

Delayed Sampling and Automatic Rao–Blackwellization of Probabilistic Programs

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Uppsala University

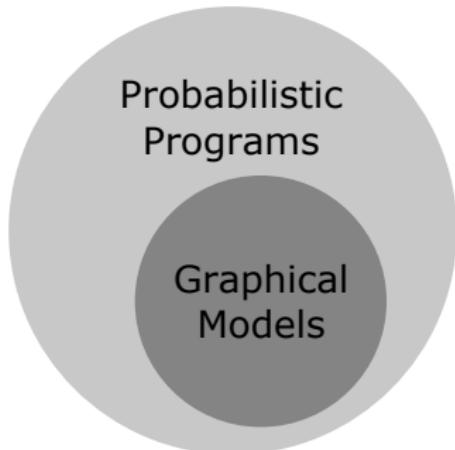
- + Daniel Lundén (KTH)
- + Jan Kudlicka (Uppsala)
- + David Broman (KTH)
- + Thomas Schön (Uppsala)

Outline

1. Brief introduction to probabilistic programming.
2. Delayed sampling to automate variance reduction techniques.
3. Examples.

What is a probabilistic program?

A **probabilistic program** encodes a **probabilistic model** according to the semantics of a particular **probabilistic programming language**.



Probabilistic programs extend graphical models with support for **stochastic branching**.

A Probabilistic Programming Language (PPL)

The particular PPL here is **Birch**, which is currently being developed at Uppsala University.

- ▶ It is the successor of **LibBi** (www.libbi.org).
- ▶ It is a probabilistic and object-oriented language.
- ▶ It compiles down to C++.
- ▶ It is not yet released, but is not a secret.
- ▶ It will be open source.

Probabilistic programs as stochastic processes

As a probabilistic program runs, its memory state evolves **dynamically** and **stochastically** in time.

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- ▶ Let $k = 1, 2, 3, \dots$ denote a sequence of **checkpoints**.
- ▶ Let $(X_{1:k})_{k=1}^{\infty}$ denote the (memory) state of the running program at these checkpoints, where $X_{1:k} \in \mathbb{X}_{1:k}$ and $\mathbb{X}_{1:k} = \mathbb{X}_k \times \mathbb{X}_{1:k-1}$.
- ▶ The state transitions according to $X_k | x_{1:k-1} \sim p_k(dx_k | x_{1:k-1})$.
- ▶ At each checkpoint we can manipulate the running program: pause execution, inspect memory state, consider distributions over that state, modify that state. This is what facilitates inference.

Two checkpoint types

A typical formulation uses two checkpoint types:

- ▶ **sample** to sample a random variable.
- ▶ **observe** to condition on a random variable having some value.

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If we also run N instances of the program simultaneously and add resampling, we have a **bootstrap particle filter**.

Canonical Example with Two Checkpoints

Program	Checkpoint
a ~ Gaussian(0.0, 1.0);	
b ~ Gaussian(a, 1.0);	
c ~ Gaussian(b, 1.0);	
d ~ Gaussian(b, 1.0);	
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stdout.print(c);	

