Reinforcement learning

Preliminaries – Probability and Markov Chains

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Objectives of these slides

- Introduce probability theory
- Define Markov chains and provide their basic properties
- Illustrate concepts through simple examples
The goal is to formally model "random" phenomena or experiments. Samples: all information you need in understanding an experiment is contained in a sample randomly selected by nature. Set of samples: $\Omega$, a sample $\omega$.

- Example 1: throwing a die, $\Omega = \{1, 2, 3, 4, 5, 6\}$
- Example 2: select a real number uniformly at random between 0 and 1, $\Omega = [0, 1]$
• A $\sigma$-algebra is a subset $\mathcal{F}$ of sets of the sample set such that:
  1. $\Omega \in \mathcal{F}$
  2. $F \in \mathcal{F} \Rightarrow cF \in \mathcal{F}$
  3. If $F_n \in \mathcal{F}$ for all $n \in \mathbb{N}$, then $\bigcup_{n \in \mathbb{N}} F_n \in \mathcal{F}$

• $\sigma$-algebra generated by a set $G$ of subsets is the smallest $\sigma$-algebra containing the subsets of $G$

• Example 1: throwing a die, $\sigma$-algebra = the set of all subsets of $\{1, 2, 3, 4, 5, 6\}$

• Example 2: select a real number uniformly at random between 0 and 1, the natural algebra is that generated by the open sets of $[0, 1]$, $\mathcal{F} = \mathcal{B}([0, 1])$
Probability Measures

• Measurable space: $(\Omega, \mathcal{F})$

• A probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$ is such that:
  1. $\mathbb{P}(\Omega) = 1$ and $\mathbb{P}(\emptyset) = 0$
  2. If $F_n \in \mathcal{F}$ for all $n \in \mathbb{N}$, and if $F_n \cap F_m = \emptyset$ when $n \neq m$, then

$$\mathbb{P}(\bigcup_{n \in \mathbb{N}} F_n) = \sum_{n \in \mathbb{N}} \mathbb{P}(F_n)$$

• Terminology: $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space, $F \in \mathcal{F}$ is an event

• Example 1: throwing a die, $\mathbb{P}(\omega) = 1/6$, for all $\omega \in \Omega$

• Example 2: select a real number uniformly at random between 0 and 1, $\mathbb{P}([0, x)) = x$, for all $x \leq 1$
Random Variables

- A random variable $X$ is a measurable function $X : \Omega \rightarrow \mathbb{R}$, i.e.,

$$\forall B \in \mathcal{B} (\mathbb{R}), \quad X^{-1}(B) \in \mathcal{F}$$

- **Example 1**: throw a die

$$X(\omega) = \begin{cases} 
0 & \text{if } \omega \text{ is even} \\
1 & \text{if } \omega \text{ is odd}
\end{cases}$$

- Interpretation: we run an experiment, and observe the value of a random variable. It provides partial information about the sample selected by nature.

- Distribution of $X$ defined by $\forall B \in \mathcal{B} (\mathbb{R}), \ \mathbb{P}[X \in B]$
Expectation

- Restrict attention to countable sample sets
- Probability space: \((\Omega, \mathcal{F}, \mathbb{P})\)
- The expectation of the r.v. \(X\) defined on \((\Omega, \mathcal{F}, \mathbb{P})\) is (if it exists):

\[
\mathbb{E}[X] = \sum_{a \in A} a \mathbb{P}[X = a],
\]

where \(A = \{X(\omega), \omega \in \Omega\}\)
Conditional Expectation

- Restrict attention to countable sample sets
- Probability space: $(\Omega, \mathcal{F}, \mathbb{P})$
- Conditional probability: for $F, G \in \mathcal{F}$,

$$
\mathbb{P}(F|G) = \frac{\mathbb{P}(F \cap G)}{\mathbb{P}(G)}
$$

- Let $X$ and $Y$ two r.v. defined on $(\Omega, \mathcal{F}, \mathbb{P})$, and let $A = X(\Omega)$ and $B = Y(\Omega)$. The conditional expectation of $X$ given $Y = b$, $b \in B$, is:

$$
\mathbb{E}[X|Y = b] = \sum_{a \in A} a \mathbb{P}(X = a|Y = b)
$$
• The r.v. \( Z = \mathbb{E}[X|Y] \) is defined by:

\[
Z(\omega) = \mathbb{E}[X|Y = b], \quad \text{if } Y(\omega) = b
\]

• Interpretation: \( Z \) is the expectation of \( X \) given that we know the value of \( Y \)

• **Example 1:** See slide 6. Define \( Y(\omega) = \omega \). Then:

\[
\mathbb{E}[Y|X] = \begin{cases} 
4 & \text{if } \omega \text{ is even} \\
3 & \text{if } \omega \text{ is odd}
\end{cases}
\]

• \( \mathbb{E}[X] = \mathbb{E}[\mathbb{E}[X|Y]] \)
\(\sigma\)-algebra Generated by r.v.

