Notes on the BENCHOP implementations for the RBF-FD method

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Abstract

This text describes the RBF-FD method and its implementation for the BENCHOP-project.

All of the problems considered may be represented in the form

\[
\frac{\partial u(s,t)}{\partial t} = Lu(s,t), \quad t \in [T,0],
\]

(1)

where \( u \) is the value of the priced contract, \( s \) represents the value of an underlying asset or assets and is treated as a spatial variable, \( t \) is an independent time variable, \( T \) is the time of option maturity and \( L \) is a corresponding operator in one or two spatial dimensions represented by \( s \).

1 Method Description

1.1 Discretization in space

The equation (1) is discretized in space by approximating the operator \( L \) on a set of \( N \) scattered nodes \( s^{(i)} \) using RBF-FD method. That means that the operator \( L \) is approximated at every node as

\[
Lu(s,t)^{(i)} \approx \sum_{k=1}^{n} w_k^{(i)} u_k^{(i)}, \quad i = 1, 2, \ldots, N;
\]

(2)

where \( w_k^{(i)} \) is a weight of a generalized finite difference stencil of size \( n \). The weights are obtained by solving the following linear systems

\[
A^{(i)} \cdot w^{(i)} = 1^{(i)},
\]

(3)

for every stencil defined around each of the nodes \( s^{(i)} \), where the matrix \( A^{(i)} \) is of the form
The stabilization method known as RBF-GA\cite{1} is used. It is highly sensitive to the choice of the shape parameter for different stencils, a stabilization method known as RBF-GA\cite{1} is used.

Since conditioning of the small linear systems by which weights \( \mathbf{w}^{(i)} \) are obtained is highly sensitive to the choice of the shape parameter for different stencils, a stabilization method known as RBF-GA\cite{1} is used.

More details will be available in \cite{2}.

\[ \mathbf{A}^{(i)} = \begin{pmatrix}
\phi(\|s_1^{(i)} - s_1^{(i)}\|) & \phi(\|s_1^{(i)} - s_1^{(i)}\|) & \cdots & \phi(\|s_1^{(i)} - s_n^{(i)}\|) \\
\phi(\|s_2^{(i)} - s_1^{(i)}\|) & \phi(\|s_2^{(i)} - s_1^{(i)}\|) & \cdots & \phi(\|s_2^{(i)} - s_n^{(i)}\|) \\
\vdots & \vdots & \ddots & \vdots \\
\phi(\|s_n^{(i)} - s_1^{(i)}\|) & \phi(\|s_n^{(i)} - s_1^{(i)}\|) & \cdots & \phi(\|s_n^{(i)} - s_n^{(i)}\|)
\end{pmatrix}, \]

while vectors \( \mathbf{w}^{(i)} \) and \( \mathbf{l}^{(i)} \) look like

\[ \mathbf{w}^{(i)} = \begin{pmatrix} w_1^{(i)} \\ w_2^{(i)} \\ \vdots \\ w_n^{(i)} \end{pmatrix}, \quad \mathbf{l}^{(i)} = \begin{pmatrix} [\mathcal{L}\phi(\|s - s_1^{(i)}\|)]_1 \\ [\mathcal{L}\phi(\|s - s_2^{(i)}\|)]_1 \\ \vdots \\ [\mathcal{L}\phi(\|s - s_n^{(i)}\|)]_1 \end{pmatrix}, \]

having \( \phi(\|s_1^{(i)} - s_1^{(i)}\|) \) as a radial basis function with shape parameter \( \varepsilon \in \mathbb{R}^+ \), collocated at \( s_1^{(i)} \).

Once all the \( \mathbf{w}^{(i)} \) vectors are obtained, they can be arranged in a \( N \times N \) matrix \( \mathbf{W} \) such that (1) is approximated by a discrete equation

\[ \frac{\partial \mathbf{u}(t)}{\partial t} \approx \mathbf{W}\mathbf{u}(t), \quad t \in [T, 0], \quad (4) \]

where \( \mathbf{u}(t) \) is a discrete solution in space.

### 1.2 Discretization in time

After discretization in space, the equation (4) is treated in a fashion of method of lines. Namely, it is discretized in time using backwards differentiation formula (BDF). First time step has been taken with BDF-1 scheme (5) and the rest of the integration in time is done by BDF-2 scheme (6), since the later needs information from two previous points in order to make a step further.

\[ \mathbf{u}_j \approx (\mathbf{I} - \Delta t\mathbf{W})^{-1}\mathbf{u}_{j-1}, \quad j = 1; \quad (5) \]

\[ \mathbf{u}_j \approx (\mathbf{I} - \Delta t\mathbf{W})^{-1}\left(\frac{4}{3}\mathbf{u}_{j-1} - \frac{1}{3}\mathbf{u}_{j-2}\right), \quad j = 2, 3, \ldots, M. \quad (6) \]

where \( j \) is indexing the discretized time domain \([T, 0]\) with a time step \( \Delta t \), consisting of \( M \) points, and \( \mathbf{I} \) is an identity matrix of size \( \mathbf{W} \).

### 1.3 Advanced treatment

Since conditioning of the small linear systems by which weights \( \mathbf{w}^{(i)} \) are obtained is highly sensitive to the choice of the shape parameter for different stencils, a stabilization method known as RBF-GA\cite{1} is used.

More details will be available in \cite{2}.
References
