A Stable and Conservative High Order Multi-block Method for the Compressible Navier-Stokes Equations

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Abstract

A stable and conservative high order multi-block method for the time-dependent compressible Navier-Stokes equations has been developed. Stability and conservation are proved using summation-by-parts operators, weak interface conditions and the energy method. This development makes it possible to exploit the efficiency of the high order finite difference method for non-trivial geometries. The computational results corroborate the theoretical analysis.

Key words: Navier-Stokes, finite difference, high order, stability, conservation

1 Introduction

The high order finite difference method in combination with summation-by-parts operators and weak boundary conditions can very efficiently and re-

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¹ This work was done while the first two authors were visiting CTR, The Center for Turbulence Research at Stanford University.

liably handle large problems on structured grids for reasonably smooth geometries. This has been shown in a sequence of papers, see for example [8,17,11,12,14,19,21]. The most recent papers ([19],[21]) on this subject discuss the specific problem with farfield and no-slip boundaries.

In this paper we conclude the development by treating the similar but not identical problem with a stable and accurate coupling of blocks. The technology in the two papers [19],[21] together with the interface treatment in this paper will conclude the development of a high order accurate and truly stable multi-block finite difference method for the Navier-Stokes equations.

In the next phase of this development we will use the coupling technique developed in this paper and combine the high order finite difference method with the finite volume method in combination with unstructured grids which can more readily handle complex geometries. That development is ongoing, see for example [13],[3] and [15].

The development in this paper is the theoretical foundation for that work. The main challenge for multi-block methods is to manage the possible instability at the interfaces or block boundaries between sub-domains. We will focus on the coupling procedure at the interface and for *the first time* prove stability and conservation of a high order accurate multi-block finite difference method applied to the Navier-Stokes equations.

The rest of the paper is organized as follows. In the next section we present the symmetric constant coefficient form of the Navier-Stokes equations followed by a short discussion of well-posedness in section 3. The formulation of the numerical method on a single domain is considered in section 4. The coupling procedure is the topic of section 5 and the numerical experiments are presented in section 6. Finally, conclusions are drawn in section 7.

2 The Navier-Stokes equations

The frozen coefficient time-dependent compressible Navier-Stokes equations in two-dimensions in non-conservative form are given by, see [1]

$$\tilde{u}_t + A\tilde{u}_x + B\tilde{u}_y = C\tilde{u}_{xx} + D\tilde{u}_{xy} + E\tilde{u}_{xy}, \quad (1)$$

where $\tilde{u} = [\tilde{\rho}, \tilde{u}_1, \tilde{u}_2, \tilde{p}]^T$ and A, B, C, D , and E are coefficient matrices. $\tilde{\rho}$ is the density, $\tilde{u}_1$ and $\tilde{u}_2$ are the velocities and $\tilde{p}$ is the pressure. To apply the energy method we must symmetrize (1). Following the technique of symmetrization developed in [1] and [16], a symmetric form of (1) is,

$$u_t + (A_1 u)_x + (A_2 u)_y = \varepsilon \left[(B_{11} u_x + B_{12} u_y)_x + (B_{21} u_x + B_{22} u_y)_y \right], \quad (2)$$

with $\varepsilon = 1/Re$, $u = (c\tilde{\rho}/(\sqrt{\gamma}\rho), \tilde{u}_1, \tilde{u}_2, \rho\tilde{T}/\sqrt{\gamma(\gamma-1)})^T$ and

$$A_1 = \begin{bmatrix} u_1 & \frac{c}{\sqrt{\gamma}} & 0 & 0 \\ \frac{c}{\sqrt{\gamma}} & u_1 & 0 & \sqrt{\frac{\gamma-1}{\gamma}}c \\ 0 & 0 & u_1 & 0 \\ 0 & \sqrt{\frac{\gamma-1}{\gamma}}c & 0 & u_1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} u_2 & 0 & \frac{c}{\sqrt{\gamma}} & 0 \\ 0 & u_2 & 0 & 0 \\ \frac{c}{\sqrt{\gamma}} & 0 & u_2 & \sqrt{\frac{\gamma-1}{\gamma}}c \\ 0 & 0 & \sqrt{\frac{\gamma-1}{\gamma}}c & u_2 \end{bmatrix},$$

$$B_{11} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{\lambda+2\mu}{\rho} & 0 & 0 \\ 0 & 0 & \frac{\mu}{\rho} & 0 \\ 0 & 0 & 0 & \frac{\gamma\mu}{Pr\rho} \end{bmatrix}, \quad B_{12} = B_{21} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\lambda+\mu}{2\rho} & 0 \\ 0 & \frac{\lambda+\mu}{2\rho} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad B_{22} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{\mu}{\rho} & 0 & 0 \\ 0 & 0 & \frac{\lambda+2\mu}{\rho} & 0 \\ 0 & 0 & 0 & \frac{\gamma\mu}{Pr\rho} \end{bmatrix}.$$

In the vectors and matrices above we have used the temperature $\tilde{T}$, the ratio of the specific heats $\gamma = c_p/c_v$, the speed of sound c , the dynamic viscosity μ , the bulk viscosity λ , the kinematic viscosity $\nu = \mu/\rho$, the Prandtl number $Pr = \nu/\alpha$ (α is the thermal diffusivity) and the Reynolds number $Re = \rho_\infty U_\infty L/\mu_\infty$. The infinity subscript denotes free stream conditions and L is a characteristic length. Note again that the form of the matrices (Jacobians) above are obtained for the symmetrized frozen coefficient version of the Navier-Stokes equations.

Equation (2) can be rewritten in conservative form as

$$u_t + F_x + G_y = 0, \quad (3)$$

where

$$\begin{aligned} F &= A_1 u - \varepsilon(B_{11}u_x + B_{12}u_y) = F^I - \varepsilon F^V, \\ G &= A_2 u - \varepsilon(B_{21}u_x + B_{22}u_y) = G^I - \varepsilon G^V. \end{aligned} \quad (4)$$

F^I and G^I contain the inviscid terms and F^V and G^V the viscous terms.

