Fractional-space diffusion equations (FDEs) are used to describe diffusion phenomena, that cannot be modeled by the second order diffusion equations. More precisely, when a fractional derivative replaces a second derivative in a diffusion model, it leads to enhanced diffusion. The FDEs are of numerical interest, since there exist only few cases in which the analytic solution is known. As a consequence, in the past ten years, many methods have been proposed for solving numerically FDEs problems. In [4, 5] Meerschaert and Tadjrean introduced an unconditionally stable method for approximating the FDEs: from a numerical linear algebra viewpoint, it is worth noticing that the resulting linear systems show a strong structure and indeed the related coefficient matrices can be seen a sum of two diagonal times Toeplitz matrices (see [13]). Exploiting such a structure, in [12] the authors employed the conjugate gradient method normal residual (CGNR) and numerically showed that its convergence is fast when the diffusion coefficients are small, that is in this case the resulting linear system is well-conditioned. On the other hand, when the diffusion coefficient are not small, the problem becomes ill-conditioned and the convergence of the CGNR method slows down. To avoid the resulting drawback, in [7] Pang and Sun proposed a multigrid method that converges very fast, even in the ill-conditioned case. The linear convergence of such a method has been proved only in the case of constant and equal diffusion coefficients. With the same purpose, Lei and Sun used the CGNR method with a circulant preconditioner and verified that it converges superlinearly (see [3]), again in the case of constant diffusion coefficients. Both methods preserve the computational cost per iteration of $O(N \log N)$ operations, typical of the CGNR method when applied to Toeplitz type structures.

While the numerical linear algebra problems have been studied, tested, and also theoretically analyzed in some detail in the one-dimensional setting, the same is not true when the domain of the FDE under consideration is multidimensional and when the diffusion coefficients are nonconstant. In more dimensions (even just in 2D), we have to expect some theoretical difficulties in generalizing the known techniques, as indicated by the theoretical barriers proven in [11, 10, 6], while for the case of nonconstant diffusion coefficients we could use the spectral results in [8, 9]. The aim of the project is to give a preliminary numerical experimentation for 2D FDEs, with special attention to the generalization of the preconditioners in connection with Krylov methods.
Problem setting

In detail for 1D problems we are interested in the following initial-boundary value problem

\[
\begin{aligned}
\frac{\partial u(x,t)}{\partial t} &= d_+(x,t) \frac{\partial^\alpha u(x,t)}{\partial x^\alpha} + d_-(x,t) \frac{\partial^\alpha u(x,t)}{\partial x^{-\alpha}} + f(x,t), \quad (x,t) \in (L, R) \times (0, T], \\
u(L, t) &= u(R, t) = 0, \\
u(x, 0) &= u_0(x),
\end{aligned}
\]

where \(\alpha \in (1, 2)\) is the fractional derivative order, \(f(x,t)\) is the source term and the nonnegative functions \(d_\pm(x,t)\) are the diffusion coefficients. The right-handed (+) and the left-handed (-) fractional derivatives in (1) are defined in Riemann-Liouville form as follows

\[
\begin{align*}
\frac{\partial^\alpha u(x,t)}{\partial_+ x^\alpha} &= \frac{1}{\Gamma(n - \alpha)} \int_L^x (x - \xi)^{\alpha-n-1} u(\xi,t) \, d\xi, \\
\frac{\partial^\alpha u(x,t)}{\partial_- x^\alpha} &= \frac{1}{\Gamma(n - \alpha)} \int_x^R (\xi - x)^{\alpha-n-1} u(\xi,t) \, d\xi,
\end{align*}
\]

where \(n\) is an integer such that \(n - 1 < \alpha \leq n\) and \(\Gamma(\cdot)\) is the gamma function. If \(\alpha = m\), with \(m \in \mathbb{N}\), the fractional derivatives reduce to the standard integer derivatives, i.e.,

\[
\frac{\partial^m u(x,t)}{\partial_+ x^m} = \frac{\partial^m u(x,t)}{\partial x^m}, \quad \frac{\partial^m u(x,t)}{\partial_- x^m} = (-1)^m \frac{\partial^m u(x,t)}{\partial x^m}.
\]

