Analyzing the parameter bias when an instrumental variable method is used with noise-corrupted data

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Abstract

When an output error model is fitted to data with noise-corrupted inputs using a prediction error method, a bias occurs. It was previously shown that the bias is of order $O(1/\delta)$ for a small pole-zero separation δ. These notes examine the same problem when an instrumental variable model is fitted. A similar result is shown to hold for the instrumental variable case.

1 Introduction

The aim of these notes is to analyze the asymptotic bias when an instrumental variable method is used, while the measured input data contains noise.

The model structure considered is

$$A(q^{-1})y(t) = B(q^{-1})u(t) + \varepsilon(t) \ , \tag{1}$$

and the parameter vector to be estimated is

$$\theta = \begin{pmatrix} a_1 & \dots & a_{n_a} & b_1 & \dots & b_{n_b} \end{pmatrix}^T \ . \tag{2}$$

Let θ_0 denote the true value of the parameter vector.

The data contains a noise-free part, and an additional measurement noise, as

$$u(t) = u_0(t) + \tilde{u}(t) \ , \tag{3}$$

$$y(t) = y_0(t) + \tilde{y}(t) \ . \tag{4}$$

Introduce also the following notations for the regressor vector:

$$\begin{aligned}
\varphi(t) &= \begin{pmatrix} -y(t-1) & \dots & -y(t-n_a) & u(t-1) & \dots & u(t-n_b) \end{pmatrix}^T && (5)\\
&= \varphi_0^T(t) + \tilde{\varphi}^T(t) && (6)\\
&= \varphi_0^T(t) + \tilde{\varphi}_y^T(t) + \tilde{\varphi}_u^T(t) \ , && (7)\\
\varphi_0(t) &= \begin{pmatrix} -y_0(t-1) & \dots & -y_0(t-n_a) & u_0(t-1) & \dots & u_0(t-n_b) \end{pmatrix}^T \ , && (8)\\
\tilde{\varphi}_y(t) &= \begin{pmatrix} -\tilde{y}(t-1) & \dots & -\tilde{y}(t-n_a) & 0 & \dots & 0 \end{pmatrix}^T \ , && (9)\\
\tilde{\varphi}_u(t) &= \begin{pmatrix} 0 & \dots & 0 & \tilde{u}(t-1) & \dots & \tilde{u}(t-n_b) \end{pmatrix}^T \ . && (10)
\end{aligned}$$

For the noise-free data it holds

$$y_0(t) = \varphi_0^T(t)\theta_0 \ , \tag{11}$$

corresponding to

$$A(q^{-1})y_0(t) = B(q^{-1})u_0(t) \ . \tag{12}$$

The parameter vector θ is assumed to be estimated using an instrumental variable method. The presence of input noise $\tilde{u}(t)$ will cause a bias in the parameter estimates. Here we examine this bias effect. The asymptotic case when the number of data points tends to infinity is considered. Special concern is given to the case of almost non-identifiable systems, where the polynomials A and B have almost a pole-zero cancellation.

2 Preliminaries

Before performing a bias analysis of applying an IV method to the model outlined in Section 1, we need some mathematical tools. They are formulated as lemmas below. Many of the results are well known, and most of those relating to properties of matrices can be found in [1].

In general terms let λ denote an eigenvalue, and σ a singular value of a matrix.

Lemma 1. Let $f_1(x) \geq 0$ and $f_2(x) \geq 0$ be two given functions. Then it holds

$$\min_x \left(f_1(x)f_2(x)\right) \geq \min_x f_1(x) \times \min_x f_2(x) \ . \tag{13}$$

Proof: Obvious and omitted.

Lemma 2. Let A be a symmetric matrix. Then it holds

$$\lambda_{\min}(A) = \min_x \frac{x^T A x}{x^T x} \ . \tag{14}$$

Proof: Omitted, the result is well known. The minimizing x in (14) is the eigenvector associated to the minimal eigenvalue.

Lemma 3. Let A be a symmetric matrix. Then it holds

$$\lambda_{\min}(A) = \frac{1}{\lambda_{\max}(A^{-1})} \ . \tag{15}$$

Proof: Omitted, the result is well known.

