A shock-capturing finite difference method for the compressible Euler equations
(Master thesis project 30 ECTS)

Project description

The compressible Euler equations model inviscid fluid flow and are of relevance in an abundance of engineering applications. The governing equations are given by the conservation of mass, momentum and energy. In one space-dimension the system of partial differential equations (PDE) reads:

\[
\begin{align*}
\partial_t \bar{q} + \partial_x F(\bar{q}) &= 0, \quad x \in \Omega, \quad t \geq 0, \\
\mathcal{L} \bar{q} &= \bar{g}, \quad x \in \partial \Omega, \quad t \geq 0, \\
\bar{q} &= \bar{q}_0, \quad x \in \Omega, \quad t = 0,
\end{align*}
\]

(1)

where

\[
\bar{q} = \begin{bmatrix} \rho \\ \rho u \\ e \end{bmatrix}, \quad F(q) = \begin{bmatrix} \rho u \\ \rho (u^2 + p) \\ (e + p)u \end{bmatrix},
\]

\[
\mathcal{L} \bar{q} = \bar{g}
\]

is a set of well-posed boundary conditions, and \( \bar{q}_0 \) is the initial condition. The unknown fields are density \( \rho \), velocity \( u \), and internal energy \( e \). The equations are closed by an equation of state, here for a perfect ideal gas with a ratio of specific heat \( \gamma \)

\[
p = (\gamma - 1)(e - \frac{1}{2}u^2).
\]

Solving (1) analytically is in most cases not tractable, and one must therefore turn to approximate solutions, obtained via numerical methods. In the present project, a special class of high-order accurate finite difference methods (FDM) referred to as summation-by-parts (SBP) FDMs will be utilized. SBP FDMs provide an efficient discretization method, in particular for hyperbolic PDEs such as (1).

A difficulty in solving (1) results from the equations being non-linear, such that shocks (discontinuities in the solution) may form. This poses a significant challenge for numerical methods, where approximating derivatives of discontinuous fields result in unphysical spurious oscillations unless treated. One way to stabilize the numerical solution is via viscous regularization, where an artificial viscous term \( \partial_x (\varepsilon \partial_x \bar{q}) \) is added to the equations such that discontinuities are diffused. Here \( \varepsilon \) denotes the artificial viscosity. The viscosity should vanish in the limit of grid refinement such that the regularized scheme is consistent with the continuous equations. Moreover, the viscosity should only be applied in the vicinity of discontinuities, such that fine features in the solution are preserved away from shocks. Discretizing (1) in space using SBP FDMs and stabilizing with artificial viscosity results in

\[
\begin{align*}
\partial_t \bar{q} + D_x F(\bar{q}) &= D_{xx}(\varepsilon) \bar{q}, \quad t \geq 0, \\
\bar{q} &= \bar{q}_0, \quad t = 0,
\end{align*}
\]

(2)

where \( \bar{q} \) is the numerical approximation of the solution vector, \( D_x \approx \partial_x \), and \( D_{xx}(\varepsilon) \approx \partial_x \varepsilon \partial_x \). In addition, one needs to impose the discrete boundary conditions \( \mathcal{L} \bar{q} = \bar{g} \) in a stable fashion, see e.g \([2, 4]\).

In this project, residual-based artificial viscosity (RV) will be employed, where \( \varepsilon \) is computed based on the PDE residuals. The RV method provides high-order accurate, and robust shock-capturing and has been successfully employed in a number of numerical methods for PDEs including finite- and spectral elements, and residual basis function methods \([1]\). For FDMs the RV method has been developed for scalar PDEs \([4]\) but has yet to be extended to systems of PDEs. The aim of the project is therefore to develop an RV FDM for the compressible Euler equations.
Project topics

The project topics are summarized below.

1. Analyze energy stability of (1) including boundary conditions using linearized theory [3].

2. Analyze semi-discrete energy stability of (2) including boundary conditions using linearized theory [3].

3. Formulate a residual-based artificial viscosity for (2).

4. Evaluate the method on 1D benchmark problems, such as the Sod shock tube problem.

5. If time permits, extend the method to a 2D multi-block finite difference setting, to study benchmark problems such as flow over forward-facing steps.

The numerical solvers will be implemented using MATLAB or python. SBP finite difference operators will be provided.

Relevant courses

The course *Advanced Numerical Methods* is strongly recommended for the present project.

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References