3 Well-posedness of the continuous problem

To keep the algebraic complexity of the analysis as low as possible, we consider rectangular domains with cartesian coordinates. Applying the energy method

to (3) on the domain $\Omega \in [-1, 1] \times [0, 1]$ we obtain

$$\iint_{\Omega} u^T u_t dx dy + \iint_{\Omega} u^T F_x dx dy + \iint_{\Omega} u^T G_y dx dy = 0. \quad (5)$$

By using the Green-Gauss theorem, equation (5) can be written as

$$\begin{aligned} \frac{d}{dt}(\|u\|^2) = & - \underbrace{\int_0^1 u^T (F^I - 2\varepsilon F^V)|_{x=1} dy}_{\text{East}} - \underbrace{\int_1^0 u^T (F^I - 2\varepsilon F^V)|_{x=-1} dy}_{\text{West}} \\ & - \underbrace{\int_{-1}^1 u^T (G^I - 2\varepsilon G^V)|_{y=0} dx}_{\text{South}} - \underbrace{\int_1^{-1} u^T (G^I - 2\varepsilon G^V)|_{y=1} dx}_{\text{North}} \\ & - 2\varepsilon \iint_{\Omega} \begin{bmatrix} u_x \\ u_y \end{bmatrix}^T \underbrace{\begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}}_B \begin{bmatrix} u_x \\ u_y \end{bmatrix} dx dy. \end{aligned} \quad (6)$$

To have a bounded energy growth, the boundary terms (East, West, North and South) must be bounded using the correct number and form of boundary conditions. That is the topic in papers [19],[21] and it is not discussed further here. The contribution from the integral term is negative semi-definite since the matrix $B = [B_{11} \ B_{12}; B_{21} \ B_{22}]$ is positive semi-definite.

We summarize the result for the continuous problem (2)-(4) in the following proposition.

Proposition 3.1 *The continuous problem (2)-(4) is well posed if i) the correct number and form of the boundary conditions are used to bound the boundary terms and ii) the matrix $B = [B_{11} \ B_{12}; B_{21} \ B_{22}]$ is positive semi-definite.*

Remark The n -dimensional Navier-Stokes equations require n boundary conditions at an inflow boundary and $n - 1$ at an outflow boundary. In this case (two-dimensions) we need four boundary conditions an inflow boundary and three at an outflow boundary, see for example [18],[6],[16].

4 Stability on a single domain

Consider the computational domain with a Cartesian mesh of $(M+1) \times (N+1)$ points. Let the k -th element of the continuous variable u at the structured grid point (x_i, y_j) be $u(i, j, k)$ ($0 \leq k \leq 3$). The finite difference approximation of $u(i, j, k)$ is collected in a global vector $\mathbf{v}$ such that $v[4i(N+1) + 4j + k] = u(i, j, k)$ ($0 \leq i \leq M, 0 \leq j \leq N$ and $0 \leq k \leq 3$). Let $\mathbf{v}_x$ and $\mathbf{v}_y$ be

approximations of u_x and u_y . By using the finite difference method developed in [8,17,11,12,14,19,21] a semi-discrete approximation of equation (3) can be written as

$$\mathbf{v}_t + D_x \mathbf{F} + D_y \mathbf{G} = 0, \quad (7)$$

where $D_x = P_x^{-1} Q_x \otimes I_y \otimes I_4$ and $D_y = I_x \otimes P_y^{-1} Q_y \otimes I_4$ are first derivative operators in x -, and y - directions, respectively. I_x and I_y are the identity matrices of size $(M+1) \times (M+1)$ and $(N+1) \times (N+1)$. Moreover,

$$\mathbf{F} = \mathbf{F}^I - \varepsilon \mathbf{F}^V, \quad \mathbf{G} = \mathbf{G}^I - \varepsilon \mathbf{G}^V,$$

with

$$\begin{aligned} \mathbf{F}^I &= (I_x \otimes I_y \otimes A_1) \mathbf{v}, & \mathbf{F}^V &= (I_x \otimes I_y \otimes B_{11}) \mathbf{v}_x + (I_x \otimes I_y \otimes B_{12}) \mathbf{v}_y, \\ \mathbf{G}^I &= (I_x \otimes I_y \otimes A_2) \mathbf{v}, & \mathbf{G}^V &= (I_x \otimes I_y \otimes B_{21}) \mathbf{v}_x + (I_x \otimes I_y \otimes B_{22}) \mathbf{v}_y, \end{aligned}$$

and $\mathbf{v}_x = D_x \mathbf{v}$, $\mathbf{v}_y = D_y \mathbf{v}$. Let $\bar{P} = P_x \otimes P_y$ and multiply equation (7) with $\mathbf{v}^T (\bar{P} \otimes I_4)$. (This is the discrete equivalent of multiplying (3) with v^T and integrating over the computational domain to get the energy estimate (6)). This leads to

$$\mathbf{v}^T (\bar{P} \otimes I_4) \mathbf{v}_t + \mathbf{v}^T (Q_x \otimes P_y \otimes I_4) \mathbf{F} + \mathbf{v}^T (P_x \otimes Q_y \otimes I_4) \mathbf{G} = 0 \quad (8)$$

By adding the transpose of equation (8) to itself and using the SBP relations

$$Q_x + Q_x^T = \text{diag}(-1, 0, \dots, 0, 1), \quad Q_y + Q_y^T = \text{diag}(-1, 0, \dots, 0, 1) \quad (9)$$

we can write the result as

$$\frac{d}{dt} (\|\mathbf{v}\|_{\bar{P} \otimes I_4}^2) = -\text{IT} + \varepsilon \text{VT}. \quad (10)$$

The inviscid term IT in (10) is

$$\begin{aligned} \text{IT} &= \mathbf{v}^T (Q_x \otimes P_y \otimes I_4) \mathbf{F}^I + (\mathbf{F}^I)^T (Q_x^T \otimes P_y \otimes I_4) \mathbf{v} + \\ &\quad \mathbf{v}^T (P_x \otimes Q_y \otimes I_4) \mathbf{G}^I + (\mathbf{G}^I)^T (P_x \otimes Q_y^T \otimes I_4) \mathbf{v} \\ &= \underbrace{\mathbf{v}_E^T (P_y \otimes I_4) \mathbf{F}_E^I}_{\text{East}} - \underbrace{\mathbf{v}_W^T (P_y \otimes I_4) \mathbf{F}_W^I}_{\text{West}} - \underbrace{\mathbf{v}_S^T (P_x \otimes I_4) \mathbf{G}_S^I}_{\text{South}} + \underbrace{\mathbf{v}_N^T (P_x \otimes I_4) \mathbf{G}_N^I}_{\text{North}} \end{aligned} \quad (11)$$