Lemma 4. Let A be an $m \times n$-dimensional matrix and B an $n \times m$-dimensional matrix. Then AB and BA have the same non-zero eigenvalues.

Proof: Let AB have an eigenvalue λ, associated with the eigenvector x. Thus $ABx = \lambda x$. Multiply from the left with B:

$$BABx = \lambda Bx \Rightarrow BA(Bx) = \lambda(Bx) \ , \tag{16}$$

showing that λ is also an eigenvalue of BA. If $m > n$, then AB is $m \times m$ and it has $m - n$ eigenvalues in zero, in addition to the eigenvalues in common with BA.

Lemma 5. Let A and B be symmetric and positive definite matrices of the same dimension. Then it holds

$$\lambda_{\min}(AB) \geq \lambda_{\min}(A)\lambda_{\min}(B) \ . \tag{17}$$

Proof: Diagonalize A. As it is positive definite, all eigenvalues are positive, and one can write

$$A = UD^2U^T \ , \tag{18}$$

where U is an orthogonal matrix and D is a diagonal matrix,

$$D = \begin{pmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{pmatrix} \ .$$

The eigenvalues of A are then precisely $d_1^2, \ldots, d_n^2$.

One can then write

$$
\begin{aligned}
\lambda_{\min}(AB) &= \lambda_{\min}(UD^2U^TB) = \lambda_{\min}(DU^TBUD) \\
&= \min_x \frac{x^TDU^TBUDx}{x^Tx} \qquad \left[\begin{array}{c} y = UDx \\ x = D^{-1}U^Ty \end{array}\right] \\
&= \min_x \frac{y^TBy}{y^Ty}\frac{y^Ty}{x^Tx} = \min_y \frac{y^TBy}{y^Ty}\frac{y^Ty}{y^TUD^{-2}U^Ty} \\
&\geq \min_y \frac{y^TBy}{y^Ty} \min_y \frac{y^Ty}{y^TUD^{-2}U^Ty} \\
&= \lambda_{\min}(B)\frac{1}{\max_y \frac{y^TUD^{-2}U^Ty}{y^Ty}} \\
&= \lambda_{\min}(B)\frac{1}{\lambda_{\max}(UD^{-2}U^T)} = \lambda_{\min}(B)\frac{1}{\lambda_{\max}(A^{-1})} \\
&= \lambda_{\min}(B)\lambda_{\min}(A) \ . \qquad (19)
\end{aligned}
$$

In the proof we used Lemma 4 in line 1, Lemma 2 in line 2, Lemma 1 in line 4, Lemma 2 in line 5 and Lemma 3 in line 7.

Lemma 6. Let the matrix

$$
\begin{pmatrix} A & B \\ B^T & C \end{pmatrix} \geq 0 \qquad (20)
$$

be symmetric and nonnegative definite. Then it holds

$$
\sigma^2_{\min}(B) \leq \lambda_{\max}(A)\lambda_{\min}(C) \ . \qquad (21)
$$

Proof. As the matrix in (20) is nonnegative definite, it holds

$$
C \geq B^TA^{-1}B \ ,
$$

and thus

$$
\begin{aligned}
\lambda_{\min}(C) &\geq \lambda_{\min}(B^TA^{-1}B) = \lambda_{\min}(BB^TA^{-1}) \\
&\geq \lambda_{\min}(BB^T)\lambda_{\min}(A^{-1}) \\
&= \sigma^2_{\min}(B)\frac{1}{\lambda_{\max}(A)} \ ,
\end{aligned}
$$

which completes the proof. In the proof we used Lemma 4 in line 1 and Lemma 5 in line 2.

Lemma 7. The two-norm of a matrix Q fulfils

$$\| Q \|_2 \stackrel{\Delta}{=} \sup_{x \neq 0} \frac{\| Qx \|_2}{\| x \|_2} = \sigma_{\max}(Q) \ . \tag{22}$$

Proof: Well-known fact, and omitted.

Lemma 8. If $\| Q \|_2 < 1$, then

$$(I - Q)^{-1} = \sum_{k=0}^{\infty} Q^k \ . \tag{23}$$

Proof: Well-known fact, and omitted.

