where the domain Ω in (22) is an arbitrary polygon. Let the sequence of the grids on Ω be obtained as follows. Some arbitrary (quite coarse) triangulation on the polygon is constructed. We refer triangles in this triangulation as base, denote the corresponding discretization matrix as A_1. Then every triangle is refined by several equal small triangles that are similar to the base one, and the number of small triangles in every base triangle is the same. In such a way we generate the matrix sequence $\{A_n\}$ (Figure 4).

By cutting (i.e., adding additional zero boundary conditions) the original polygon along the boundaries of the base triangles, and then renumbering the vertices so that inside each such triangle the numbering is continuous, we obtain that, up to low-rank symmetric transformation and similarity transformation, the sequence of the discretization matrices of the original polygon is

$$\begin{bmatrix} \widehat{B}_n^{(1)} & 0 & \dots & 0 & 0 \\ 0 & \widehat{B}_n^{(2)} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \widehat{B}_n^{(k-1)} & 0 \\ 0 & 0 & \dots & 0 & \widehat{B}_n^{(k)} \end{bmatrix}, \tag{62}$$

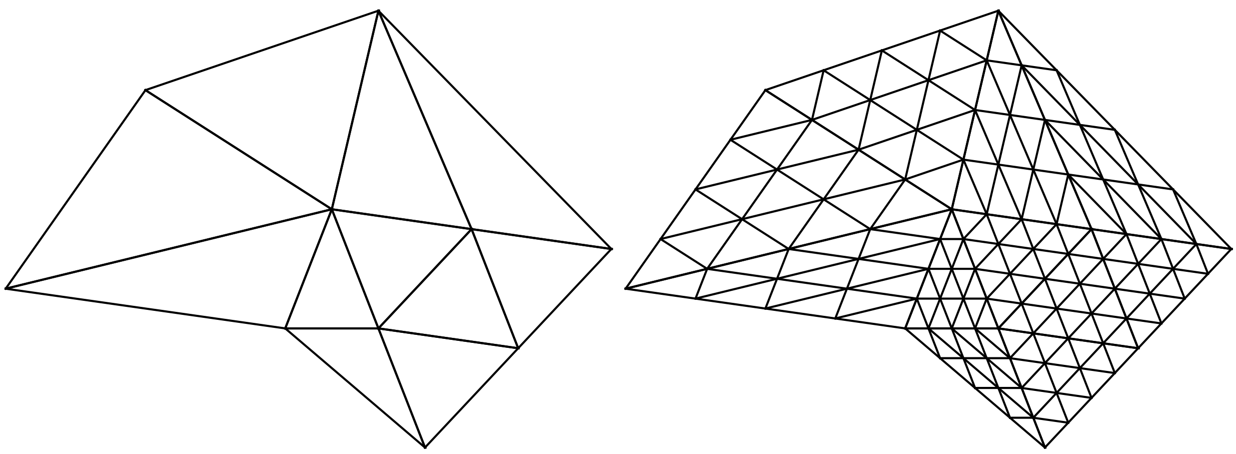

Figure 4: The examples of grids on polygonal domains.

where $\widehat{B}_n^{(j)}$ corresponds to the discretization matrices on the triangles, the sequences of which belong to GLT_P^2 with the known distributions. Moreover, all matrices have the same number of eigenvalues and singular values, so the distributions of spectra of $\widehat{B}_n^{(j)}$ characterize well the spectrum distribution of the sequence of discretization matrices on the polygon. In particular, the following is true