The viscous term VT in (10) can be written as

$$\begin{aligned}
\text{VT} = & \mathbf{v}^T(Q_x \otimes P_y \otimes I_4)\mathbf{F}^V + (\mathbf{F}^V)^T(Q_x^T \otimes P_y \otimes I_4)\mathbf{v} + \\
& \mathbf{v}^T(P_x \otimes Q_y \otimes I_4)\mathbf{G}^V + (\mathbf{G}^V)^T(P_x \otimes Q_y \otimes I_4)^T\mathbf{v} \\
= & 2\underbrace{\mathbf{v}_E^T(P_y \otimes I_4)\mathbf{F}_E^V}_{\text{East}} - 2\underbrace{\mathbf{v}_W^T(P_y \otimes I_4)\mathbf{F}_W^V}_{\text{West}} - 2\underbrace{\mathbf{v}_S^T(P_x \otimes I_4)\mathbf{G}_S^V}_{\text{South}} \\
& + 2\underbrace{\mathbf{v}_N^T(P_x \otimes I_4)\mathbf{G}_N^V}_{\text{North}} - 2 \begin{bmatrix} \mathbf{v}_x \\ \mathbf{v}_y \end{bmatrix}^T \begin{bmatrix} \bar{P} \otimes B_{11} & \bar{P} \otimes B_{12} \\ \bar{P} \otimes B_{21} & \bar{P} \otimes B_{22} \end{bmatrix} \begin{bmatrix} \mathbf{v}_x \\ \mathbf{v}_y \end{bmatrix}.
\end{aligned} \tag{12}$$

An expanded version of equation (10) using the relations above becomes

$$\begin{aligned}
\frac{d}{dt}(\|\mathbf{v}\|_{\bar{P} \otimes I_4}^2) = & - \underbrace{\mathbf{v}_E^T(P_y \otimes I_4)(\mathbf{F}_E^I - 2\varepsilon\mathbf{F}_E^V)}_{\text{East}} + \underbrace{\mathbf{v}_W^T(P_y \otimes I_4)(\mathbf{F}_W^I - 2\varepsilon\mathbf{F}_W^V)}_{\text{West}} \\
& + \underbrace{\mathbf{v}_S^T(P_x \otimes I_4)(\mathbf{G}_S^I - 2\varepsilon\mathbf{G}_S^V)}_{\text{South}} - \underbrace{\mathbf{v}_N^T(P_x \otimes I_4)(\mathbf{G}_N^I - 2\varepsilon\mathbf{G}_N^V)}_{\text{North}} \\
& - 2\varepsilon \begin{bmatrix} \mathbf{v}_x \\ \mathbf{v}_y \end{bmatrix}^T \begin{bmatrix} \bar{P} \otimes B_{11} & \bar{P} \otimes B_{12} \\ \bar{P} \otimes B_{21} & \bar{P} \otimes B_{22} \end{bmatrix} \begin{bmatrix} \mathbf{v}_x \\ \mathbf{v}_y \end{bmatrix}.
\end{aligned} \tag{13}$$

Note that for square matrices $\bar{P}$ and B_{11} (or B_{12} , B_{21} and B_{22}) the Kronecker product $\bar{P} \otimes B_{11}$ and $B_{11} \otimes \bar{P}$ are even permutation similar, that is, there exists a permutation matrix Φ such that $\bar{P} \otimes B_{11} = \Phi^T(B_{11} \otimes \bar{P})\Phi$, see [7] for details. Equation (13) can therefore be written

$$\frac{d}{dt}(\|\mathbf{v}\|_{\bar{P} \otimes I_4}^2) = \text{BT} - 2\varepsilon \begin{bmatrix} \mathbf{w}_x \\ \mathbf{w}_y \end{bmatrix}^T \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \otimes \bar{P} \begin{bmatrix} \mathbf{w}_x \\ \mathbf{w}_y \end{bmatrix} \tag{14}$$

where BT collect all the boundary terms in (13) and $\mathbf{w}_x = \Phi\mathbf{v}_x$, $\mathbf{w}_y = \Phi\mathbf{v}_y$.

Exactly similar to the continuous case, a bounded energy growth in (14) require boundedness in terms of given data of the boundary terms (East, West, North and South). Again, that is dealt with in the papers [19],[21]. The contribution from the integral term in (14) is negative semi-definite since the matrices $\bar{P}$ and $B = [B_{11} \ B_{12}; B_{21} \ B_{22}]$ are positive semi-definite.

Exactly similar to the continuous case, we summarize the result for the semi-discrete single domain problem (7) in the following proposition.

Proposition 4.1 *The semi-discrete problem (7) is stable if i) the boundary terms BT are limited by given data and ii) the matrices $\bar{P}$ and the matrix $B = [B_{11} \ B_{12}; B_{21} \ B_{22}]$ are positive semi-definite.*

5 Stable and conservative interface conditions

We consider a computational domain consisting of two sub-domains and a common interface at $x = 0$, see Figure 1. Let $\mathbf{u}$ and $\mathbf{v}$ be the unknowns in the left and right sub-domain, respectively, and introduce the superscripts L and R to identify the left and right sub-domains.

