Notes on the BENCHOP implementations for the RBF-FD method

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Abstract

This text describes the RBF-FD method and its implementation for the BENCHOP-project.

All of the problems considered may be represented in the form

$$\frac{\partial u(s,t)}{\partial t} = \mathcal{L}u(s,t), \quad t \in [T, 0], \tag{1}$$

where u is the value of the priced contract, s represents the value of an underlying asset or assets and is treated as a spatial variable, t is an independent time variable, T is the time of option maturity and $\mathcal{L}$ is a corresponding operator in one or two spatial dimensions represented by s.

1 Method Description

1.1 Discretization in space

The equation (1) is discretized in space by approximating the operator $\mathcal{L}$ on a set of N scattered nodes $s^{(i)}$ using RBF-FD method. That means that the operator $\mathcal{L}$ is approximated at every node as

$$\mathcal{L}u(s,t)^{(i)} \approx \sum_{k=1}^{n} w_k^{(i)} u_k^{(i)}, \quad i = 1, 2, \ldots, N; \tag{2}$$

where $w_k^{(i)}$ is a weight of a generalized finite difference stencil of size n. The weights are obtained by solving the following linear systems

$$\mathbf{A}^{(i)} \cdot \mathbf{w}^{(i)} = \mathbf{l}^{(i)}, \tag{3}$$

for every stencil defined around each of the nodes $s^{(i)}$, where the matrix $\mathbf{A}^{(i)}$ is of the form

$$\mathbf{A}^{(i)} = \begin{pmatrix} \phi(\varepsilon\|s_1^{(i)} - s_1^{(i)}\|) & \phi(\varepsilon\|s_1^{(i)} - s_2^{(i)}\|) & \dots & \phi(\varepsilon\|s_1^{(i)} - s_n^{(i)}\|) \\ \phi(\varepsilon\|s_2^{(i)} - s_1^{(i)}\|) & \phi(\varepsilon\|s_2^{(i)} - s_2^{(i)}\|) & \dots & \phi(\varepsilon\|s_2^{(i)} - s_n^{(i)}\|) \\ \vdots & \vdots & \ddots & \vdots \\ \phi(\varepsilon\|s_n^{(i)} - s_1^{(i)}\|) & \phi(\varepsilon\|s_n^{(i)} - s_2^{(i)}\|) & \dots & \phi(\varepsilon\|s_n^{(i)} - s_n^{(i)}\|) \end{pmatrix},$$

while vectors $\mathbf{w}^{(i)}$ and $\mathbf{l}^{(i)}$ look like

$$\mathbf{w}^{(i)} = \begin{pmatrix} w_1^{(i)} \\ w_2^{(i)} \\ \vdots \\ w_n^{(i)} \end{pmatrix}, \quad \mathbf{l}^{(i)} = \begin{pmatrix} [\mathcal{L}\phi(\varepsilon\|s - s_1^{(i)}\|)]_i \\ [\mathcal{L}\phi(\varepsilon\|s - s_2^{(i)}\|)]_i \\ \vdots \\ [\mathcal{L}\phi(\varepsilon\|s - s_n^{(i)}\|)]_i \end{pmatrix},$$

having $\phi(\varepsilon\|s_k^{(i)} - s_j^{(i)}\|)$ as a radial basis function with shape parameter $\varepsilon \in \mathbb{R}^+$, collocated at $s_j^{(i)}$ and estimated at $s_k^{(i)}$.

Once all the $\mathbf{w}^{(i)}$ vectors are obtained, they can be arranged in a $N \times N$ matrix $\mathbf{W}$ such that (1) is approximated by a discrete equation

$$\frac{\partial \mathbf{u}(t)}{\partial t} \approx \mathbf{W}\mathbf{u}(t), \quad t \in [T, 0], \tag{4}$$

where $\mathbf{u}(t)$ is a discrete solution in space.

1.2 Discretization in time

After discretization in space, the equation (4) is threated in a fashion of method of lines. Namely, it is discretized in time using backwards differentiation formula (BDF). First time step has been taken with BDF-1 scheme (5) and the rest of the integration in time is done by BDF-2 scheme (6), since the later needs information from two previous points in order to make a step further.

$$\mathbf{u}_j \approx (\mathbf{I} - \Delta t\mathbf{W})^{-1}\mathbf{u}_{j-1}, \quad j = 1; \tag{5}$$

$$\mathbf{u}_j \approx (\mathbf{I} - \Delta t\mathbf{W})^{-1}(\frac{4}{3}\mathbf{u}_{j-1} - \frac{1}{3}\mathbf{u}_{j-2}), \quad j = 2, 3, \dots, M. \tag{6}$$

where j is indexing the discretized time domain $[T, 0]$ with a time step Δt, consisting of M points, and $\mathbf{I}$ is an identity matrix of size $\mathbf{W}$.

1.3 Advanced treatment

Since conditioning of the small linear systems by which weights $\mathbf{w}^{(i)}$ are obtained is highly sensitive to the choice of the shape parameter for different stencils, a stabilization method known as RBF-GA[1] is used.

More details will be available in [2].

References

[1] B. Fornberg, E. Lehto, and C. Powell. Stable calculation of Gaussian-based RBF-FD stencils. *Computers & Mathematics with Applications*, 65(4):627 – 637, 2013.

[2] S. Milovanović and L. von Sydow. Radial Basis Function generated Finite Differences for Pricing Basket Options. *Manuscript in preparation*, 2016.