Theorem 11. *Let $\{A_n\}$ be a matrix sequence obtained by the discretization of the problem (22) by FEM, where the domain Ω is a polygon and the grid is obtained in a way described above. Let every sequence of matrices $\{B_n^{(j)}\}_n$ correspond to the sequence of discretization matrices on basic triangles and has the GLT_P^2-distribution with symbol $f_j(\boldsymbol{x}, \boldsymbol{\theta})$ on the domain $[0,1]^2 \times [-\pi, \pi]^2$ and the basic triangulation consists of k triangles. Then the sequence of matrices $\{A_n\}_n$ has the distribution of eigenvalues and singular values with symbol $f(\boldsymbol{x}, \boldsymbol{\theta})$ on the domain $[0,1]^2 \times [-\pi, \pi]^2$, where*

$$f(\boldsymbol{x}, \theta_1, \theta_2) = f_j(\boldsymbol{x}, k(\theta_1 + \pi) - 2\pi(j-1) - \pi, \theta_2), \tag{63}$$

$$\theta_1 \in \Big[-\pi + \frac{2\pi(j-1)}{k}, -\pi + \frac{2\pi j}{k}\Big],\ j = 1, \dots, k. \tag{64}$$

Proof. We prove this theorem for the case of eigenvalues distribution. From the above it follows that the distribution of eigenvalues of $\{A_n\}_n$ coincides with the distribution of eigenvalues of matrix sequence of the form (62). Let matrix A_n have the dimension $N_P(n) \times N_P(n)$, and every matrix $B_n^{(j)}$ has the dimension $N_T(n) \times N_T(n)$, $j = 1, \dots, k$. Then

$$\lim_{n\to\infty}\frac{1}{N_P(n)}\sum_{i=1}^{N_P(n)}F(\lambda_i(A_n))=\lim_{n\to\infty}\frac{1}{kN_T(n)}\sum_{j=1}^{k}\sum_{i=1}^{N_T(n)}F(\lambda_i(B_n^{(j)}))=$$

$$\frac{1}{k}\sum_{j=1}^{k}\lim_{n\to\infty}\frac{1}{N_T(n)}\sum_{i=1}^{N_T(n)}F(\lambda_i(B_n^{(j)}))=\frac{1}{k}\sum_{j=1}^{k}\frac{1}{\mu(D)}\int\limits_D F(f_j(\boldsymbol{x},\boldsymbol{\theta}))d\boldsymbol{x}d\boldsymbol{\theta}=$$

$$\frac{1}{k}\sum_{j=1}^{k}\frac{1}{\mu(D)}\int\limits_{[0,1]^2}\int\limits_{[-\pi,\pi]}\int\limits_{[-\pi,\pi]}F(f_j(\boldsymbol{x},\theta_1,\theta_2))d\theta_1d\theta_2d\boldsymbol{x}=$$

$$\frac{1}{\mu(D)}\int\limits_{[0,1]^2}\int\limits_{[-\pi,\pi]}\frac{1}{k}\sum_{j=1}^{k}\Big(\int\limits_{[-\pi,\pi]}F(f_j(\boldsymbol{x},\theta_1,\theta_2))d\theta_1\Big)d\theta_2d\boldsymbol{x}=$$

$$\frac{1}{\mu(D)}\int\limits_{[0,1]^2}\int\limits_{[-\pi,\pi]}\frac{1}{k}\sum_{j=1}^{k}\Big(k\int\limits_{[-\pi+\frac{2\pi(j-1)}{k},-\pi+\frac{2\pi j}{k}]}F(f_j(\boldsymbol{x},k(\theta_1+\pi)-2\pi(j-1)-\pi,\theta_2))d\theta_1\Big)d\theta_2d\boldsymbol{x}=$$

$$\frac{1}{\mu(D)}\int\limits_{[0,1]^2}\int\limits_{[-\pi,\pi]}\int\limits_{[-\pi,\pi]}F(f(\boldsymbol{x},\theta_1,\theta_2))d\theta_1d\theta_2d\boldsymbol{x}=\frac{1}{\mu(D)}\int\limits_D F(f(\boldsymbol{x},\boldsymbol{\theta}))d\boldsymbol{x}d\boldsymbol{\theta}. \quad (65)$$