The semi-discrete approximation of (2) on the two sub-domains with an interface can be written

$$\mathbf{u}_t + D_x^L \mathbf{F}^L + D_y^L \mathbf{G}^L = (M^L)^{-1} (\bar{\Sigma}_1^L [\mathbf{u}_I - \mathbf{v}_I] + \bar{\Sigma}_2^L [(\mathbf{F}^V)_I^L - (\mathbf{F}^V)_I^R]), \quad (15a)$$

$$\mathbf{v}_t + D_x^R \mathbf{F}^R + D_y^R \mathbf{G}^R = (M^R)^{-1} (\bar{\Sigma}_1^R [\mathbf{v}_I - \mathbf{u}_I] + \bar{\Sigma}_2^R [(\mathbf{F}^V)_I^R - (\mathbf{F}^V)_I^L]), \quad (15b)$$

where the matrices E^L, E^R picks out the parts of the vectors residing at the interface such that for example $\mathbf{u}_I = E^L \mathbf{u}$, $\mathbf{v}_I = E^R \mathbf{v}$. In the following, the subscript I indicates that the quantity resides on the interface. We also have the definitions:

$$\begin{aligned} D_x^L &= (P_x^L)^{-1} Q_x^L \otimes I_y^L \otimes I_4, & D_y^L &= I_x^L \otimes (P_y^L)^{-1} Q_y^L \otimes I_4, \\ D_x^R &= (P_x^R)^{-1} Q_x^R \otimes I_y^R \otimes I_4, & D_y^R &= I_x^R \otimes (P_y^R)^{-1} Q_y^R \otimes I_4, \\ M^L &= P_x^L \otimes P_y^L \otimes I_4, & M^R &= P_x^R \otimes P_y^R \otimes I_4, \\ \bar{\Sigma}_1^L &= (E^L)^T P_y^L \otimes \Sigma_1^L, & \bar{\Sigma}_2^L &= (E^L)^T P_y^L \otimes \Sigma_2^L, \\ \bar{\Sigma}_1^R &= (E^R)^T P_y^R \otimes \Sigma_1^R, & \bar{\Sigma}_2^R &= (E^R)^T P_y^R \otimes \Sigma_2^R. \end{aligned}$$

Note that the outer boundary conditions are neglected in this analysis, or phrased differently, they are considered separately, see [19],[21].

Applying the energy method to (15a) and (15b) yields

$$\frac{d}{dt} (\|\mathbf{u}\|_{M^L}^2 + \|\mathbf{v}\|_{M^R}^2) + 2\varepsilon \text{Diss} = \mathbf{w}_I^T M \mathbf{w}_I \quad (16)$$

where $\mathbf{w}_I = [\mathbf{u}_I, \mathbf{v}_I, (\mathbf{u}_x)_I, (\mathbf{v}_x)_I, (\mathbf{u}_y)_I, (\mathbf{v}_y)_I]^T$ and

$$\begin{aligned} \text{Diss} &= \begin{bmatrix} \mathbf{u}_x \\ \mathbf{u}_y \end{bmatrix}^T \begin{bmatrix} P_x^L \otimes P_y^L \otimes B_{11} & P_x^L \otimes P_y^L \otimes B_{12} \\ P_x^L \otimes P_y^L \otimes B_{21} & P_x^L \otimes P_y^L \otimes B_{22} \end{bmatrix} \begin{bmatrix} \mathbf{u}_x \\ \mathbf{u}_y \end{bmatrix} + \\ &\quad \begin{bmatrix} \mathbf{v}_x \\ \mathbf{v}_y \end{bmatrix}^T \begin{bmatrix} P_x^R \otimes P_y^R \otimes B_{11} & P_x^R \otimes P_y^R \otimes B_{12} \\ P_x^R \otimes P_y^R \otimes B_{21} & P_x^R \otimes P_y^R \otimes B_{22} \end{bmatrix} \begin{bmatrix} \mathbf{v}_x \\ \mathbf{v}_y \end{bmatrix}. \end{aligned}$$

The matrix M in (16) determines the stability of the interface treatment. M is

symmetric and the elements of M are

$$\begin{aligned}
M_{11} &= P_y^L \otimes (-A_1 + \Sigma_1^L + (\Sigma_1^L)^T), & M_{12} &= P_y^L \otimes -\Sigma_1^L + P_y^R \otimes -(\Sigma_1^R)^T, \\
M_{13} &= P_y^L \otimes (\varepsilon I_4 + \Sigma_2^L) B_{11}, & M_{14} &= P_y^L \otimes -\Sigma_2^L B_{11}, \\
M_{15} &= P_y^L \otimes (\varepsilon I_4 + \Sigma_2^L) B_{12}, & M_{16} &= P_y^L \otimes -\Sigma_2^L B_{12}, \\
M_{22} &= P_y^R \otimes (A_1 + \Sigma_1^R + (\Sigma_1^R)^T), & M_{23} &= P_y^R \otimes -\Sigma_2^R B_{11}, \\
M_{24} &= P_y^L \otimes (-\varepsilon I_4 + \Sigma_2^R) B_{11}, & M_{25} &= P_y^L \otimes -\Sigma_2^R B_{12}, \\
M_{26} &= P_y^R \otimes (-\varepsilon I_4 + \Sigma_2^R) B_{12}, & M_{33} &= M_{34} = M_{35} = M_{36} = 0, \\
M_{44} &= M_{45} = M_{46} = 0, & M_{55} &= M_{56} = M_{66} = 0.
\end{aligned}$$

Notice that the matrix M in its present form is indefinite.

In order to construct a symmetric semi-definite negative matrix on the right hand side of equation (16) we must “borrow” interface terms from Diss on the left hand side, see [2]. The term Diss can be written as

$$\begin{aligned}
\text{Diss} &= \widetilde{\text{Diss}} + \alpha^L p^L \begin{bmatrix} (\mathbf{u}_x)_I \\ (\mathbf{u}_y)_I \end{bmatrix}^T \begin{bmatrix} P_y^L \otimes B_{11} & P_y^L \otimes B_{12} \\ P_y^L \otimes B_{21} & P_y^L \otimes B_{22} \end{bmatrix} \begin{bmatrix} (\mathbf{u}_x)_I \\ (\mathbf{u}_y)_I \end{bmatrix} \\
&\quad + \beta^R p^R \begin{bmatrix} (\mathbf{v}_x)_I \\ (\mathbf{v}_y)_I \end{bmatrix}^T \begin{bmatrix} P_y^R \otimes B_{11} & P_y^R \otimes B_{12} \\ P_y^R \otimes B_{21} & P_y^R \otimes B_{22} \end{bmatrix} \begin{bmatrix} (\mathbf{v}_x)_I \\ (\mathbf{v}_y)_I \end{bmatrix}
\end{aligned}$$