- Family of random variables on \((\Omega, \mathcal{F})\): \((X_i, i \in I)\)
- The \(\sigma\)-algebra generated by \((X_i, i \in I)\) is the smallest \(\sigma\)-algebra \(\mathcal{G}\) containing \(X_i^{-1}(B)\). \(\mathcal{G} = \sigma(X_i, i \in I)\)
- Interpretation: We run an experiment. Nature selects a sample \(\omega\). \(\mathcal{G}\) consists of those events \(F\) for which for all sample, you are able to decide whether \(F\) occurred or not by observing \((X_i(\omega), i \in I)\).
- **Example 1:** See Slide 6. \(\sigma(X) = \{\emptyset, \Omega, \{1, 3, 5\}, \{2, 4, 6\}\}\)
- \(\mathbb{E}[X|Y]\) is \(\sigma(Y)\)-measurable, i.e., for all \(B \in \mathcal{B}(\mathbb{R})\), \(\mathbb{E}[X|Y]^{-1}(B) \in \sigma(Y)\)
Independent Events

• Two events $A$ and $B$ are independent if,

$$P(A \cap B) = P(A) \cdot P(B)$$

or equivalently when $P(A) > 0$, if $P(B|A) = P(B)$
Independent Events

- Two events $A$ and $B$ are independent if,

\[ \mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B) \]

or equivalently when $\mathbb{P}(A) > 0$, if $\mathbb{P}(B|A) = \mathbb{P}(B)$

- Interpretation: The information about the outcome of $A$ does not help us to predict the outcome of $B$. 
Example: Consider throwing a die and flipping a coin simultaneously. Note that

\[ \Omega = \{1, 2, \ldots, 6\} \times \{H, T\}. \]

Define

\[ A = \text{die’s outcome is even} \]
\[ B = \text{coin’s flip is tail} \]

We have \( \mathbb{P}(B|A) = \mathbb{P}(A) \), so \( A \) and \( B \) are independent.
Independent Events

• Example. Consider an urn containing 5 white balls and 5 black balls. Pick two balls sequentially \textit{with replacement}. Define

\begin{align*}
A &= \text{the first ball is white} \\
B &= \text{the second ball is white}
\end{align*}

We have \( \mathbb{P}(B|A) = \mathbb{P}(B) \), so \( A \) and \( B \) are independent events.
Independent Events

- **Example.** Consider an urn containing 5 white balls and 5 black balls. Pick two balls sequentially *with replacement*. Define

  \[ A = \text{the first ball is white} \]
  \[ B = \text{the second ball is white} \]

  We have \( \mathbb{P}(B|A) = \mathbb{P}(B) \), so \( A \) and \( B \) are independent events.

- **Example.** Now pick two balls *without replacement*. Now \( \mathbb{P}(B|A) = \frac{4}{4+5} \) but \( \mathbb{P}(B) = \frac{1}{2} \). So \( A \) and \( B \) are dependent.
Independent Events

- A collection of events $A_1, \ldots, A_n$ are independent if for any subset $I \in \{1, \ldots, n\}$,

\[ \mathbb{P} \left( \bigcap_{i \in I} A_i \right) = \prod_{i \in I} \mathbb{P}(A_i). \]
Independent Events

- A collection of events $A_1, \ldots, A_n$ are independent if for any subset $I \in \{1, \ldots, n\}$,
  \[ P\left( \bigcap_{i \in I} A_i \right) = \prod_{i \in I} P(A_i). \]

- Events can be pairwise independent but not independent!
- **Example:** Flip two coins. Let

  \[
  A = \text{1st flip is tail}, \\
  B = \text{2nd flip is tail} \\
  C = \text{both flips are the same}
  \]

  Show that these events are pairwise independent but not jointly independent.
Independent Random Variables

- Two r.v. $X$ and $Y$ are *independent* if $\{X \in A\}$ and $\{Y \in B\}$ are independent events for all Borel sets $A$ and $B$.
- If $X$ and $Y$ are independent, then for each possible pair of values $a$ and $b$,

$$\mathbb{P}(X = a, Y = b) = \mathbb{P}(X = a) \mathbb{P}(Y = b).$$
Random variables $X_1, \ldots, X_n$ are mutually independent if for all $x_1, x_2, \ldots, x_n$,

$$
\mathbb{P}\left( \bigcap_{i=1}^{n} X_i = x_i \right) = \prod_{i=1}^{n} P(X_i = x_i).
$$

Moreover

$$
\mathbb{E}\left( \prod_{i=1}^{n} X_i \right) = \prod_{i=1}^{n} \mathbb{E}(X_i).
$$
A Useful Property for $\mathbb{E}(X)$

- If r.v. $X$ only takes non-negative values, then

$$\mathbb{E}(X) = \sum_{i=0}^{\infty} \mathbb{P}(X \geq i).$$
Markov Chains

- A stochastic dynamical system
- Probability space: $\left( \Omega, \mathcal{F}, \mathbb{P} \right)$
- Finite state space: $S = \{1, \ldots, |S|\}$
- A sequence of r.v.’s with values in $S$ is a Markov chain iff for all $n \geq 1$ and all $j, i_1, \ldots, i_n \in S$

$$\mathbb{P}(X_{n+1} = j | X_1 = i_1, \ldots, X_n = i_n) = \mathbb{P}(X_{n+1} = j | X_n = i_n)$$