□

Remark. The following question may arise here. Consider a polygon with a given eigenvalue problem on it, for certainty, let it be an eigenvalue problem for the Laplace operator. We consider two different triangulations on the domain and construct two sequences of the grids, according to the scheme above, and solve the problem of approximation of eigenvalues of the Laplace operator on the polygon using FEM. The eigenvalues of the continuous operator are approximated by the eigenvalues of the matrices $M_n^{-1}A_n$, where M_n is the mass matrix and A_n is the stiffness matrix. However, the eigenvalues of such a sequence will tend to infinity with the growth of n, therefore, it is impossible to write down the distribution of eigenvalues of this sequence. This problem can be solved by normalizing, i.e., by studying the distribution of eigenvalues of $\frac{1}{n^2}M_n^{-1}A_n$. This sequence does not have to be symmetric, which makes it impossible to apply a general theory about the distribution of eigenvalues. The difficulty can be overcome by moving to a similar sequence of matrices $\frac{1}{n^2}(M_n)^{-1/2}A_n(M_n)^{-1/2}$, for which the presented theory allows us to find the distribution of eigenvalues of the resulting sequence. The question is whether the distributions of the eigenvalues of such matrices are the same for different triangulations, since the eigenvalues of both matrix sequences must approximate the spectrum of the continuous Laplace operator on a polygon regardless of which triangulation we consider. The answer to this question is: no, they are not.

Firstly, we can verify this numerically. Consider the following two polygons and triangulations (Figure 5), and let us construct the grids, according to the scheme described above

(Figure 6). We numerically compute the eigenvalues of the resulting matrices for a large n and construct a piecewise linear function that takes on the values of the sorted eigenvalues on a uniform grid (Figure 7). The figure shows that eigenvalues are quite different, so their distributions are not the same for different triangulations.

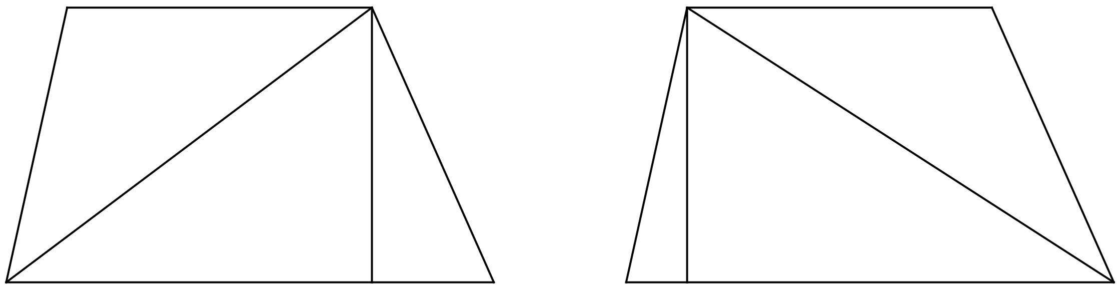

Figure 5: Different basic triangulations of the trapezoid.

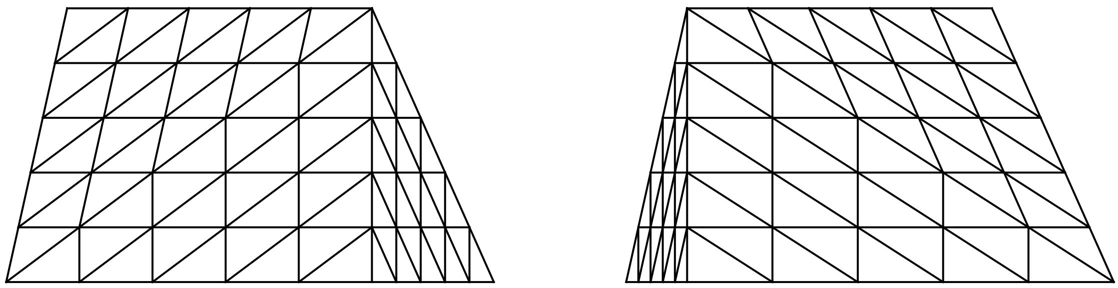

Figure 6: The grids inducing on the trapezoids.