where $p^L = (P_x^L)_{M,M}$, $p^R = (P_x^R)_{1,1}$ and

$$\begin{aligned}
\widetilde{\text{Diss}} &= \begin{bmatrix} \mathbf{u}_x \\ \mathbf{u}_y \end{bmatrix}^T \begin{bmatrix} \widetilde{P}_x^L \otimes P_y^L \otimes B_{11} & \widetilde{P}_x^L \otimes P_y^L \otimes B_{12} \\ \widetilde{P}_x^L \otimes P_y^L \otimes B_{21} & \widetilde{P}_x^L \otimes P_y^L \otimes B_{22} \end{bmatrix} \begin{bmatrix} \mathbf{u}_x \\ \mathbf{u}_y \end{bmatrix} + \\
&\quad \begin{bmatrix} \mathbf{v}_x \\ \mathbf{v}_y \end{bmatrix}^T \begin{bmatrix} \widetilde{P}_x^R \otimes P_y^R \otimes B_{11} & \widetilde{P}_x^R \otimes P_y^R \otimes B_{12} \\ \widetilde{P}_x^R \otimes P_y^R \otimes B_{21} & \widetilde{P}_x^R \otimes P_y^R \otimes B_{22} \end{bmatrix} \begin{bmatrix} \mathbf{v}_x \\ \mathbf{v}_y \end{bmatrix}.
\end{aligned}$$

The modified norms in $\widetilde{\text{Diss}}$ are $\widetilde{P}_x^L = P_x^L - \text{diag}(0, \dots, \alpha^L p^L)$ and $\widetilde{P}_x^R = P_x^R - \text{diag}(\beta^R p^R, 0, \dots, 0)$. Note that with $0 < \alpha^L, \beta^R \leq 1$, then $\widetilde{P}_x^L \geq 0$ and $\widetilde{P}_x^R \geq 0$ and hence $\widetilde{\text{Diss}} \geq 0$.

As a result, the modified version of equation (16) can be written as

$$\frac{d}{dt} \left(\|\mathbf{u}\|_{M^L}^2 + \|\mathbf{v}\|_{M^R}^2 \right) + 2\varepsilon \widetilde{\text{Diss}} = \mathbf{w}_I^T \widetilde{M} \mathbf{w}_I, \quad (17)$$

where $\widetilde{M}$ plays the role of M except that the zero elements in M are replaced by

$$\begin{aligned}
M_{33} &= -2\varepsilon \alpha^L p^L P_y^L \otimes B_{11}, & M_{35} &= -2\varepsilon \alpha^L p^L P_y^L \otimes B_{12}, \\
M_{44} &= -2\varepsilon \beta^R p^R P_y^R \otimes B_{11}, & M_{46} &= -2\varepsilon \beta^R p^R P_y^R \otimes B_{12}, \\
M_{55} &= -2\varepsilon \alpha^L p^L P_y^L \otimes B_{22}, & M_{66} &= -2\varepsilon \beta^R p^R P_y^R \otimes 2B_{22}, \\
M_{53} &= M_{35}^T, & M_{64} &= M_{46}^T.
\end{aligned}$$

5.1 Conservation conditions

Before considering the stability, we investigate the conservation properties at the interface. Let φ be a smooth test function ($\varphi_I^L = \varphi_I^R = \varphi_I$). Multiplying equations (15a) and (15b) by $(\varphi^L)^T M^L$ and $(\varphi^R)^T M^R$ respectively and using the SBP relations (9) leads to

$$\begin{aligned} & (\varphi^L)^T M^L \mathbf{u}_t + (\varphi^R)^T M^R \mathbf{v}_t - (\varphi_x^T M \mathbf{F} + \varphi_y^T M \mathbf{G})^L - (\varphi_x^T M \mathbf{F} + \varphi_y^T M \mathbf{G})^R = \\ & \varphi_I^T \left[- (P_y^L \otimes A_1) \mathbf{u}_I + (P_y^R \otimes A_1) \mathbf{v}_I + (P_y^L \otimes \Sigma_1^L - P_y^R \otimes \Sigma_1^R) (\mathbf{u}_I - \mathbf{v}_I) \right] + \\ & \varphi_I^T \left[(P_y^L \otimes \varepsilon I_4) (\mathbf{F}_I^V)^L - (P_y^R \otimes \varepsilon I_4) (\mathbf{F}_I^V)^R + (P_y^L \otimes \Sigma_1^L - P_y^R \otimes \Sigma_1^R) ((\mathbf{F}_I^V)^L - (\mathbf{F}_I^V)^R) \right] \end{aligned} \quad (18)$$

As usual we have neglected the boundary terms. If $P_y^L = P_y^R$, the conditions

$$\Sigma_1^R = \Sigma_1^L - A_1, \quad \Sigma_2^R = \Sigma_2^L + \varepsilon I_4, \quad (19)$$

cancel the interface terms (the right-hand side) in (18) and lead to a conservative scheme.

Remark The conservation conditions (19) are a subset of the resulting stability conditions, see [12,14] for more details.

5.2 Stability conditions

Inserting $P_y^L = P_y^R = P_y$ and the conservation conditions (19) into (17) results in

$$\begin{aligned} & \frac{d}{dt} \left(\|\mathbf{u}\|_{M^L}^2 + \|\mathbf{v}\|_{M^R}^2 \right) + 2\varepsilon \widetilde{\text{Diss}} \\ & = - \begin{bmatrix} \Phi \mathbf{u} \\ \Psi \mathbf{v} \\ \Phi \mathbf{u}_x \\ \Psi \mathbf{v}_x \\ \Phi \mathbf{u}_y \\ \Psi \mathbf{v}_y \end{bmatrix}_I^T \begin{bmatrix} N_{11} & -N_{11} & N_{13} & N_{14} & N_{15} & N_{16} \\ -N_{11} & N_{11} & -N_{13} & -N_{14} & -N_{15} & -N_{16} \\ N_{13}^T & -N_{13}^T & N_{33} & 0 & N_{35} & 0 \\ N_{14}^T & -N_{14}^T & 0 & N_{44} & 0 & N_{46} \\ N_{15}^T & -N_{15}^T & N_{35}^T & 0 & N_{55} & 0 \\ N_{16}^T & -N_{16}^T & 0 & N_{46}^T & 0 & N_{66} \end{bmatrix} \otimes P_y \begin{bmatrix} \Phi \mathbf{u} \\ \Psi \mathbf{v} \\ \Phi \mathbf{u}_x \\ \Psi \mathbf{v}_x \\ \Phi \mathbf{u}_y \\ \Psi \mathbf{v}_y \end{bmatrix}_I \\ & = - \mathbf{x}^T (N \otimes P_y) \mathbf{x}. \end{aligned} \quad (20)$$

