Stochastic and local volatility benchmark problems

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1 SABR stochastic-local volatility model

The Stochastic Alpha Beta Rho (SABR) model [6] is an established SDE system which is often used for interest rates and FX modeling in practice. The SABR model is based on a toletric local volatility component in terms of a model toleter, $\beta$. The formal definition of the SABR model reads

$$\begin{align*}
\frac{dS(t)}{S(t)} &= \sigma(t)S^\beta(t)dW_S(t), & S(0) &= S_0 \exp(rT), \\
\frac{d\sigma(t)}{\sigma(t)} &= \alpha\sigma(t)dW_\sigma(t), & \sigma(0) &= \sigma_0.
\end{align*}$$

where $S(t) = \bar{S}(t)\exp(r(T - t))$ denotes the forward value of the underlying asset $\bar{S}(t)$, with $r$ the interest rate, $S_0$ the spot price and $T$ the contract’s final time. Quantity $\sigma(t)$ denotes the stochastic volatility, $W_S(t)$ and $W_\sigma(t)$ are two correlated Brownian motions with constant correlation coefficient $\rho$ (i.e. $W_SW_\sigma = \rho t$). The open model toleters are $\alpha > 0$ (the volatility of the volatility), $0 \leq \beta \leq 1$ (the elasticity) and $\rho$ (the correlation coefficient). The corresponding PDE for the valuation of options is given by:

$$\frac{\partial u}{\partial t} + \frac{1}{2}\sigma^2 S^{2\beta} \frac{\partial^2 u}{\partial S^2} + \rho \alpha S^\beta \sigma^2 \frac{\partial^2 u}{\partial \sigma \partial S} + \frac{1}{2}\alpha^2 \sigma^2 \frac{\partial^2 u}{\partial \sigma^2} - ru = 0$$

for $S > 0$, $\sigma > 0$ and $0 \leq t < T$.

Two toleter sets:

- Set I ([4]): $T = 2$, $r = 0.0$, $S_0 = 0.5$, $\sigma_0 = 0.5$, $\alpha = 0.4$, $\beta = 0.5$ and $\rho = 0$.
- Set II ([2]): $T = 10$, $r = 0.0$, $S_0 = 0.07$, $\sigma_0 = 0.4$, $\alpha = 0.8$, $\beta = 0.5$ and $\rho = -0.6$.

European call option payoff $\max(S(T) - K_i(T), 0)$ with three strikes

$$K_i(T) = S(0)\exp(0.1 \times \sqrt{T} \times \delta_i),$$

$\delta_i = -1.0, 0.0, 1.0$.

- Output: $u$ for three strikes and two toleter sets.
• Benchmark: Error in the solution as a function of CPU-time.

• Headings:

\[ \text{[U]} = \text{SABReuCallI}_M\text{T}_H(\text{tol}) \]
\[ \text{[U]} = \text{SABReuCallII}_M\text{T}_H(\text{tol}) \]

$M\text{T}_H$ should be replaced with a three/four/five-letter method-specific code.
$\text{tol}$ controls the computational effort and consequently the accuracy and CPU-time. The absolute error in the computed solution should be in the order of $\text{tol}$. $\text{U}$ should be a row vector with three elements $(U(1) U(2) U(3))$.

Notes:

• Consider implied volatilities next to option prices, for comparison purposes.
• For $\rho = 0$, there is a formula for the exact simulation of the SABR model [7].
• The use of time discretization MC schemes can give a loss of the martingale property. A correction must be introduced then.

## 2 Quadratic local stochastic volatility model

*In the following $\tau$ denotes forward time and $t$ backward time.*

See e.g. [8]:

\[
\begin{cases}
  dS_\tau = rS_\tau d\tau + \sqrt{V_\tau} f(S_\tau) dW^1_\tau, \\
  dV_\tau = \kappa(\eta - V_\tau) d\tau + \sigma \sqrt{V_\tau} dW^2_\tau,
\end{cases}
\]

with $f(s) = \frac{1}{2} \alpha s^2 + \beta s + \gamma$. Select

• Heston: $\alpha = 0$, $\beta = 1$, $\gamma = 0$.
• QLSV: $\alpha = 0.02$, $\beta = 0$, $\gamma = 0$.

PDE:

\[
\frac{\partial u}{\partial t} + \frac{1}{2} f(s)^2 v \frac{\partial^2 u}{\partial s^2} + \rho \sigma f(s) v \frac{\partial^2 u}{\partial s \partial v} + \frac{1}{2} \sigma^2 v \frac{\partial^2 u}{\partial v^2} + rs \frac{\partial u}{\partial s} + \kappa(\eta - v) \frac{\partial u}{\partial v} - ru = 0
\]

for $s > 0$, $v > 0$ and $0 < t \leq T$.