Secondly, the theoretical explanation can be given. Eigenvalues of $M_n^{-1}A_n$ converge to the eigenvalues of the Laplace operator pointwise, with k-th smallest eigenvalue of $M_n^{-1}A_n$ approximating the k-th eigenvalue of the Laplace operator. Accordingly, k-th smallest eigenvalue of the matrix $\frac{1}{n^2}M_n^{-1}A_n$ approximates $\frac{1}{n^2}\lambda_k$, where λ_k is the k-th smallest eigenvalue of the Laplace operator. For simplicity, we assume that the sequence of matrices $\frac{1}{n^2}M_n^{-1}A_n$ is distributed with non-decreasing symbol f on the domain $[0, 2\pi]$. Then the k-th eigenvalue of the matrix $\frac{1}{n^2}M_n^{-1}A_n$ is approximately equal to $f\left(\frac{2\pi k}{n+1}\right)$, that is the arguments of the

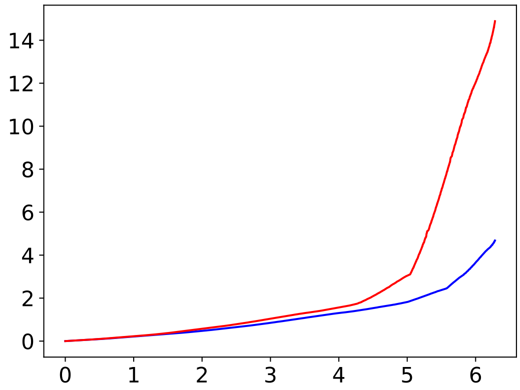

Figure 7: The distributions of the eigenvalues of the matrices $\frac{1}{n^2}M_n^{-1}A_n$.

function at which its values approximate the eigenvalues of the operator tend to zero with the increasing dimension of the matrices. As can be seen from Figure 7, the distribution functions for different triangulations are really very close in the neighborhood of 0, while far from zero they can differ quite strongly. Hence, the possible intuitive feeling about the equality of distributions actually has no theoretical basis and is misleading.

7. Application to preconditioning

One of practical applications of the spectral theory of matrix sequences is the construction of preconditioners for solving systems of linear algebraic equations. The rate of convergence of conjugate gradient method directly depends on eigenvalue distribution. In particular, if the eigenvalues are clustered at one point, then the convergence is especially fast. Preconditioners are used to improve the convergence rate of this method. The popular kind of preconditioners is circulant preconditioners [20, 21, 22] provided by many operations such as inversion or multiplication by a vector can be implemented very fast. However it is know that circulant preconditioners cannot provide a proper cluster for multilevel matrices [23].

Consider a system

$$Ax = f. \tag{66}$$

A well known method for solving such systems using preconditioning is the preconditioned conjugate gradient method (PCG). The convergence of PCG with preconditioner C is equivalent to application of the conjugate gradient method without preconditioning to the system

$$C^{1/2}AC^{1/2}C^{-1/2}x = C^{1/2}f \tag{67}$$

We further describe the process of constructing the preconditioner C. We will use PCG method with preconditioner C for numerical computations and formulation (67) for the theoretical justification of eigenvalue clusterization.

Let $W_n = [w_{i-j}]_{i,j=1}^n$ be a Toeplitz matrix. Then a simple circulant C_n for it is a matrix

$$C_n = \begin{cases} \operatorname{circ}(w_0, w_1, \dots, w_m, w_{-m}, \dots, w_{-1}), & n = 2m \\ \operatorname{circ}(w_0, w_1, \dots, w_{m-1}, 0, w_{-m+1}, \dots, w_{-1}), & n = 2m-1 \end{cases}, \tag{68}$$

where

$$\operatorname{circ}(c_0, \dots, c_{n-1}) = [c_{i-j (\operatorname{mod} n)}]_{i,j=1}^n. \tag{69}$$

Similarly, multilevel simple circulants are defined and the same properties are fulfilled for blocks, as well as within each block. It can be proved [4] that if $\{W_n\}_n$ is a sequence of Hermitian multilevel Toeplitz matrices, and $\{C_n\}_n$ is a sequence of corresponding simple circulants, then $\{W_n\}_n$ and $\{C_n\}_n$ have the same distribution of eigenvalues.

Consider the problem (22) again. The discretization matrix of this problem by FEM has the form (39). We describe the procedure for a possible effective preconditioning in this case.