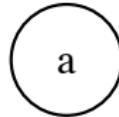
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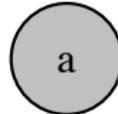
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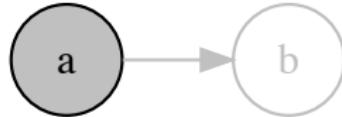
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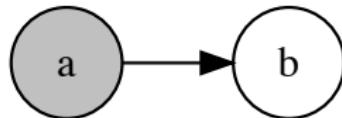
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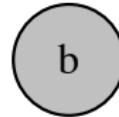
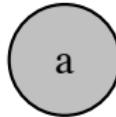
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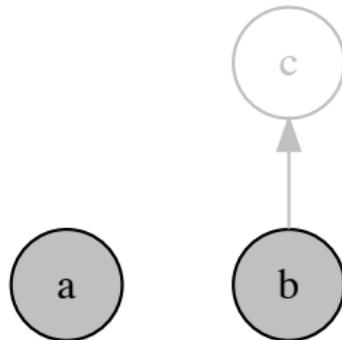
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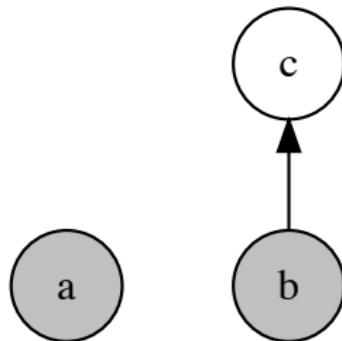
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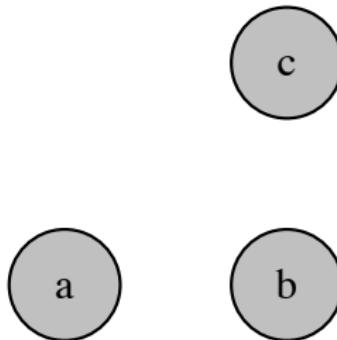
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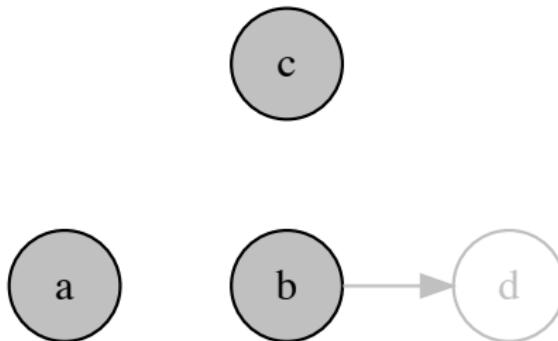
