Aim: Illustrate the use of “locally optimal” proposals in the auxiliary particle filter (= fully adapted PF)

Outline:

1. Locally optimal proposals
2. When can they be computed?
3. Numerical illustration of fully adapted PF
Fully adapted particle filter
With the choices

**Resampling weights:** $\nu^i_{t-1} \propto w^i_{t-1} \ p(y_t | x^i_{t-1}), \ i = 1, \ldots, N$

**Propagation proposal:** $q(x_t | x_{t-1}, y_t) = p(x_t | x_{t-1}, y_t)$

we obtain weights $\tilde{w}^i_t = \text{const.} \Rightarrow w^i_t = \frac{1}{N}, \ i = 1, \ldots, N$

Referred to as the **fully adapted particle filter (FAPF)**
Locally optimal proposals

With the choices

**Resampling weights:** $\nu_{t-1}^i \propto p(y_t | x_{t-1}^i), i = 1, \ldots, N$

**Propagation proposal:** $q(x_t | x_{t-1}, y_t) = p(x_t | x_{t-1}, y_t)$

we obtain weights $\tilde{w}_t^i = \text{const.} \Rightarrow w_t^i = \frac{1}{N}, i = 1, \ldots, N$

Referred to as the **fully adapted particle filter (FAPF)**
Locally optimal proposals

**Bootstrap particle filter**

\[ w_{t-1}^i \quad p(x_t | x_{t-1}^i) \quad p(y_t | x_t^i) \]

**Fully adapted particle filter**

\[ p(y_t | x_{t-1}^i) \quad p(x_t | x_{t-1}^i, y_t) \quad \frac{1}{N} \]
ex) 1st order autoregressive conditional heteroskedasticity (ARCH) model:

\[ X_t = \sqrt{1 + 0.5X_{t-1}^2} V_t, \quad V_t \sim \mathcal{N}(0, 1), \]
\[ Y_t = X_t + E_t, \quad E_t \sim \mathcal{N}(0, r). \]

We simulate a data set and compare the bootstrap particle filter with the fully adapted particle filter, both using \( N = 100 \) particles.

**Evaluation criteria:** Estimator variance for the test function \( \varphi(x_t) = x_t, \)
\( t = 1, \ldots, 100. \)
Data set with $r = 1$
ex) ARCH model

Data set with $r = 0.1$ (high signal-to-noise ratio)
Data set with $r = 10$ (low signal-to-noise ratio)
Partially adapted particle filter
Partial adaptation

**Non-conjugate models:** approximate \( \bar{p}(x_t | x_{t-1}, y_t) \approx p(x_t | x_{t-1}, y_t) \) and \( \bar{p}(y_t | x_{t-1}) \approx p(y_t | x_{t-1}) \). E.g., local linearization, variational approximation, ...  

Approximate model used only to define the proposal!
Partial adaptation

Non-conjugate models: approximate $\bar{p}(x_t | x_{t-1}, y_t) \approx p(x_t | x_{t-1}, y_t)$ and $\bar{p}(y_t | x_{t-1}) \approx p(y_t | x_{t-1})$. E.g., local linearization, variational approximation, . . .

Approximate model used only to define the proposal!

Care needs to be taken so that the approximations are suitable to use as importance sampling proposals. (Heavier tails than target.)
Locally optimal proposals: Proposals that take all the available information in the current measurement $y_t$ into account.

Fully adapted particle filter: An auxiliary variable that uses locally optimal proposals both for the ancestor indices (auxiliary variables) and for the state variable.

Partially adapted particle filter: An auxiliary particle filter that uses some suboptimal proposals (e.g. an approximation of the locally optimal ones) which still take the current measurement $y_t$ into account.