Notes on the BENCHOP implementation of Finite difference method on uniform grid with Rannacher smoothed Crank-Nicolson scheme

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Abstract

This text describes the Finite difference method on uniform grid with Rannacher smoothed Crank-Nicolson scheme and its implementation for the BENCHOP-project.

1 Mathematical formulation

Before exercise date and between dividend dates option price u satisfies Black-Scholes Partial Differential Equation (BS PDE) (Wilmott, 2006):

$$\frac{\partial u}{\partial t} + (r-q)\, s \frac{\partial u}{\partial s} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 u}{\partial s^2} - ru = 0, \tag{1}$$

where s is the price of underlying asset, r is the interest rate, q is the constant dividend yield, σ is volatility.

The dividends are taken into consideration by applying jump conditions:

$$u\left(s, t_i^{d-}\right) = u\left(s - d_i\left(s\right), t_i^{d+}\right). \tag{2}$$

Here $d_i\left(s\right)$ is the dividend payment.

We supported three dividend policies in the code:

1. "liquidator" model for fixed (absolute) dividends: $d_i\left(s\right) = \min\left(\mathrm{D_i}, \mathrm{s}\right)$, D_i is the dividend amount (Haug et al., 2003);

2. proportional (percentage) dividends: $d_i\left(s\right) = p_i s$, p_i is the fraction of the price s;

3. mixed (capped) dividends: $d_i\left(s\right) = \min\left(\mathrm{p_i s}, \mathrm{D_i}\right)$.

Boundary conditions for European call option are written in the following form:

$$\begin{aligned} u\left(0, t\right) &= 0, \\ \left.\frac{\partial^2 u}{\partial s^2}\right|_{s\to\infty} &= 0. \end{aligned} \tag{3}$$

The final condition is given by the payoff function $P_{\text{call}}(s)$:

$$u(s,T) = P_{\text{call}}(s) = \max(\mathrm{s - X, 0}), \tag{4}$$

X is the strike price.

For European put option these conditions are the following:

$$\begin{gathered} u(0,t) = Xe^{-r(T-t)}, \\ u(s \to \infty, t) = 0 \end{gathered} \tag{5}$$

and

$$u(s,T) = P_{\text{put}}(s) = \max(\mathrm{X - s, 0}). \tag{6}$$

$P_{\text{put}}(s)$ is the respective payoff function.

Value of American option cannot be lower than the intrinsic value:

$$u(s,t) \geq P(s). \tag{7}$$

On dividend dates jump conditions (2) are combined with (7) and give:

$$u\left(s, t_i^{d-}\right) = \max\left[\mathrm{u\left(s - d_i(s), t_i^{d+}\right), P(s)}\right]. \tag{8}$$

2 Spatial and time discretization

Equation (1) is solved on the finite rectangular domain in $S \times t$ space covered by rectangular mesh. Uniform grid for underlying prices is introduced as:

$$s_i = s_{\min} + i\Delta s, \Delta s = (s_{\max} - s_{\min})/I,$$

where $s_{\min} = 0$, $s_{\max} = mX$, the multiplier m from range $[1.2; 12]$ is the input parameter that affects accuracy, Δs is the grid step, I is the number of steps. And backward time grid between the final time moment $T_{\max}$ coincident with the maturity date T, where the final condition is applied, and the evaluation date $t = 0$:

$$t^{k+1} = t^k - \Delta t^{k+1}, t^0 = T_{\max}, t^K = 0,$$

where K is the number of time steps, Δt^k is the piecewise constant time step. The time grid is uniform over intervals between successive time moments: $t = 0$, ex-dividend dates t_i^d and the final moment $T_{\max}$. By applying jump conditions (2) we perform transition from one interval to another.

On each interval of continuity we use the following scheme:

$$\begin{aligned} &\frac{u_i^k - u_i^{k+1}}{\Delta t^{k+1}} + \omega(r-q)s_i\frac{u_{i+1}^{k+1} - u_{i-1}^{k+1}}{2\Delta s} + (1-\omega)(r-q)s_i\frac{u_{i+1}^k - u_{i-1}^k}{2\Delta s} + \\ &\quad + \frac{1}{2}\omega\sigma^2 s_i^2\frac{u_{i-1}^{k+1} - 2u_i^{k+1} + u_{i+1}^{k+1}}{\Delta s^2} + \frac{1}{2}(1-\omega)\sigma^2 s_i^2\frac{u_{i-1}^k - 2u_i^k + u_{i+1}^k}{\Delta s^2} - \\ &\qquad - \omega r u_i^{k+1} - (1-\omega) r u_i^k = 0, \quad k = 0, ..., K-1. \end{aligned} \tag{9}$$

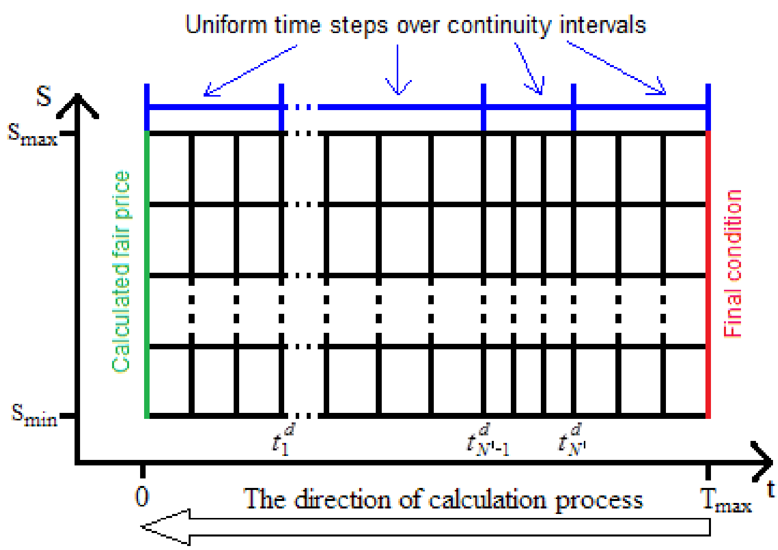

Figure 1: Finite difference method grid

We do $m \geq 2$ sub-steps of fully implicit Backward Euler scheme over the first time step of each interval ($\omega = 1$) and then switch to Crank-Nicolson scheme ($\omega = 1/2$). This procedure is called Rannacher time stepping (Rannacher, 1984; Giles and Carter, 2006).

The system of linear equations given by (9) is solved by LU decomposition.

Boundary and final conditions are projected to the finite domain. For European call option it gives:

$$\begin{aligned} u_0^k &= 0, \\ u_{I+1}^k = 2u_I^k &- u_{I-1}^k, k = 1, ..., K; \\ u_i^0 &= \max\left(\mathrm{s_i} - \mathrm{X}, 0\right). \end{aligned}$$

For European put option we get:

$$\begin{aligned} u_0^k &= Xexp\left[-r\left(T - t^k\right)\right], \\ u_{I+1}^k &= 0, k = 1, ..., K; \\ u_i^0 &= \max\left(\mathrm{X} - \mathrm{s_i}, 0\right). \end{aligned}$$

For American options we need to solve free boundary problem by combining backward substitution in LU decomposition with the analysis of inequality (7) and the intrinsic value given by payoff function (4,6) as proposed by Ikonen and Toivanen (2007). For put options we use UL decomposition instead of LU to keep standard indexing in the data vector.

References

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