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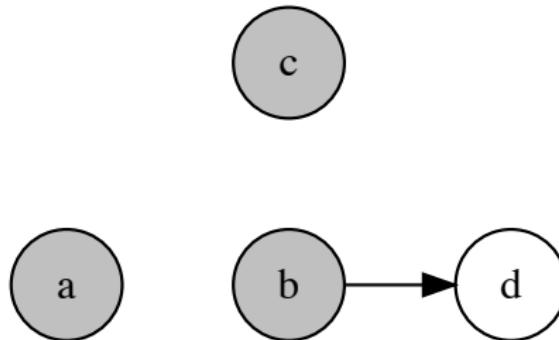
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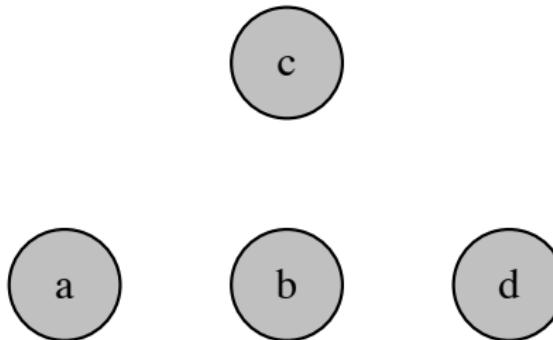
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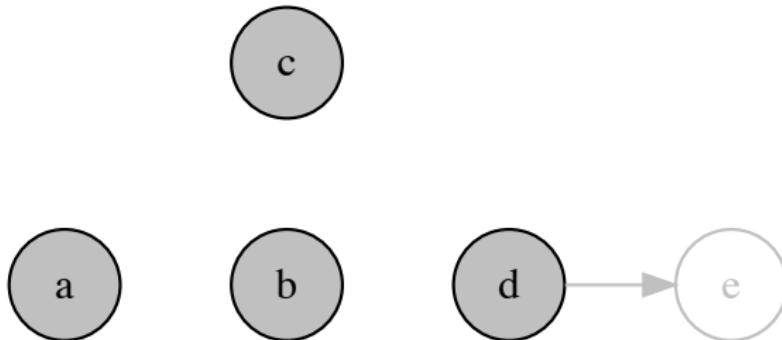
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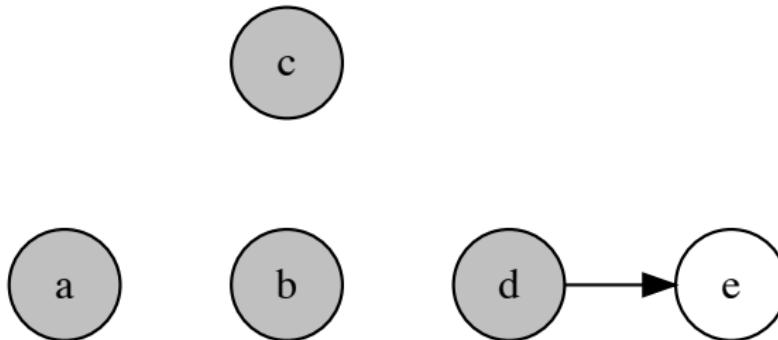
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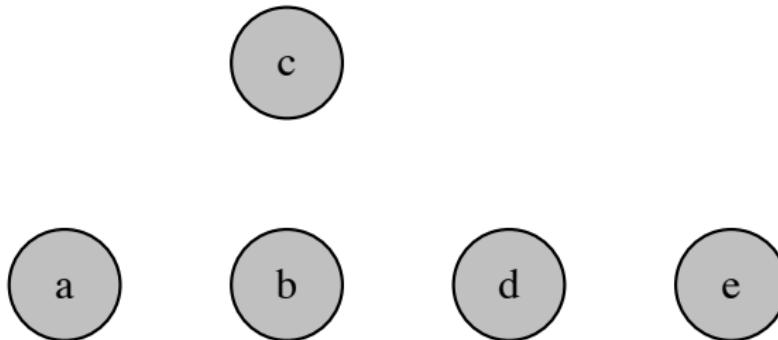
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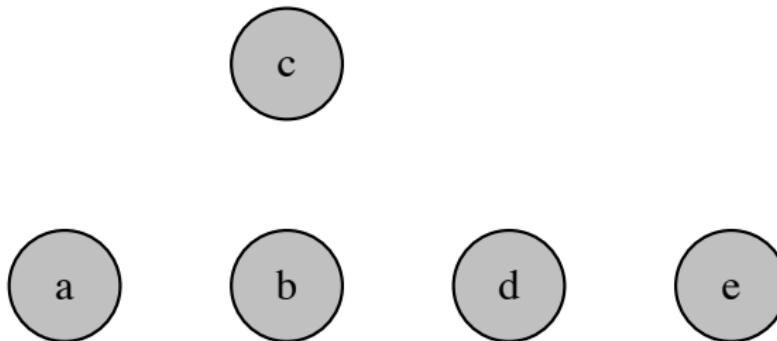
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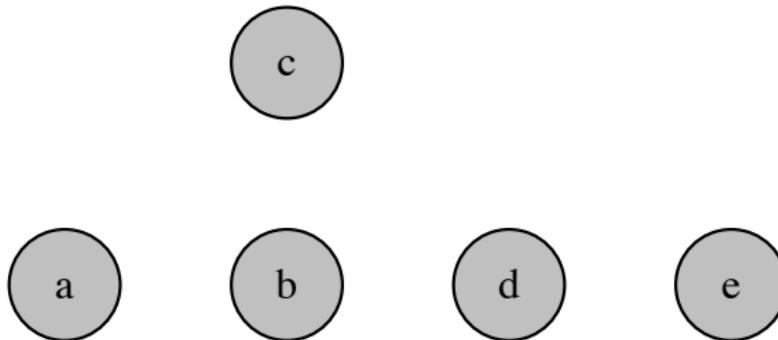
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This **eager sampling** approach misses opportunities for variance reduction such as Rao–Blackwellization, variable elimination, and locally-optimal proposals.