- Transition matrix for homogenous Markov chains: $P$

$$P_{i,j} = \mathbb{P}(X_{n+1} = j | X_n = i), \quad \forall i, j \in S$$
Example: Reflected random walk

- $S = \{0, 1, \ldots, N\}$
- $P_{0,1} = P_{N,N-1} = 1,$
  $\forall i \neq 0, N, P_{i,i+1} = P_{i,i-1} = 1/2$
Kolmogorov Equations

- The distribution of the state at time $n$ is described by a row vector $\mu_n \in [0, 1]^{|S|}$

- Kolmogorov equation: $\mu_{n+1} = \mu_n P$

- $m$-steps transition: $\mu_{n+m} = \mu_n P^m$

\[(P^m)_{i,j} := p^m(i,j) = \mathbb{P}(X_{n+m} = j | X_n = i)\]

- Accessibility, Communication:

\[(i \rightarrow j) \iff (\exists m : p^m(i,j) > 0)\]

\[(i \leftrightarrow j) \iff (i \rightarrow j \text{ and } j \rightarrow i)\]
Communication Classes, Irreducibility

- By definition: each state communicates with itself
- Communication is an equivalence class
- Two communicating states are said to belong to the same communication class
- A finite Markov chain is irreducible iff there is a unique communication class
- A state constituting
Transition Graph

1 \rightarrow 2: p_0, 1 - p_1
2 \rightarrow 1: p_1, p_2
2 \rightarrow 3: 1 - p_1
3 \rightarrow 2: 1 - p_2
1 \rightarrow 3: 1 - p_0
State Classification

- Time to reach $i$: $\tau_i = \inf(n \geq 1 : X_n = i)$
- Recurrent state: $\mathbb{P}_i(\tau_i < \infty) = 1$ where $\mathbb{P}_i(\cdot) = \mathbb{P}(\cdot | X_0 = i)$
- Positive recurrent state: $\mathbb{E}_i(\tau_i) < \infty$
- Transient state: $\mathbb{P}_i(\tau_i < \infty) < 1$
- Recurrence is a class property:
  \[
i \leftrightarrow j \implies (i, j \text{ are both recurrent or transient})\]
- Number of visits: $N_i = \sum_{n \geq 1} 1_{X_n = i}$
  \[
  \mathbb{P}_i(\tau_i < \infty) = 1 \iff \mathbb{P}_i(N_i = \infty) = 1
  \]
In an irreducible finite Markov chain, all states are positive recurrent.
Periodicity

- The period of state $i$ is defined by: $\gcd\{n > 0 : p^n(i, i) > 0\}$
- A state is aperiodic if its period is equal to 1
- In an irreducible Markov, all states have the same period
- An irreducible Markov chain with period $d$ has a cyclic structure:

$$\exists S_0, \ldots , S_{d-1} : \bigcup_k S_k = S, S_d = S_0$$

$$\forall k, \forall i \in S_k, \sum_{j \in S_{k+1}} p(i, j) = 1$$
An irreducible Markov chain with period $d$ has a cyclic structure: for instance, order the states so that we get in order $S = S_0, \ldots, S_3$, then

$$P = \begin{pmatrix}
0 & A_0 & 0 & 0 \\
0 & 0 & A_1 & 0 \\
0 & 0 & 0 & A_2 \\
A_3 & 0 & 0 & 0
\end{pmatrix}$$
Stationary Distribution

- A distribution $\pi$ is stationary if: $\pi = \pi P$

- Global balance equations: $\pi$ is stationary iff:

  $$\forall i \in S, \quad \pi_i = \sum_j P_{j,i} \pi_j$$

- A finite irreducible Markov chain has a unique stationary distribution

  $$\forall i \in S, \quad \pi(i) = \frac{\mathbb{E}_0 \left[ \sum_{n \geq 1} 1_{X_n = i} 1_{n \leq \tau_0} \right]}{\mathbb{E}_0[\tau_0]}$$

  $$\pi_i = \frac{1}{\mathbb{E}_i[\tau_i]}$$
Ergodicity

• For a finite irreducible Markov chain:

$$\forall f : S \rightarrow \mathbb{R} : \sum_{i \in S} |f(i)| \pi_i < \infty$$

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} f(X_k) = \sum_{i \in S} f(i) \pi_i$$
Example: Reflected random walk

- What are the communication classes?
- Compute $\mathbb{E}_i[\tau_i]$
- Compute the stationary distribution
References

- Finite Markov chains and Algorithmic applications, O. Häggstrom, Cambridge Univ. Press, 2002
- Markov chains and Mixing Times, D. Levin, Y. Peres, E. Wilmer, AMS 2009
- Network Performance Analysis (Chapters 1 - 6), T. Bonald, M. Feuillet, Wiley, 2011