Lemma 9. Let $A(\varepsilon) = A + \varepsilon B(\varepsilon) = A + \varepsilon B + O(\varepsilon^2)$ be a symmetric matrix. Assume A has a distinct eigenvalue λ, with an associated eigenvector x. Then $A(\varepsilon)$ has an eigenvalue in

$$\lambda(\varepsilon) = \lambda + \varepsilon \frac{x^T B x}{x^T x} + O(\varepsilon^2) \ . \tag{24}$$

Proof. See [4] or [5].

3 Analysing the instrumental variable estimate

3.1 The basic IV model

The model (1) can be written as

$$y(t) = \varphi^T(t)\theta + \varepsilon(t) \ . \tag{25}$$

A general instrumental variable estimate of θ in its basic version takes the form

$$\frac{1}{N} \sum_{t=1}^{N} z(t)\varphi^T(t)\hat{\theta} = \frac{1}{N} \sum_{t=1}^{N} z(t)y(t) \ . \tag{26}$$

The vector $z(t)$ of instruments can be chosen in different ways. Write (26) for short as

$$\hat{R}_{z\varphi}\hat{\theta} = \hat{r}_{zy} \ . \tag{27}$$

3.2 The extended IV model

In case an overdetermined system of equations is used in the IV estimate (meaning that the dimension of $z(t)$ is higher than $n_a + n_b$), the general form for the IV estimate is

$$\hat{\theta} = \left(\hat{R}_{z\varphi}^T W \hat{R}_{z\varphi}\right)^{-1} \hat{R}_{z\varphi}^T W \hat{r}_{zy} \ , \tag{28}$$

where W is a positive definite weighting matrix.

3.3 General assumptions

Several assumptions have to be applied in the analysis:

- The unperturbed input signal $u_0(t)$ is persistenly exciting, at least of order $n_a + n_b$. This assumption is needed to guarantee identifiability.
- The polynomials A and B are coprime. This assumption is also needed to guarantee identifiability.
- The three signals $u_0(t)$, $\tilde{u}(t)$ and $\tilde{y}(t)$ are uncorrelated. This is a convenient and mild assumption.
- The elements of the instrumental variable vector $z(t)$ is uncorrelated with the output noise $\tilde{y}(t)$. A natural way to achieve this is to let the elements of $z(t)$ consists of delayed and/or filtered values of the input.
- The asymptotic case with an infinite amount of data is considered, that is $N \to \infty$. Then due to ergodicity one can replace normalized sums by expectations. More specifically,

$$\hat{R}_{z\varphi} \to R_{z\varphi} = E\left\{z(t)\varphi^T(t)\right\} , \quad \hat{r}_{zy} \to r_{zy} = E\left\{z(t)y(t)\right\} \ . \tag{29}$$

4 The asymptotic bias

Let $\hat{\theta}$ be the solution to (26), and derive an equation for the bias $\tilde{\theta} = \hat{\theta} - \theta_0$.

$$\begin{aligned} E\left\{z(t)\varphi^T(t)\right\}\tilde{\theta} &= E\left\{z(t)\varphi^T(t)\right\}\hat{\theta} - E\left\{z(t)\varphi^T(t)\right\}\theta_0 \\ &= E\left\{z(t)\left[y(t) - \varphi^T(t)\theta_0\right]\right\} \\ &= E\left\{z(t)\left[y_0(t) + \tilde{y}(t) - \varphi_0^T(t)\theta_0 - \tilde{\varphi}_y^T(t)\theta_0 - \tilde{\varphi}_u^T(t)\theta_0\right]\right\} \\ &= -E\left\{z(t)\tilde{\varphi}_u^T(t)\theta_0\right\} \ . \end{aligned} \tag{30}$$

Here we used (7), (11) and the fact that $z(t)$ and $\tilde{y}(t)$ are uncorrelated processes.

One can conclude so far that the right hand side of (30) is proportional to λ_u^2, and thus that the bias for small values of the input noise variance also is proportional to λ_u^2. This can be expressed as

$$\tilde{\theta} = O(\lambda_u^2) \ , \tag{31}$$

which is a result of the same sort as when a prediction error method is used, cf. [2], [3].

When $z(t)$ has the same dimension as θ, the bias reads (in the asymptotic case)

$$\tilde{\theta} = -R_{z\varphi}^{-1} r_{z\tilde{\varphi}_u\theta_0} \ . \tag{32}$$

The size of the bias is therefore inversely proportional to the smallest singular value of $R_{z\varphi}$.