Let us observe that when \(\alpha = 2\) the equation in (1) reduces to a parabolic partial differential equation (PDE), while when \(\alpha = 1\) it becomes a hyperbolic PDE. From a numerical point of view, an interesting definition of the fractional derivatives is the shifted Grünwald definition given by

\[
\begin{align*}
\frac{\partial^\alpha u(x,t)}{\partial_+ x^\alpha} &= \lim_{\Delta x \to 0^+} \frac{1}{\Delta x^\alpha} \sum_{k=0}^{\lfloor (x-L)/\Delta x \rfloor} g_k^{(\alpha)} \ u(x - (k-1)\Delta x, t), \\
\frac{\partial^\alpha u(x,t)}{\partial_- x^\alpha} &= \lim_{\Delta x \to 0^+} \frac{1}{\Delta x^\alpha} \sum_{k=0}^{\lfloor (R-x)/\Delta x \rfloor} g_k^{(\alpha)} \ u(x + (k+1)\Delta x, t),
\end{align*}
\]

where \(\lfloor \cdot \rfloor\) is the floor function, while \(g_k^{(\alpha)}\) are the alternating fractional binomial coefficients defined as

\[
g_k^{(\alpha)} = (-1)^k \binom{\alpha}{k} = \frac{(-1)^k}{k!} (\alpha - 1) \cdots (\alpha - k + 1) \quad k = 0, 1, \ldots
\]

with the formal notation \(\binom{\alpha}{0} = 1\). The shifted Grünwald formulas are numerically relevant since, from (2), we can define the following estimates of the left and right-handed fractional derivatives

\[
\begin{align*}
\frac{\partial^\alpha u(x,t)}{\partial_+ x^\alpha} &= \frac{1}{\Delta x^\alpha} \sum_{k=0}^{\lfloor (x-L)/\Delta x \rfloor} g_k^{(\alpha)} \ u(x - (k-1)\Delta x, t) + O(\Delta x), \\
\frac{\partial^\alpha u(x,t)}{\partial_- x^\alpha} &= \frac{1}{\Delta x^\alpha} \sum_{k=0}^{\lfloor (R-x)/\Delta x \rfloor} g_k^{(\alpha)} \ u(x + (k+1)\Delta x, t) + O(\Delta x).
\end{align*}
\]

In [4] Meerschaert and Tadjrean proved that the implicit Euler method based on the shifted Grünwald formula is consistent and unconditionally stable. Let us fix two positive integers \(N, M\), and define the following partition of \([L, R] \times [0, T]\), i.e.,

\[
\begin{align*}
x_i &= L + i\Delta x, \quad \Delta x = \frac{(R-L)}{N+1}, \quad i = 0, \ldots, N + 1, \\
t_m &= m\Delta t, \quad \Delta t = \frac{T}{M}, \quad m = 0, \ldots, M.
\end{align*}
\]
More in detail, the idea that underlies the Meerschaert-Tadjrean method is to combine a discretization in time of equation (1) by an implicit Euler method, with a discretization in space of the fractional derivatives by a shifted Grünwald estimate, i.e.,
\[
\frac{u(x_i, t_m) - u(x_i, t_{m-1})}{\Delta t} = d_{\pm,i}^{(m)} \frac{\partial^\alpha u(x_i, t_m)}{\partial x^\alpha} + d_{\pm,i}^{(m)} \frac{\partial^\alpha u(x_i, t_m)}{\partial x^\alpha} + f_i^{(m)} + O(\Delta t),
\]
where \(d_{\pm,i}^{(m)} := d_{\pm}(x_i, t_m)\), \(f_i^{(m)} := f(x_i, t_m)\) and
\[
\frac{\partial^\alpha u(x_i, t_m)}{\partial x^\alpha} = \frac{1}{\Delta x^\alpha} \sum_{k=0}^{i+1} g_k^{(a)} u(x_{i-k+1}, t_m) + O(\Delta x),
\]
\[
\frac{\partial^\alpha u(x_i, t_m)}{\partial x^\alpha} = \frac{1}{\Delta x^\alpha} \sum_{k=0}^{N-i+2} g_k^{(a)} u(x_{i+k-1}, t_m) + O(\Delta x).
\]

