RBF approximations for solving Navier-Stokes equations

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Project description

The Navier-Stokes equations describe the motion of viscous fluid substances in a medium. These equations are classified as parabolic PDEs which are obtained by applying the Newton’s second law to fluid motion, together with the assumption that the stress in the fluid is the sum of a diffusing viscous term (proportional to the gradient of velocity) and a pressure term. For velocity vector solution $\mathbf{u} \equiv \mathbf{u}(x,t)$ and pressure function $p \equiv p(x,t)$ at position $x \in \mathbb{R}^d$ ($d = 2, 3$) and time $t \geq 0$, the incompressible Navier-Stokes equation reads as

\[
\begin{align*}
\nabla \cdot \mathbf{u} &= 0, \quad \text{(continuity equation)} \\
\rho \left( \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} \right) &= -\nabla p + \nu \nabla^2 \mathbf{u} + \mathbf{f} \quad \text{(momentum equation)}
\end{align*}
\]

with initial conditions $\mathbf{u}(x,0) = \mathbf{u}_0(x)$ and $p(x,0) = p_0(t)$, where $\mathbf{u}$ is the velocity vector (flow velocity), $p$ is the pressure, $\rho$ is the density, $\nu$ is the viscosity constant, and $\mathbf{f}$ is any external force. The continuity equation represents the conservation of mass while the second equation represents the conservation of momentum. On a bounded domain $\Omega \subset \mathbb{R}^d$ as a medium, proper boundary conditions should also be imposed for both $\mathbf{u}$ and $p$. In the momentum equation, the first term $\frac{\partial \mathbf{u}}{\partial t}$ describes the change of velocity in time, the second term, $(\mathbf{u} \cdot \nabla) \mathbf{u}$ is the convective term. Indeed, $\rho \left( \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} \right) = \rho \frac{D\mathbf{u}}{Dt} = ma$ where $m$ is the mass and $a$ is the acceleration. Here $\frac{D}{Dt}$ is the total (or material) derivative. The right-hand side of the momentum equations then serves for sum of internal and external forces, i.e., $\sum F$, as the Newton second law $ma = \sum F$ suggests. The term $-\nabla p$ ensures that fluid flows in the direction of largest change in pressure. For a Newtonian fluid the viscosity operates as a diffusion of momentum, so we also have the term $\nu \nabla^2 \mathbf{u}$. Finally, the force term $\mathbf{f}$ can also be written as $\mathbf{f} = \rho \mathbf{g}$ where $\mathbf{g}$ is a body acceleration acting on the continuum, for example gravity, any inertial acceleration, electrostatic acceleration, and so on.

Finite elements and finite difference methods are widely used for solving the incompressible Navier-Stokes equation. See Figure 1 for surface plot of the velocity magnitude and streamlines (top), and surface plot of the pressure (bottom) which are computed by the finite element method in COMSOL.

The streamlines of a vector field are the paths along which a massless fluid particle would
The aim of this project is to employ radial basis function (RBF) approximations to discretize the momentum equation by imposing the continuity equation (divergence free property) directly into the RBF approximation space. RBF approximations are based on scattered points on the medium $\Omega$ instead of a background triangulation or a grid mesh which are used in finite element and finite difference methods. The RBF approximation of velocity $u$ is written as

$$u(x, t) \approx s(x, t) = \sum_{j=1}^{N} \Phi_{\text{div}}(\| x - x_j \|_2) c_j(t)$$

where $\Phi_{\text{div}} \in \mathbb{R}^{d \times d}$ is a matrix-valued RBF with divergence-free columns, $\{x_1, x_2, \ldots, x_N\}$ is a set of scattered points in $\Omega$ and $c_j \in \mathbb{R}^d$ are unknown coefficients. Then we can simply show that $\nabla \cdot s = 0$. The coefficient vector $c(t)$ at each time step $t$ is obtained such that $s$ (approximately) satisfies the momentum equation and the prescribed boundary conditions. We also approximate the pressure by

$$p(x, t) = \sum_{j=1}^{N} b_j(t) \phi(\| x - x_j \|_2)$$

where $\phi$ is a scaler valued RBF and $b_j(t)$ are unknown coefficients in time level $t$. Implementation details, time integration schemes, stability analyses, and possible extensions of the proposed method will be considered in the project.