The permutation matrices Φ and Ψ are defined in Section 4 and

$$\begin{aligned} N_{11} &= A_1 - \Sigma_1^L - (\Sigma_1^L)^T, & N_{13} &= -(\varepsilon I_4 + \Sigma_2^L) B_{11}, & N_{14} &= \Sigma_2^L B_{11}, \\ N_{15} &= -(\varepsilon I_4 + \Sigma_2^L) B_{12}, & N_{16} &= \Sigma_2^L B_{12}, & N_{33} &= 2\varepsilon \alpha^L p^L B_{11}, \\ N_{35} &= 2\varepsilon \alpha^L p^L B_{12}, & N_{44} &= 2\varepsilon \beta^R p^R B_{11}, & N_{46} &= 2\varepsilon \beta^R p^R B_{12}, \\ N_{55} &= 2\varepsilon \alpha^L p^L B_{22}, & N_{66} &= 2\varepsilon \beta^R p^R B_{22}. \end{aligned}$$

Remark The condition $P_y^L = P_y^R$ implies that the same SBP operators should be used in the y -direction in both sub-domains. This restriction can be removed, and that will be the topic in a future paper.

A bounded energy require a positive semi-definite matrix N . To simplify the algebra we introduce a transformation matrix S such that $S^T S = I$ and

$$S = \begin{bmatrix} \frac{1}{\sqrt{2}}I_4 & \frac{1}{\sqrt{2}}I_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & I_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_4 & 0 \\ 0 & 0 & 0 & I_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_4 \\ \frac{1}{\sqrt{2}}I_4 & -\frac{1}{\sqrt{2}}I_4 & 0 & 0 & 0 & 0 \end{bmatrix}, \hat{N} = S N S^T = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & N_{33} & N_{35} & 0 & 0 & \sqrt{2}N_{13} \\ 0 & N_{35}^T & N_{55} & 0 & 0 & \sqrt{2}N_{15} \\ 0 & 0 & 0 & N_{44} & N_{46} & \sqrt{2}N_{14} \\ 0 & 0 & 0 & N_{46}^T & N_{66} & \sqrt{2}N_{16} \\ 0 & \sqrt{2}N_{13}^T & \sqrt{2}N_{15}^T & \sqrt{2}N_{14}^T & \sqrt{2}N_{16}^T & 2N_{11} \end{bmatrix}$$

To simplify the matrix $\hat{N}$ we introduce

$$\alpha = \alpha^L p^L, \quad \beta = \beta^R p^R, \quad \Sigma_2^L = -\varepsilon \Delta, \quad \Sigma_1^L = \Sigma_{1I}^L + \varepsilon \Sigma_{1V}^L, \quad (21)$$

where we choose Δ to be diagonal. This means that $\hat{N}$ can be split into an inviscid part $\hat{N}_I$ and a viscous part $\hat{N}_V$, we get

$$\hat{N} = \underbrace{\begin{bmatrix} \mathbf{0}_{20,20} & \mathbf{0}_{20,4} \\ \mathbf{0}_{4,20} & 2(A_1 - \Sigma_{1I}^L - (\Sigma_{1I}^L)^T) \end{bmatrix}}_{\hat{N}_I} + \varepsilon \underbrace{\begin{bmatrix} \mathbf{0}_{4,4} & \mathbf{0}_{4,8} & \mathbf{0}_{4,8} & \mathbf{0}_{4,4} \\ \mathbf{0}_{8,4} & 2\alpha K_{11} & \mathbf{0}_{8,8} & \sqrt{2}K_{13} \\ \mathbf{0}_{8,4} & \mathbf{0}_{8,8} & 2\beta K_{11} & \sqrt{2}K_{23} \\ \mathbf{0}_{4,4} & \sqrt{2}K_{13}^T & \sqrt{2}K_{23}^T & 2K_{33} \end{bmatrix}}_{\hat{N}_V} \quad (22)$$

where $K_{33} = -(\Sigma_{1V}^L + (\Sigma_{1V}^L)^T)$ and

$$K_{11} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}, K_{13} = \begin{bmatrix} (\Delta - I_4)B_{11} \\ (\Delta - I_4)B_{12} \end{bmatrix}, K_{23} = \begin{bmatrix} -\Delta B_{11} \\ -\Delta B_{12} \end{bmatrix}.$$

The subscripts on $\mathbf{0}$ in (22) indicate the size of the block.

The condition for $\hat{N}_I$ in equation (22) to be positive semi-definite is

$$A_1 - \Sigma_{1I}^L - (\Sigma_{1I}^L)^T \geq 0. \quad (23)$$

If A_1 is rewritten as $A_1 = X^T \Lambda X = X^T \Lambda^+ X + X^T \Lambda^- X = A_1^+ + A_1^-$ where $\Lambda^+ = \text{diag}(\max(\lambda_i, 0))$, $\Lambda^- = \text{diag}(\min(\lambda_i, 0))$ and λ_i are the eigenvalues of A_1 , we find that (23) is satisfied if

$$\Sigma_{1I}^L + (\Sigma_{1I}^L)^T \leq A_1^-. \quad (24)$$

Next we turn to the more difficult analysis of the definiteness of $\hat{N}_V$. The dimensions of $\hat{N}_V$ and the matrices K_{11}, K_{13}, K_{23} and K_{33} are given in (22). Note that since the matrices B_{ij} all lack the first row and column, the only non-zero part of $\hat{N}_V$ that we need to consider for definiteness is the condensed version (we neglect the rows and columns that consist of zeros) of the lower 3×3 block in (22).