In case an overdetermined system of equations are used in the IV estimate (meaning that the dimension of $z(t)$ is higher than $n_a + n_b$), the analysis has to be modified a bit. Instead of (32) the bias will now be written as

$$\tilde{\theta} = -\left(R_{z\varphi}^T W R_{z\varphi}\right)^{-1} R_{z\varphi}^T W r_{z\tilde{\varphi}_u\theta_0} \ . \tag{33}$$

Note that it is not restrictive to set $W = I$ in this analysis (as the vector $z(t)$ of instruments is not specified). By making the transformation $z(t) \to W^{1/2} z(t)$ in (28) the estimate takes the form with $W = I$. Then note from (33) that the bias takes the form, cf. (32),

$$\tilde{\theta} = -\left(R_{z\varphi}^T R_{z\varphi}\right)^{-1} R_{z\varphi}^T r_{z\tilde{\varphi}_u\theta_0} = -R_{z\varphi}^{\dagger} r_{z\tilde{\varphi}_u\theta_0} \ . \tag{34}$$

where $R_{z\varphi}^{\dagger}$ is the pseudoinverse of $R_{z\varphi}$. Apparently the size of $R_{z\varphi}^{\dagger}$ is indeed of importance for the size of the bias $\tilde{\theta}$. It holds

$$R_{z\varphi} = R_{z\varphi_0} + R_{z\tilde{\varphi}_u} \ . \tag{35}$$

One may think that the bias gets larger when the second term in (35) is neglected. (Including this term seems like increasing the denominator in a ratio.) This specific aspect will be further investigated later in this report, see Sections 5 and 6.

Indeed, the term $R_{z\tilde{\varphi}_u}$ in (35) is proportional to λ_u^2. When λ_u^2 is small, the bias $\tilde{\theta}$, see (34), may therefore be approximated by $\tilde{\theta}_{\text{app}}$ as

$$\tilde{\theta} = \underbrace{R_{z\varphi_0}^{\dagger} r_{z\tilde{\varphi}_u\theta_0}}_{\tilde{\theta}_{\text{app}}} + O(\lambda_u^4) \tag{36}$$

In Section 5 we examine $\tilde{\theta}_{\text{app}}$ for a small pole-zero separation δ, and show that $\tilde{\theta}_{\text{app}}$ is $O(1/\delta)$.

The analysis in Section 6 deals with the full bias term $\tilde{\theta}$, and it will be shown that $\tilde{\theta}$ *is not* $O(1/\delta)$.

5 Bias for the case of small pole-zero separation

Based on the analysis of PEM, see [2], [3], one can expect that the case of almost pole-zero separation is of particular concern.

Assume that there is a minimal pole-zero separation equal to δ in the system. When $\delta \to 0$, the system thus become unidentifiable. Further, in that case the covariance matrix R_{φ_0} will become singular as found in the analysis of PEM, see [2], [3]. More specifically, its smallest eigenvalue is of order $O(\delta^2)$.

Then we can write

$$\begin{aligned} A(q^{-1}) &= \overline{A}(q^{-1})(1 + aq^{-1}) \ , && (37) \\ B(q^{-1}) &= \overline{B}(q^{-1})(1 + aq^{-1} + \delta q^{-1}) \ , && (38) \\ \overline{A}(q^{-1}) &= 1 + \overline{a}_1 q^{-1} + \ldots \overline{a}_{na-1} q^{-na+1} \ , && (39) \\ \overline{B}(q^{-1}) &= \overline{b}_1 q^{-1} + \ldots \overline{b}_{nb-1} q^{-nb+1} \ . && (40) \end{aligned}$$

The polynomials $\overline{A}$ and $\overline{B}$ are assumed to be coprime.