- ▶ We would like to apply such improvements in an automatic way that does not require modification of the probabilistic program.
- ▶ But we don't know the model structure until we execute the program.
- ▶ So some heuristic decisions must be made without complete knowledge of the model structure.

Delayed sampling

We use three checkpoint types:

- ▶ **assume** to initialize a random variable with some distribution.
- ▶ **observe** to condition, given some value for a random variable.
- ▶ **value** when an actual variate is required.

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Between **assume** and **value**, the distribution for a random variable can be updated using analytical relationships, such as conjugate priors.

The graph

- ▶ Alongside the state X , we maintain a graph $G = (V, E)$ that evolves dynamically as the program executes. At any time it represents only a fraction of the full model.
- ▶ We partition V into three disjoint sets according to three states:

$I \subseteq V$, the set of nodes in an  initialized state,

$M \subseteq V$, the set of nodes in a  marginalized state, and

$R \subseteq V$, the set of nodes in a  realized state.

- ▶ Each node transitions through the states in this order.

The graph

- ▶ **Initialize** inserts a new node v into the graph. If v has a parent, then the edge (u, v) is inserted.

This moves v to the  initialized state.

- ▶ **Marginalize** updates the distribution of node v by marginalizing over its parent.

This moves v to the  marginalized state.

- ▶ **Sample** or **Observe** assigns a value to v by either sampling or observing, respectively.

This moves v to the  realized state.