Let us denote the condensed version of the lower 3×3 block in $\hat{N}_V$ with $\tilde{N}$ and use a similar notation also for the rest of the matrices. That means that we should consider definiteness of

$$\tilde{N} = \begin{bmatrix} 2\alpha\tilde{K}_{11} & \mathbf{0}_{6,6} & \sqrt{2}\tilde{K}_{13} \\ \mathbf{0}_{6,6} & 2\beta\tilde{K}_{11} & \sqrt{2}\tilde{K}_{23} \\ \sqrt{2}\tilde{K}_{13}^T & \sqrt{2}\tilde{K}_{23}^T & 2\tilde{K}_{33} \end{bmatrix}, \quad \tilde{K}_{33} = -(\Sigma + \Sigma)^T \quad (25)$$

$$\tilde{K}_{11} = \begin{bmatrix} \tilde{B}_{11} & \tilde{B}_{12} \\ \tilde{B}_{21} & \tilde{B}_{22} \end{bmatrix}, \quad \tilde{K}_{13} = \begin{bmatrix} (\Delta - I_3)\tilde{B}_{11} \\ (\Delta - I_3)\tilde{B}_{12} \end{bmatrix}, \quad \tilde{K}_{23} = \begin{bmatrix} -\Delta\tilde{B}_{11} \\ -\Delta\tilde{B}_{12} \end{bmatrix}. \quad (26)$$

Note again that we have now replaced all 4×4 matrices with the corresponding 3×3 ones. We have also kept the notation Δ and changed Σ_{1V} to Σ .

We find that a sufficient condition for positive semi-definiteness of $\tilde{N}$ is

$$\tilde{K}_{11} > 0 \text{ and } -(\Sigma + \Sigma^T) = \tilde{K}_{33} \geq \frac{1}{2\alpha}\tilde{K}_{13}^T\tilde{K}_{11}^{-1}\tilde{K}_{13} + \frac{1}{2\beta}\tilde{K}_{23}^T\tilde{K}_{11}^{-1}\tilde{K}_{23}, \quad (27)$$

because we can factorize $\tilde{N}$ as $\tilde{N} = \varepsilon LDL^T$ with

$$D = \begin{bmatrix} 2\alpha\tilde{K}_{11} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 2\beta\tilde{K}_{11} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & D_{33} \end{bmatrix}, \quad L = \begin{bmatrix} I & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & I & \mathbf{0} \\ \frac{1}{\sqrt{2\alpha}}\tilde{K}_{13}^T\tilde{K}_{11}^{-1} & \frac{1}{\sqrt{2\beta}}\tilde{K}_{23}^T\tilde{K}_{11}^{-1} & I \end{bmatrix}, \quad (28)$$

and $D_{33} = 2\tilde{K}_{33} - \frac{1}{\alpha}\tilde{K}_{13}^T\tilde{K}_{11}^{-1}\tilde{K}_{13} - \frac{1}{\beta}\tilde{K}_{23}^T\tilde{K}_{11}^{-1}\tilde{K}_{23}$.

The conditions (19), (23) and (27) make the matrix $\hat{N}$ positive semi-definite, which implies that matrix N is positive semi-definite, since for a arbitrary vector $\mathbf{y}$,

$$\mathbf{y}^T N \mathbf{y} = \mathbf{y}^T S^T \hat{N} S \mathbf{y} = \hat{\mathbf{y}}^T \hat{N} \hat{\mathbf{y}} \geq 0.$$

The matrix $\tilde{K}_{11}^{-1}$ can be written in block matrix form as

$$\tilde{K}_{11}^{-1} = \begin{bmatrix} \tilde{B}_{11}^{-1} + \tilde{B}_{11}^{-1}\tilde{B}_{12}\tilde{D}^{-1}\tilde{B}_{21}\tilde{B}_{11}^{-1} & -\tilde{B}_{11}^{-1}\tilde{B}_{12}\tilde{D}^{-1} \\ -\tilde{D}^{-1}\tilde{B}_{21}\tilde{B}_{11}^{-1} & \tilde{D}^{-1} \end{bmatrix},$$

with $\tilde{D} = \tilde{B}_{22} - \tilde{B}_{21}\tilde{B}_{11}^{-1}\tilde{B}_{12}$. The choice $\Delta = \delta I_3$ ($\delta \in \mathcal{R}$) simplifies the algebra considerably and leads to

$$\tilde{K}_{13}^T\tilde{K}_{11}^{-1}\tilde{K}_{13} = (1 - \delta)^2\tilde{B}_{11}, \quad \text{and} \quad \tilde{K}_{23}^T\tilde{K}_{11}^{-1}\tilde{K}_{23} = \delta^2\tilde{B}_{11}.$$

That means that the last condition in (27) together with the assumption that Σ is symmetric leads to

$$\Sigma \leq -\frac{[\beta(1-\delta)^2 + \alpha\delta^2]\varepsilon}{4\alpha\beta}\tilde{B}_{11}. \quad (29)$$

It is easy to verify that the right hand side of (29) has the least restrictive value $-\varepsilon\tilde{B}_{11}/(4(\alpha+\beta))$ when $\delta = \beta/(\alpha+\beta)$.

Now we have done all the necessary derivations and we can summarize the result in the following proposition.

Proposition 5.1 *If the conditions*

$$\Sigma_{1I}^L \leq A_1^-/2, \quad (\text{inviscid stability}) \quad (30a)$$

$$\Sigma_{1V}^L \leq -\varepsilon\tilde{B}_{11}/4(\alpha+\beta), \quad (\text{viscous stability}) \quad (30b)$$

$$\Sigma_2^L = -\varepsilon\beta I_4/(\alpha+\beta), \quad (\text{viscous stability}) \quad (30c)$$

$$\Sigma_1^R = \Sigma_1^L - A_1, \quad (\text{inviscid conservation}) \quad (30d)$$

$$\Sigma_2^R = \Sigma_2^L + \varepsilon I_4. \quad (\text{viscous conservation}) \quad (30e)$$

are satisfied, then the scheme (15) is stable and conservative.

Remark Recall that $\alpha = \alpha^L p^L$ and $\beta = \beta^R p^R$ ($0 \leq \alpha^L, \beta^R \leq 1$) where

$$p^L = \Delta x^L \cdot \begin{cases} \frac{1}{2} & \text{2nd order SBP,} \\ \frac{17}{48} & \text{4th order SBP,} \\ \frac{13649}{43200} & \text{6th order SBP,} \end{cases} \quad p^R = \Delta x^R \cdot \begin{cases} \frac{1}{2} & \text{2nd order SBP,} \\ \frac{17}{48} & \text{4th order SBP,} \\ \frac{13649}{43200} & \text{6th order SBP,} \end{cases}$$

In order to limit the spectral radius of the problem, the values of α^L and β^R should be chosen as large as possible, that is $\alpha^L = \beta^R = 1$.