Further, introduce the vector

$$\overline{\theta} = \begin{pmatrix} 1 & \overline{a}_1 & \ldots & \overline{a}_{na-1} & 0 & \overline{b}_1 & \ldots & \overline{b}_{nb-1} \end{pmatrix}^T \ . \tag{41}$$

Then it holds

$$\begin{aligned} \varphi_0(t)\overline{\theta} &= -y_0(t-1) - \overline{a}_1 y_0(t-2) - \cdots - \overline{a}_{na-1} y_0(t-n_a) \\ &\quad + \overline{b}_1 u_0(t-2) + \cdots + \overline{b}_{nb-1} u_0(t-n_b) \\ &= -\overline{A}(q^{-1}) y_0(t-1) + \overline{B}(q^{-1}) u_0(t-1) \\ &= -\overline{A}(q^{-1}) \frac{B(q^{-1})}{A(q^{-1})} u_0(t-1) + \overline{B}(q^{-1}) u_0(t-1) \\ &= -\frac{\overline{A}(q^{-1}) B(q^{-1}) - A(q^{-1}) \overline{B}(q^{-1})}{A(q^{-1})} u_0(t-1) \\ &= \frac{\overline{A}(q^{-1}) \overline{B}(q^{-1}) \delta q^{-1}}{A(q^{-1})} u_0(t-1) \\ &= O(\delta) \ . \end{aligned} \tag{42}$$

Therefore,

$$R_{z\varphi_0}\overline{\theta} = O(\delta) \ . \tag{43}$$

and the first column of $R_{z\varphi_0}$ is a linear combination of columns $2, \ldots, n_a + n_b$ plus a term that is $O(\delta)$. This in turn leads to

$$\det\left(R_{z\varphi_0}\right) = O(\delta) \ , \tag{44}$$

From Lemma 6 it follows

$$\sigma^2_{\min}\left(R_{z\varphi_0}\right) \leq \lambda_{\max}(R_z)\lambda_{\min}(R_{\varphi_0}) \ . \tag{45}$$

Recall from [2], [3] that

$$R_{\varphi_0} = \mathcal{S}^T(A, -B) R_u \mathcal{S}(A, -B) \ , \tag{46}$$

where

$$\mathcal{S}(A, -B) = \begin{pmatrix} 0 & & 0 & 1 & 0 & 0 \\ -b_1 & & & a_1 & & \\ \vdots & \ddots & & \vdots & \ddots & 0 \\ -b_{n_b} & & & a_{n_a} & & \\ 0 & \ddots & & 0 & \ddots & \\ 0 & 0 & -b_{n_b} & 0 & 0 & a_{n_a} \end{pmatrix} \ . \tag{47}$$

The Sylvester matrix $\mathcal{S}(A, -B)$ is known to be almost singular, to have one small eigenvalue, and a determinant that is $O(\delta)$, see [2], [3]. One can therefore conclude that the smallest eigenvalue of $\mathcal{S}(A, -B)$ is $O(\delta)$, and then from (46) that the smallest eigenvalue of R_{φ_0} is $O(\delta^2)$. Inserting these findings in (66) may lead to

$$\| \ \tilde{\theta}_{\text{app}} \ \| = O(1/\delta) \ . \tag{48}$$

However, (45) implies $\sigma_{\min}(R_{z\varphi_0}) \leq O(\delta)$ rather than $\sigma_{\min}(R_{z\varphi_0}) = O(\delta)$.

To proceed one need to find a more accurate expression for $\sigma_{\min}(R_{z\varphi_0})$. Then proceed as follows.

Rewrite the regressor vector $\varphi_0(t)$:

$$\begin{aligned} \varphi_0^T(t) &= \begin{pmatrix} -y_0(t-1) & \ldots & -y_0(t-n_a) & u_0(t-1) & \ldots & u_0(t-n_b) \end{pmatrix} \\ &= \underbrace{\begin{pmatrix} w(t-1) & \ldots & w(t-n_a-n_b) \end{pmatrix}}_{\psi^T(t)} \mathcal{S}(A, -B) \ , \end{aligned} \tag{49}$$

$$w(t) = \frac{1}{A} u_0(t) \ . \tag{50}$$

Thus

$$
\begin{aligned}
R_{z\varphi_0} &= R_{z\psi}\mathcal{S}(A,-B) \ , && (51)\\
\sigma^2_{\min}(R_{z\varphi_0}) &= \lambda_{\min}(R_{\varphi_0 z}R_{z\varphi_0}) \\
&= \lambda_{\min}\left(\mathcal{S}^T(A,-B)R_{\psi z}R_{z\psi}\mathcal{S}(A,-B)\right) \ . && (52)
\end{aligned}
$$