1. We construct from the truncated Toeplitz matrix A of the form (39) a full Toeplitz matrix $\widehat{A}$. This can always be done uniquely for sufficiently large matrices due to the fact the basis functions have compact support. Moreover, a two-level Toeplitz matrix and a truncated Toeplitz matrix of order N are uniquely determined by $O(N)$ elements (the first rows and columns in the first row and column of blocks), and one set of elements can be converted to another in $O(N)$ operations;
2. We construct from the full Toeplitz matrix $\widehat{A}$ a simple circulant S. It is also easy to do in $O(N)$ operations (every circulant, in particular multilevel, uniquely given by its first column) based on formulas (68);
3. In practical tasks a circulant is often singular. In this case it can be replaced with the so-called improved circulant [24], in which all zero eigenvalues are replaced by $\delta > 0$. One can build an improved circulant in $O(N \log N)$ operations. Let us denote a new circulant as $\widehat{S}$;
4. We invert the circulant $\widehat{S}$ obtaining $\widehat{S}^{-1}$ which takes $O(N \log N)$ operations;
5. Below we will show that

$$\widehat{S} = P \begin{bmatrix} A^{-1} & 0 & 0 \\ 0 & A^{-1} & 0 \\ 0 & 0 & I \end{bmatrix} P^* + \widehat{L}_n = P \begin{bmatrix} A^{-1} + R^{(1)} & R^{(2)} & R^{(5)} \\ R^{(3)} & A^{-1} + R^{(4)} & R^{(6)} \\ R^{(7)} & R^{(8)} & R^{(9)} \end{bmatrix} P^*, \tag{70}$$

   where $R^{(k)}$ are low-rank matrices. So we take the block $A^{-1} + R^{(1)}$ as a preconditioner C. However will not construct and store $A^{-1} + R^{(1)}$ explicitly. In practice we only compute $\widehat{S}^{-1}$ and later we will show that it is enough for computing the product of $A^{-1} + R^{(1)}$ by a vector.

Now we show that the sequence of matrices $C_n^{1/2} A_n C_n^{1/2}$, where C_n is obtained from A_n in the way described above, has a distribution of eigenvalues with symbol 1. Since the sequence of matrices $\{A_n\}_n$ arises while discretizing the problem (22), it has a GLT_P^2-distribution with some symbol f. Then $\widehat{A}_n$ looks like

$$\widehat{A}_n = P_n \begin{bmatrix} A_n & 0 & 0 \\ 0 & A_n & 0 \\ 0 & 0 & I_n \end{bmatrix} P_n^* + M_n, \tag{71}$$

where M_n is a self-adjoint matrix of small rank. Indeed, here we do not remove the nodes corresponding to the diagonal of the parallelogram as in Section 4.3, but instead renumber in such a way to move them to the last positions in the matrix. Then we get

$$\widehat{A}_n = P_n \begin{bmatrix} A_n & 0 & \widehat{R}_n^{(1)} \\ 0 & A_n & \widehat{R}_n^{(2)} \\ \widehat{R}_n^{(3)} & \widehat{R}_n^{(4)} & \widehat{R}_n^{(5)} \end{bmatrix} P_n^*, \tag{72}$$

where the size of block $\widehat{R}_n^{(5)}$ is small relatively to the size of matrix. Then there is a low-rank transformation M_n such that (71) holds.

Note that the circulant S_n for a two-level Toeplitz matrix of the form (38) is obtained from $\widehat{A}_n$ by self-adjoint low-rank transformation, hence,

$$S_n = P_n \begin{bmatrix} A_n & 0 & 0 \\ 0 & A_n & 0 \\ 0 & 0 & I_n \end{bmatrix} P_n^* + \widetilde{R}_n, \tag{73}$$