Remark Note again that the conservation conditions are a subset of the total number of conditions required for stability.

6 Numerical experiments

The derivation of the stability and conservation properties as expressed in Proposition 5.1 was done for the constant coefficient problem. We now verify that the result of the linear analysis is valid for the full non-linear Navier-Stokes equations.

6.1 Verification of accuracy and stability for the Navier-Stokes equations

We consider a calculation on two sub-domains coupled at an interface, see Figure 1. A stationary viscous shock problem where the middle of shock is located at the

interface is calculated. This problem has an analytical solution (for Prandtl number $Pr = 3/4$) which means that we have full control of the errors. The Mach number in front of the shock in the reference frame of the shock is 2.0 and the angle of the shock relative to the Cartesian frame is 15° . The Reynolds number $Re = 50.0$ is based on the Mach number of shock. The penalty terms in (30) are chosen by the minimum required values. We integrate the solution to steady-state using the third order low storage explicit time advancement scheme of Le and Moin [9].

In the hybrid scheme, the second derivative SBP operator is constructed with $2p$ -th ($p = 1, 2, \dots$) order accuracy internal and $(p-1)$ -th order at the boundary by using a diagonal norm. It was proved in [20] that if the solution is pointwise bounded, the accuracy of the scheme is two orders higher than the accuracy of the second derivative approximation at the boundaries. The convergence rates for the second-, fourth-, sixth- and eighth- order schemes are thus 2, 3, 4 and 5, respectively. Since the errors for all variables (density, velocities and energy) are very similar, only the density errors are shown in our calculations. The accuracy is shown in Tables 1 and Table 2. The results are in agreement with the theory, see [4,5,20].

Points/Block	2nd order		4th order		6th order		8th order	
	Err	q	Err	q	Err	q	Err	q
17×17	-1.29	—	-1.47	—	-1.14	—	—	—
33×33	-1.89	1.99	-2.24	2.56	-1.90	2.52	-1.89	—
65×65	-2.55	2.18	-3.14	3.00	-2.92	3.41	-3.03	3.77
129×129	-3.18	2.11	-4.12	3.24	-4.01	3.59	-4.39	4.53
257×257	-3.80	2.03	-5.06	3.15	-5.11	3.66	-5.90	5.01

Table 1

The convergence rates of density on two uniform sub-domains.

Points (left)+(right)	2nd order		4th order		6th order		8th order	
	Err	q	Err	q	Err	q	Err	q
$33 \times 33 + 17 \times 33$	-1.43	—	—	—	-1.30	—	-1.39	—
$65 \times 65 + 33 \times 65$	-2.02	1.94	-2.35	2.59	-2.06	2.52	-2.04	—
$129 \times 129 + 65 \times 129$	-2.65	2.09	-3.23	2.91	-3.02	3.21	-3.14	3.68
$257 \times 257 + 129 \times 257$	-3.26	2.05	-4.13	2.98	-4.09	3.53	-4.48	4.43
$513 \times 513 + 257 \times 513$	-3.88	2.05	-5.02	2.98	-5.22	3.76	-5.92	4.80

Table 2

The convergence rates of density on two non-uniform subdomains.

In the next calculation we consider the solution computed on the mesh in Figure 1. Figure 2(a) shows the density isolines using the 4th order discretization. The

corresponding cut at $y = 0$ can be found in Figure 2(b). The distribution of density close to the interface $x = 0$ is very smooth, which illustrates that the interface does not introduce large reflections and oscillations.

Fig. 1. A hybrid mesh of $65 \times 65 + 33 \times 65$ grid points.

(a) The whole computational domain

(b) A cut at $y = 0$

Fig. 2. Density isolines for a 4th order calculation. 65×65 grid points are used in both subdomains.

The density errors at $y = 0$ with SBP operators of different order are shown in Figures 3 and 4. Figure 3 shows the result for two uniform meshes, while in Figure

(a) 65×65 points/block

(b) 129×129 points/block

Fig. 3. The errors in density at $y = 0$ with SBP operators of different orders.

(a) 65×65 + 33×65

(b) 129×129 + 65×129 points/block

Fig. 4. The errors in density at $y = 0$ with SBP operators of different orders.

4 the right block is twice as coarse in the x -direction as the left block. Figures 3(a) and 4(a) show that the higher order schemes have rather large errors, comparable to the lower order schemes close to the interface $x = 0$ for the coarse mesh. However, when the mesh is refined, (129×129 and 65×129 , respectively) the higher order schemes outperform the lower order schemes (see Figures 3(b) and 4(b)). Tables 1-2 and Figures 3-4 illustrate that the interface treatment is stable and accurate for all orders of accuracy.

To further illustrate the necessity of having correct penalty terms we neglect the viscous penalty term completely. This leads to a complete failure for all schemes (blow up in a couple of time steps), see Figure 5.

(a) The density

(b) The error in density

Fig. 5. A 4th order calculation without the necessary viscous penalty terms.

Finally, we demonstrate the multi-block method on a moving shock problem. The unsteady computation has been carried out on a uniform grid of 65×65 in each block in combination with the the 4th order accurate SBP operator. All penalty parameters have the same values as for the previous steady case. The shock moves at Mach=0.15 under 45° degrees. Snapshots of the solution between $t = 0.0$ and $t = 8.0$ are shown in Figure 6. The shape of the shock through the interface $x = 0$ remains intact, and the corresponding errors are small, see Figure 7.

Density snapshots of the moving viscous shock problem

Fig. 6. Density isolines, 4th order accuracy for the unsteady shock problem.

Density error snapshots of the moving viscous shock problem

Fig. 7. The error in density, 4th order accuracy for the unsteady shock problem.

7 Conclusions

We have proved stability and conservation of a high order accurate multi-block finite difference method applied to the Navier-Stokes equations. As theoretical tools we have used difference operators of SBP type, a penalty technique for the interface conditions and the energy method.

The stability and conservation conditions are derived without approximations. This indicates that the derived conditions are sharp. That conclusion is supported by the numerical calculations which show that instabilities occur if the conditions are violated.

Mesh refinement studies for a steady viscous shock and computations of a moving viscous shock has been performed. The numerical experiments support the theoretical conclusions and show that the interface coupling is stable and converge at the correct order.

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