Next set

$$
\begin{aligned}
F(q^{-1}) &\triangleq -\overline{A}B + A\overline{B} && (53)\\
&= f_1 q^{-1} + \cdots + f_{n_a+n_b} q^{-n_a-n_b} \\
&= \overline{AB}\left(-1 - aq^{-1} - \delta q^{-1} + 1 + aq^{-1}\right) = -\delta q^{-1}\overline{AB} = O(\delta) \ , && (54)\\
\mathbf{f} &= \begin{pmatrix} f_1 & \dots & f_{n_a+n_b} \end{pmatrix}^T \ , && (55)
\end{aligned}
$$

and note that (53) can be written in matrix form as

$$
\mathcal{S}(A,-B)\overline{\theta} = \mathbf{f} \ . \tag{56}
$$

Next set

$$
S(\delta) = \mathcal{S}(A,-B) \ , \tag{57}
$$

and note that $S(\delta)$ is an affine function of δ. This means precisely that it has one constant term and one linear term, so it can be written as

$$
S(\delta) = S_0 + \delta S_1 \ . \tag{58}
$$

Note further that

$$
S_0\overline{\theta} = 0 \ , \tag{59}
$$

as $F = O(\delta)$. Next set

$$
\begin{aligned}
R_\psi &= R_{\psi z}R_{z\psi} \ , && (60)\\
R(\delta) &= S^T(\delta)R_\psi S(\delta) \ . && (61)
\end{aligned}
$$

Note that $R(0)$ has a distinct eigenvalue $\lambda = 0$ with the associated eigenvector being $\overline{\theta}$, cf. (59).

Using now (52), (61) we seek

$$
\begin{aligned}
\sigma^2_{\min}(R_{z\varphi_0}) &= \lambda_{\min}(R(\delta)) \ , && (62)\\
R(\delta) &= \left(S_0^T + \delta S_1^T\right) R_\psi \left(S_0 + \delta S_1\right) \\
&= S_0^T R_\psi S_0 + \delta\left(S_0^T R_\psi S_1 + S_1^T R_\psi S_0 + \delta S_1^T R_\psi S_1\right) \ . && (63)
\end{aligned}
$$

Using Lemma 9 finally gives

$$\begin{aligned}\lambda_{\min}(R(\delta)) &= 0+\delta\frac{\overline{\theta}^T\left(S_0^TR_\psi S_1+S_1^TR_\psi S_0+\delta S_1^TR_\psi S_1\right)\overline{\theta}}{\overline{\theta}^T\overline{\theta}} \\ &= \delta^2\frac{\overline{\theta}^TS_1^TR_\psi S_1\overline{\theta}}{\overline{\theta}^T\overline{\theta}}=O(\delta^2)\ . \end{aligned} \tag{64}$$

This shows therefore that

$$\sigma_{\min}\left(R_{z\varphi_0}\right)=\sqrt{\lambda_{\min}(R(\delta))}=O(\delta)\ . \tag{65}$$

We then have for small values of δ

$$\begin{aligned}\|\ \tilde{\theta}_{\text{app}}\ \| &\le \|\ R_{z\varphi_0}^{\dagger}\ \|\|\ r_{z\tilde{\varphi}_{0u}\theta_0}\ \| \\ &= \sigma_{\max}\left(R_{z\varphi_0}^{\dagger}\right)\times O(1)=\frac{1}{\sigma_{\min}\left(R_{z\varphi_0}\right)}\times O(1)\ . \\ &= O(1/\delta)\ . \end{aligned} \tag{66}$$

6 The bias in the general case

It was shown in Section 5 that $\tilde{\theta}_{\text{app}}$ is of order $O(1/\delta)$. What about the full bias $\tilde{\theta}$ for a general input noise variance? Following (32) it is important to then consider the matrix $R_{z\varphi}$ (instead of the 'noise-free part' $R_{z\varphi_0}$) and in particular its determinant. Here we have the following result.

Lemma. *It does not hold that*

$$\det(R_{z\varphi})=O(\delta)\ . \tag{67}$$

Proof. The proof is by contradiction. If (67) was to hold, then $\delta=0$ would imply $\det(R_{z\varphi})=0$.