Further, any circulant matrix can be represented as $\frac{1}{N} F_n^* \Lambda_n F_n$. If the matrix S_n has an independent of n number of zero eigenvalues, in order to improve the circulant S_n, it is necessary to add to it a matrix of the form $\frac{1}{N} F_n^* \Lambda_n F_n$, where Λ is a diagonal matrix and has an independent of n number of nonzero diagonal elements. Hence, transition from S_n to $\widehat{S}_n$ is again low-rank self-adjoint transformation. As a result, the matrix

$$\widehat{S}_n = P_n \begin{bmatrix} A_n & 0 & 0 \\ 0 & A_n & 0 \\ 0 & 0 & I_n \end{bmatrix} P_n^* + \widehat{R}_n, \tag{74}$$

where $\widehat{R}_n$ is a self-adjoint low-rank matrix, which implies that $\{\widehat{S}_n\}_n$ is a GLT-sequence with the same symbol f. Then, from a general theory, the sequence of matrices $\{\widehat{S}_n^{-1}\}_n$ belongs to the GLT class with symbol $1/f$. Also applying Woodbury matrix identity we obtain

$$\widehat{S} = P \begin{bmatrix} A_n^{-1} & 0 & 0 \\ 0 & A_n^{-1} & 0 \\ 0 & 0 & I_n \end{bmatrix} P^* + \widehat{L}_n = P \begin{bmatrix} A_n^{-1} + R_n^{(1)} & R_n^{(2)} & R_n^{(5)} \\ R_n^{(3)} & A_n^{-1} + R_n^{(4)} & R_n^{(6)} \\ R_n^{(7)} & R_n^{(8)} & R_n^{(9)} \end{bmatrix} P^*, \tag{75}$$

where again $\widehat{L}_n$ is a low-rank matrix as well as $R_n^{(k)}, k = 1, \dots, 9$.

As it was said above we take $C_n = A_n^{-1} + R_n^{(1)}$. From the definition of GLT_P^2 we deduce that $\{C_n\}_n$ belongs to GLT_P^2 with symbol $1/f$, which implies that the matrix sequence $\{C_n^{1/2} A_n C_n^{1/2}\}_n$ belongs to GLT_P^2 with symbol 1.

So, we have proved that the above preconditioning procedure gives a cluster of eigenvalues in 1. One more ingredient which we need for efficient preconditioning procedure is the fast iterations of CG method. For this purpose we need to show how we can efficient multiply C_n by a vector.

As is was already said we do not construct C_n explicitly. The only matrix we need to build is $\widehat{S}_n^{-1}$, which can be done in $O(N \log N)$ operations. We also need the permutation matrix P_n, but obviously it can be constracted in $O(N)$ operations following the reasonings presented in Section 4.3. Now let we need to multiply $C_n = A_n^{-1} + R_n^{(1)}$ by a vector v. We define $\widehat{v} = [v, 0, 0]^T$. Then

$$\begin{bmatrix} A_n^{-1} + R_n^{(1)} & R_n^{(2)} & R_n^{(5)} \\ R_n^{(3)} & A_n^{-1} + R_n^{(4)} & R_n^{(6)} \\ R_n^{(7)} & R_n^{(8)} & R_n^{(9)} \end{bmatrix} \begin{bmatrix} v \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} C_n v \\ R_n^{(3)} v \\ R_n^{(7)} v \end{bmatrix} = P_n^* \widehat{S}_n^{-1} P_n \widehat{v}. \tag{76}$$

Multiplication by a permutation matrix requires $O(N)$ operations and multiplication by a circulant requires $O(N \log N)$ operations, so one step of PCG can be done in $O(N \log N)$ operations.

Thus, an efficient preconditioning scheme was constructed, namely, the construction of the preconditioner requires $O(N \log N)$ operations, one iteration of PCG method requires $O(N \log N)$ operations (given that the discretization matrix is highly sparse and multiplying by it takes $O(N)$ operations), and the matrix after preconditioning has a cluster of eigenvalues in 1.