One toleter set (see [8]):

\[ T = 1, \ r = 0, \ \kappa = 2.58, \ \eta = 0.043, \ \sigma = 1, \ \rho = -0.36. \]

Consider
• European call option payoff \( \max(s - K, 0) \) with \( K = 100 \).

• Double-no-touch option paying 1 if \( L < S_\tau < U \) (for all \( \tau \)) and 0 else with \( L = 50, \ U = 150 \).

Three spot values: \((S_0, V_0) = (S_0, 0.114)\) for \( S_0 = 75, 100, 125 \).

• Output: \( u \) for three spot values, two models (Heston and QLSV) and two pay-offs (European call and Double-no-touch).

• Benchmark: Error in the solution as a function of CPU-time.

• Headings:

\[
[U]=\text{HSTeuCall}_\text{MTH}(tol) \\
[U]=\text{HSTdnTouch}_\text{MTH}(tol) \\
[U]=\text{QLSveuCall}_\text{MTH}(tol) \\
[U]=\text{QLSVdnTouch}_\text{MTH}(tol)
\]

\$\text{MTH}\$ should be replaced with a three/four/five-letter method-specific code.  
\( \text{tol} \) controls the computational effort and consequently the accuracy and CPU-time.  
The absolute error in the computed solution should be in the order of \( \text{tol} \).  
\( U \) should be a row vector with three elements \((U(1) \ U(2) \ U(3))\).

Notes:

• Feller condition is violated.

• If \( \alpha = 0, \beta = 1, \gamma = 0, \rho = 0 \) there are semi-closed analytic formulas for both options.

3 Heston–Hull–White model

The Heston–Hull–White model is a hybrid asset price model combining the Heston stochastic volatility and Hull–White stochastic interest rate models, see e.g. [3, 5].

HHW SDE:
\[
\begin{align*}
    dS_\tau &= R_\tau S_\tau \, d\tau + \sqrt{V_\tau} S_\tau \, dW^1_\tau, \\
    dV_\tau &= \kappa (\eta - V_\tau) \, d\tau + \sigma_1 \sqrt{V_\tau} \, dW^2_\tau, \\
    dR_\tau &= a(b(\tau) - R_\tau) \, d\tau + \sigma_2 \, dW^3_\tau,
\end{align*}
\]
HHW PDE:

\[
\frac{\partial u}{\partial t} + \frac{1}{2}s^2 v \frac{\partial^2 u}{\partial s^2} + \frac{1}{2}\sigma_1^2 v \frac{\partial^2 u}{\partial v^2} + \frac{1}{2}\sigma_2^2 \frac{\partial^2 u}{\partial r^2} \\
+ \rho_{12}\sigma_1 s v \frac{\partial^2 u}{\partial s \partial v} + \rho_{13}\sigma_2 s \sqrt{v} \frac{\partial^2 u}{\partial s \partial r} + \rho_{23}\sigma_1 \sigma_2 \sqrt{v} \frac{\partial^2 u}{\partial v \partial r} \\
+ rs \frac{\partial u}{\partial s} + \kappa(\eta - v) \frac{\partial u}{\partial v} + a(b(T - t) - r) \frac{\partial u}{\partial r} - ru = 0
\]

for \( s > 0, v > 0, -\infty < r < \infty \) and \( 0 < t \leq T \).

Two tolerer sets (cf. [1, 5]):

\[
T = 10, \ \kappa = 0.5, \ \eta = 0.04, \ \sigma_1 = 1, \ \sigma_2 = 0.09, \ \rho_{12} = -0.9, \ \rho_{13} = 0.6 \ (0), \ \rho_{23} = -0.7 \ (0), \ \\
a = 0.08 \text{ and } b(\tau) \equiv 0.10.
\]

European call option payoff \( \max(s - K, 0) \) with \( K = 100 \).

Three spot values: \((S_0, V_0, R_0) = (S_0, 0.04, 0.10)\) for \( S_0 = 75, 100, 125 \).

- Output: \( u \) for three spot values and two tolerer sets.
- Benchmark: Error in the solution as a function of CPU-time.
- Headings:

\[
[U]=\text{HHWeuCallI}\_\text{SMTH}(\text{tol})  \\
[U]=\text{HHWeuCallII}\_\text{SMTH}(\text{tol})
\]

\text{SMTH} should be replaced with a three/four/five-letter method-specific code.  
\text{tol} controls the computational effort and consequently the accuracy and CPU-time.  
The absolute error in the computed solution should be in the order of \( \text{tol} \).  
\( U \) should be a row vector with three elements \((U(1) \ U(2) \ U(3))\).

Notes:

- Feller condition is violated.
- For \( \rho_{13} = \rho_{23} = 0 \) there is semi-closed analytic formula akin to Heston.
- Transformation to 2D PDE with time-dependent coefficients is possible. Hence, if numerical PDE approach is followed, indicate which PDE is solved (2D or 3D).
References