As a counterexample to (67), consider the case $\delta=0$, $n_a=1, n_b=2$. Note that then $y_0(t)=b_1u_0(t-1)$. Set

$$r_k=E\left\{u_0(t)u_0(t-k)\right\},\ k\ge 0\ .$$

Let the instrumental variable vector be

$$z(t)=\begin{pmatrix} u(t-1) & u(t-2) & u(t-3)\end{pmatrix}^T\ ,$$

Under these assumptions the matrix $R_{z\varphi}$ becomes

$$R_{z\varphi}=\begin{pmatrix} -b_1r_1 & r_0+\lambda_u^2 & r_1 \\ -b_1r_0 & r_1 & r_0+\lambda_u^2 \\ -b_1r_1 & r_2 & r_1\end{pmatrix}\ ,$$

and one finds that its determinant can be written as

$$\begin{aligned}
\det(R_{a\varphi}) &= -b_1 r_1^3 - b_1 r_1 (r_0 + \lambda_u^2)^2 - b_1 r_0 r_1 r_2 \\
&\quad + b_1 r_1^3 + b_1 r_0 r_1 (r_0 + \lambda_u^2) + b_1 r_1 r_2 (r_0 + \lambda_u^2) \\
&= b_1 r_1 \left[(r_0 + r_2)(r_0 + \lambda_u^2) - r_0 r_2 - (r_0 + \lambda_u^2)^2 \right] \\
&= b_1 r_1 \lambda_u^2 \left[-r_0 + r_2 - \lambda_u^2 \right] \ .
\end{aligned}$$

which in general is nonzero. This completes the proof of contradiction.

Apparently, the bias $\tilde{\theta}$ and its approximation $\tilde{\theta}_{\text{app}}$ do only partly behave in similar ways. The previous analysis has established the following:

- For small values of the input noise variance λ_u^2 it holds

$$\tilde{\theta} = O(\lambda_u^2), \quad \tilde{\theta}_{\text{app}} = O(\lambda_u^2) \tag{68}$$

$$\tilde{\theta} = \tilde{\theta}_{\text{app}} + O(\lambda_u^4) \tag{69}$$

- For small values of a pole-zero separation δ it holds

$$\tilde{\theta}_{\text{app}} = O(1/\delta) \tag{70}$$

$$\tilde{\theta} = O(1) \text{ in general} \tag{71}$$

Another way of expressing these finding is

$$\lim_{\lambda_u^2 \to 0} \lim_{\delta \to 0} \frac{1}{\lambda_u^2} \tilde{\theta} = O(1) \tag{72}$$

$$\begin{aligned}
\lim_{\delta \to 0} \lim_{\lambda_u^2 \to 0} \frac{1}{\lambda_u^2} \tilde{\theta} &= \lim_{\delta \to 0} \lim_{\lambda_u^2 \to 0} \frac{1}{\lambda_u^2} \tilde{\theta}_{\text{app}} \\
&= \lim_{\delta \to 0} O(1/\delta) = \infty
\end{aligned} \tag{73}$$

7 Comparing the bias term for the IV case and the output error case

The bias terms both in the IV case for small values of the input noise, that is $\tilde{\theta}_{\text{app}}$, and the general bias term in the output error model case using a prediction error method are of the order $O(1/\delta)$. Can these two bias expressions be further compared?

Consider the IV estimate in its basic form, where the number of instruments is equal to the number of parameters, $n_a + n_b$.

It holds, cf. (51),

$$\begin{aligned}
\tilde{\theta}_{\text{app}} &= -R_{z\varphi_0}^{-1} r_{z\tilde{\varphi}_u\theta_0} \\
&= -\left[R_{z\psi}\mathcal{S}(A,-B)\right]^{-1} E\left\{z(t)B(q^{-1})\tilde{u}(t)\right\} \\
&= -\mathcal{S}^{-1}(A,-B)R_{z\psi}^{-1} r_{\text{IV}} \ , \qquad (74) \\
r_{\text{IV}} &= E\left\{z(t)B(q^{-1})\tilde{u}(t)\right\} \ . \qquad (75)
\end{aligned}$$