Numerical computations were carried out, according to the scheme described above. The matrix A of the dimension 5050×5050 obtained by discretizing the problem (22) in the domain that is an isosceles rectangular triangle was considered. Consequently, the matrix A exactly has the form of reduced Toeplitz matrix (39). At first we implemented the preconditioner construction scheme described above and built the precondioner C explicitly. Then we computed the matrix $C^{1/2} A C^{1/2}$ as well as numerically computed the eigenvalues of A and $C^{1/2} A C^{1/2}$. Figure 8 shows the eigenvalues of the matrix A before and after preconditioning. At the point k each curve takes on a value equal to the k-th smallest eigenvalue of the corresponding matrix. This experiment confirms that $C^{1/2} A C^{1/2}$ has the cluster of eigenvalues in 1. Then we implemented the conjugate gradient method for solving the system of linear algebraic equations with matrix A. In this case we do not build matrix C, but only the circulant $\widehat{S}^{-1}$. We generated a random right hand side from standard normal distribution and applied the CG method and PCG method with preconditioner C. Figure 9 shows the residual norm at various steps of the conjugate gradient method. This log-scale plot shows extremely superior convergence of PCG with the proposed preconditioner.

Remark. In addition, it is worth noting that the proposed preconditiong procedure can be directly extended to the case of discretization matrices for polygonal domains. If we use numeration in which the vertices of one triangle are numbered at first, then the second

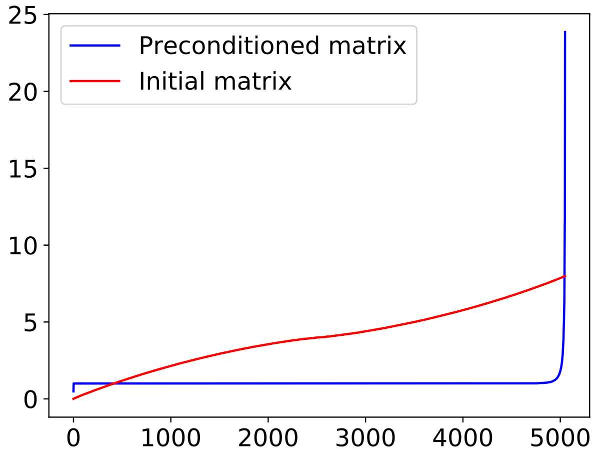

Figure 8: The distribution of eigenvalues with and without preconditioning.

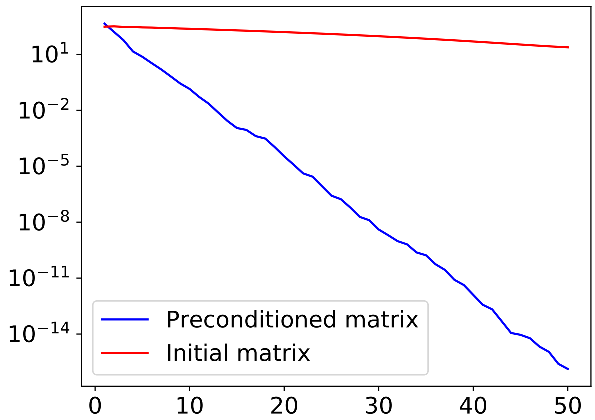

Figure 9: The values of the residual norm with and without preconditioning in the conjugate gradients method.

triangle is numbered, third, and so on, that is, the numbers of inner nodes of each triangle correspond to the certain segment of consecutive positive integers, then for such a problem, the procedure of efficient preconditioning is correct, namely, it is necessary to construct an appropriate preconditioner for each basic triangle and to use as a preconditioner for the problem on the polygon a block-diagonal matrix with blocks on the diagonal, corresponding to the preconditioners for the triangle problem. It is easy to see that such a preconditioner also guarantees the clustering of eigenvalues as well as enabling efficient construction and multiplication by a vector.

8. Conclusion

In this paper there was considered a problem of finding the spectral distribution of the matrix sequences arising in a discretization of differential problems defined on polygonal domains. It was shown that at least in case of a triangular domain (and more generally for non rectangular domains) the GLT-theory is insufficient for finding the spectral distributions, that is, the FEM discretization generates sequences, which do not belong to GLT class. On the other hand, it was proposed a generalization of GLT-sequences that enables to cope with wide range of PDE discretization problems defined on polygnal domains. Using the presented theory, an efficient preconditioning method for systems with truncated Toeplitz matrices was proposed and numerically justified.

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