The resulting finite difference approximation scheme is then
\[
\frac{u_i^{(m)} - u_i^{(m-1)}}{\Delta t} = \frac{d_{\pm,i}^{(m)} + i+1}{\Delta x^\alpha} \sum_{k=0}^{i+1} g_k^{(a)} u_i^{(m)} + \frac{d_{\pm,i}^{(m)} N-i+2}{\Delta x^\alpha} \sum_{k=0}^{N-i+2} g_k^{(a)} u_i^{(m)} + f_i^{(m)}
\]
where by \(u_i^{(m)}\) we denote a numerical approximation of \(u(x_i, t_m)\). The previous approximation scheme can be written in matrix form as (see [13])
\[
\left(\nu_{M,N} I + D_+^{(m)} T_{\alpha,N} + D_-^{(m)} T^T_{\alpha,N}\right) u^{(m)} = \nu_{M,N} u^{(m-1)} + \Delta x^\alpha f^{(m)},
\]
where \(\nu_{M,N} = \frac{\Delta x^\alpha}{\Delta t} \), \(u^{(m)} = [u_1^{(m)}, \ldots, u_N^{(m)}]^T\), \(f^{(m)} = [f_1^{(m)}, \ldots, f_N^{(m)}]^T\), \(D_\pm^{(m)} = \text{diag}(d_{\pm,1}^{(m)}, \ldots, d_{\pm,N}^{(m)})\), \(I\) is the identity matrix of order \(N\) and
\[
T_{\alpha,N} = -\begin{bmatrix}
    g_0^{(a)} & g_1^{(a)} & 0 & \ldots & 0 & 0 \\
    g_2^{(a)} & g_1^{(a)} & g_0^{(a)} & \ldots & 0 \\
    \vdots & \ddots & \ddots & \ddots & \vdots \\
    \vdots & \ddots & \ddots & \ddots & 0 \\
    g_N^{(a)} & \cdots & \cdots & g_2^{(a)} & g_1^{(a)} \\
    g_N^{(a)} & g_{N-1}^{(a)} & \cdots & \cdots & g_2^{(a)} & g_1^{(a)} \end{bmatrix}_{N \times N}
\]
is a lower Hessenberg Toeplitz matrix. The fractional binomial coefficients \(g_k^{(a)}\) satisfy few properties, summarized in the following proposition (see [4, 5, 13]).

\textbf{Proposition 1.} Let \(\alpha \in (1, 2)\) and \(g_k^{(a)}\) be defined as in (3). Then we have
\[
\left\{\begin{array}{ll}
g_0^{(a)} = 1, & g_1^{(a)} = -\alpha, \quad g_0^{(a)} > g_2^{(a)} > g_3^{(a)} > \ldots > 0, \\
\sum_{k=0}^{\infty} g_k^{(a)} = 0, & \sum_{k=0}^{n} g_k^{(a)} < 0, \quad n \geq 1.
\end{array}\right.
\]

From here onwards, we denote the coefficient matrix of the linear system (3) by \(M_{\alpha,N}^{(m)}\), that is
\[
M_{\alpha,N}^{(m)} = \nu_{M,N} I + D_+^{(m)} T_{\alpha,N} + D_-^{(m)} T^T_{\alpha,N}.
\]

Using Proposition 1, it can be shown that \(M_{\alpha,N}^{(m)}\) is strictly diagonally dominant and then non singular (see [13]), for every choice of the parameters \(m \geq 0, N \geq 1, \alpha \in (1, 2)\). A spectral analysis of the matrix \(M_{\alpha,N}^{(m)}\) has been provided in [1] and several preconditioning strategies has been investigated [1, 2, 3, 4, 5]: as already mentioned, the thesis project will concern a detailed analysis of the problem in the multidimensional setting, just starting in the case of 2D FDEs, and considering, if time permits, also variable coefficients FDEs.
References