Similar expressions for the bias in the output error model case were derived in [2], [3]:

$$\begin{aligned}
\tilde{\theta}_{\text{OE}} &= -\left[\mathcal{S}^T(-A,B)P_{\varphi_u}\mathcal{S}(-A,B)\right]^{-1}\mathcal{S}^T(-A,B)r_{\text{OE}} \\
&= \mathcal{S}^{-1}(A,-B)P_{\varphi_u}^{-1} r_{\text{OE}} \ , \qquad (76) \\
r_{\text{OE}} &= E\left\{\frac{B}{A}\tilde{u}(t)\varphi_{\tilde{u}}\right\} \ . \qquad (77)
\end{aligned}$$

Note that both $\tilde{\theta}_{\text{app}}$ and $\tilde{\theta}_{\text{OE}}$ are of order $O(1/\delta)$. This is due to the fact that both expressions (74) and (76) start with the factor $\mathcal{S}^{-1}(A,-B)$, and that the determinant of $\mathcal{S}(A,-B)$ is known to be of order $O(\delta)$.

It can also be noted that both the terms r_{IV} in (75) and r_{OE} in (77) depend linearly on the input noise variance λ_u^2.

A conjectory is that for any given system and noise levels, it is possible to find a vector of instruments $z(t)$, so that the bias terms $\tilde{\theta}_{\text{app}}$ and $\tilde{\theta}_{OE}$ are the same. Note in this context that the IV and OE estimates are constructed using quite different principles. While the OE estimate is unique with no user variables/choices, for the IV estimate the vector $z(t)$ of instruments has not been specified, and it can be viewed as a set of user choices.

Example. Define

$$\mathbf{h} = \begin{pmatrix} h_1 & \dots & h_{n_a+n_b} \end{pmatrix}^T = R_{z\psi}^{-1} r_{\text{IV}} \ . \qquad (78)$$

Then one can see that $\tilde{\theta}_{\text{app}} = \tilde{\theta}_{\text{OE}}$ is equivalent to

$$E\left\{z(t)\psi^T(t)\mathbf{h}\right\} = E\left\{z(t)B(q^{-1})\tilde{u}(t)\right\} \ . \qquad (79)$$

Here $A, B, H, u_0, \tilde{u}$ are to be regarded as given quantities, while the instrumental vector z is to be found to fulfil (79). Note that

$$\psi^T(t)\mathbf{h} = \frac{H}{A}u_0(t) \ , \qquad (80)$$

where $H(q^{-1}) = h_1 q^{-1} + \cdots + h_{n_a+n_b} q^{-n_a-n_b}$. Now set, for an arbitrary element of $z(t)$,

$$z(t) = F(q^{-1})u(t) \ . \tag{81}$$

Then (79) is equivalent to

$$E\left\{ F u_0(t) \frac{H}{A} u_0(t) \right\} = E\left\{ F\tilde{u}(t) B\tilde{u}(t) \right\} \ , \tag{82}$$

where F is the unknown quantity to be determined. One (out of many!) possibility is to choose, for element $k = 1, \ldots, n_a + n_b$ of $z(t)$,

$$F(q^{-1}) = q^{-k}(1 + f_k q^{-1}) \ . \tag{83}$$

The choice to let the delay q^{-k} in (83) vary with k will ensure that R_z becomes positive definite, which is a necessary condition for $R_{z\varphi_0}$ to be of full rank. Equation (82) with the filter F given by (83) now contains only the single scalar unknown f_k. The equation can be rewritten as

$$\left[E\left\{ u_0(t-k-1) \frac{H}{A} u_0(t) - \tilde{u}(t-k-1) B\tilde{u}(t) \right\} \right] f_k = \tag{84}$$

$$- \left[E\left\{ u_0(t-k) \frac{H}{A} u_0(t) - \tilde{u}(t-k) B\tilde{u}(t) \right\} \right] \ , \tag{85}$$

which completes the construction.

8 Conclusions

When an output error model is used with the prediction error method, for a model with a small pole-zero separation δ, the obtained parameter bias was shown in [2] to be of order $O(1/\delta)$.

When an instrumental variable estimate is used and the input noise variance λ_u^2 is very small, the bias is also of order $O(1/\delta)$. When λ_u^2 is moderate, this result no longer applies. Thus the bias term of an IV estimate is often smaller than that of a prediction error method used with an output error model.

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