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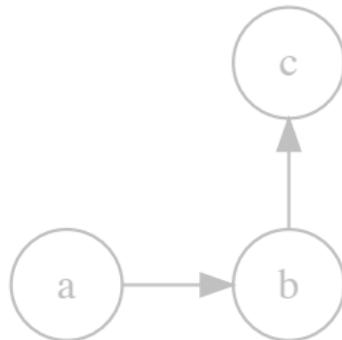
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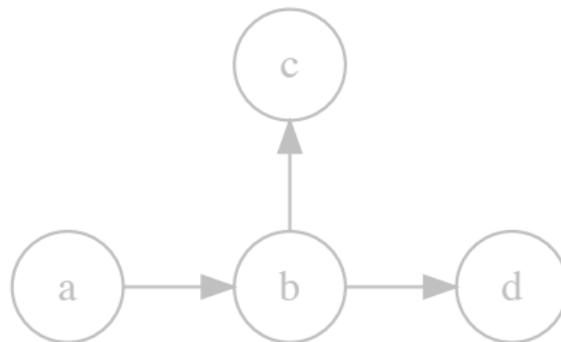
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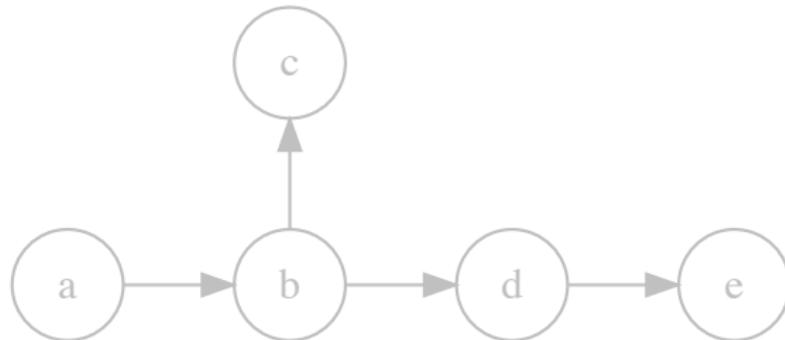
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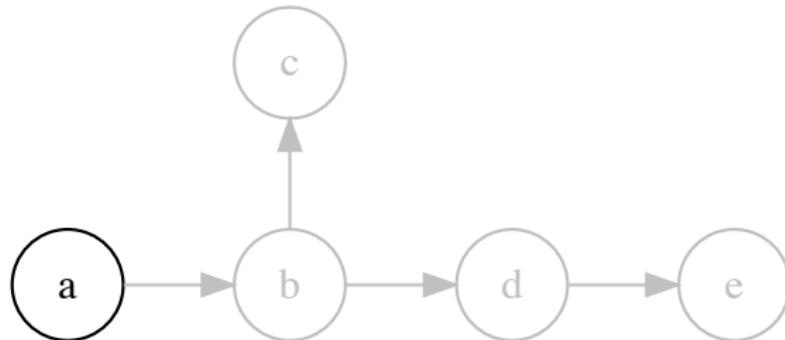
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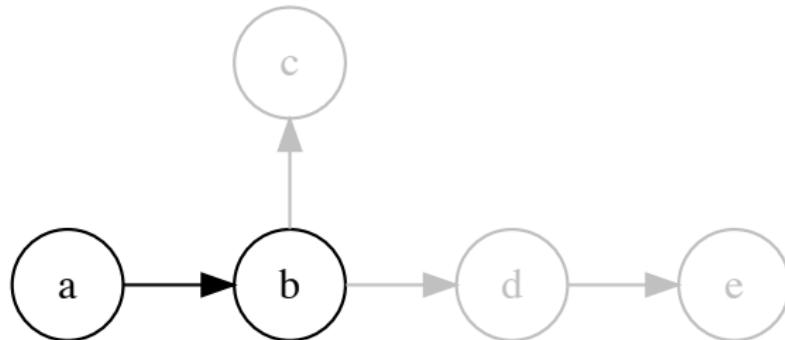
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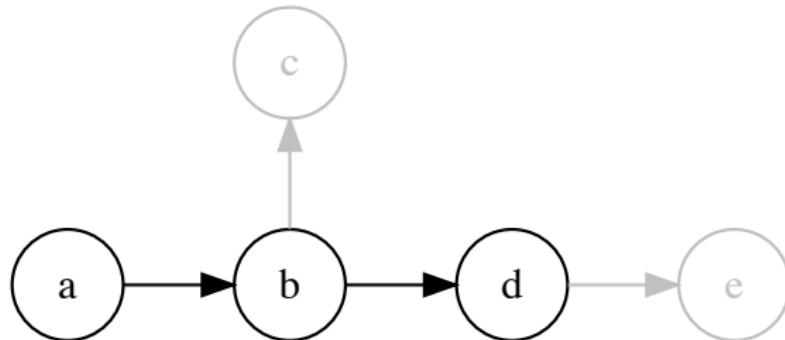
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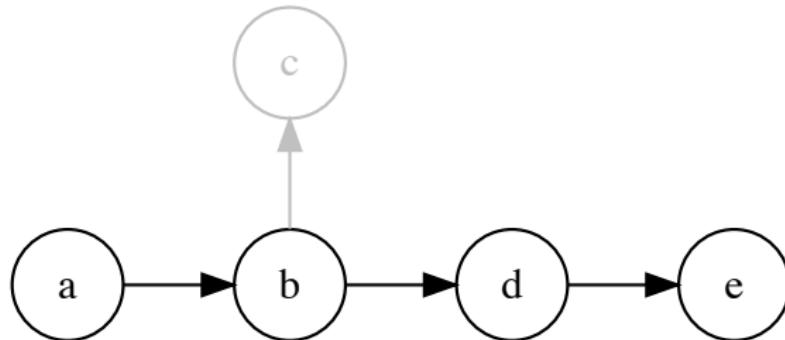
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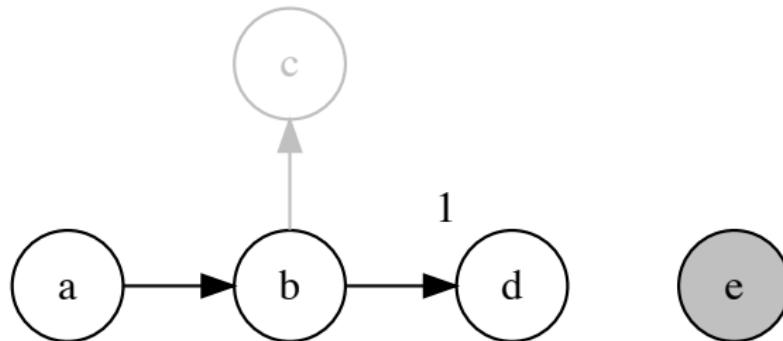
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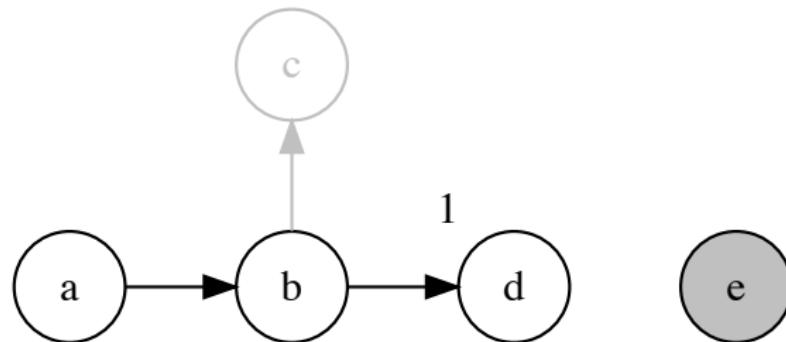
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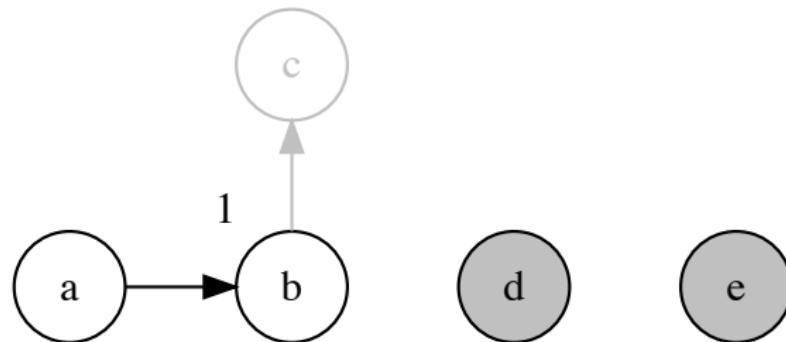
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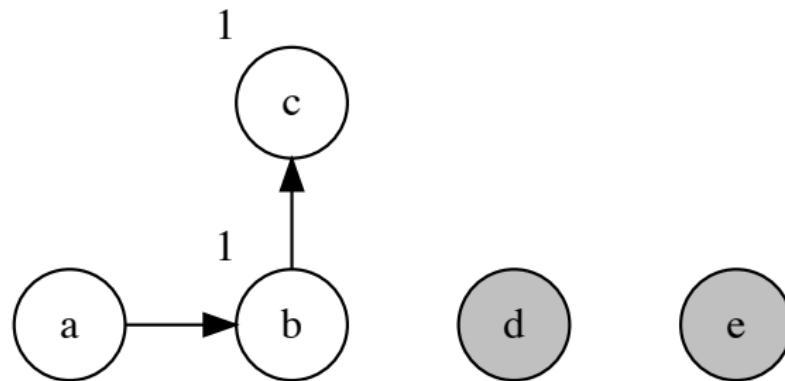
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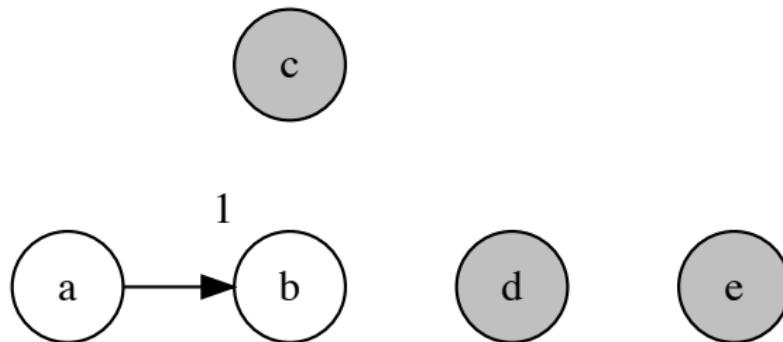
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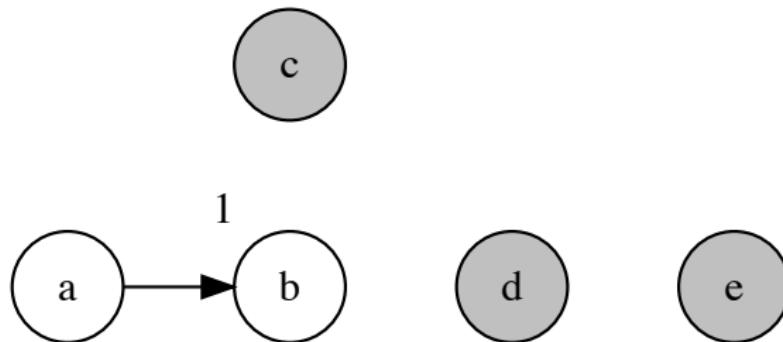
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Invariant conditions

1. The graph is a forest of disjoint trees.

2. If a node is  marginalized

then its parent is  marginalized

3. A node has at most one child that is  marginalized

These imply that the marginalized nodes form a path: from the root to a tip.

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These imply that the marginalized nodes form a path: from the root to a tip.

We refer to this path as the **M-path**, and the deepest node on this path as the **terminal node**.

The rules

We should not **Sample** or **Observe** just any node.

We should only **Sample** or **Observe** the **terminal** node of the **M-path**.

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To ensure this, we define two new operations:

- ▶ **Prune** to shorten the M-path by backward sampling.
- ▶ **Graft** to extend the M-path by forward marginalization.

We can use these operations to turn any node into the terminal node before we **Sample** or **Observe** it.

Canonical Example Revisited

Code

```
a ~ Gaussian(0.0, 1.0);  
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Checkpoint

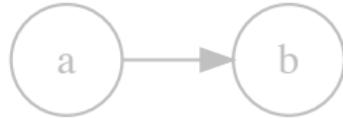
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e ~ Gaussian(d, 1.0);	
stdout.print(c);	



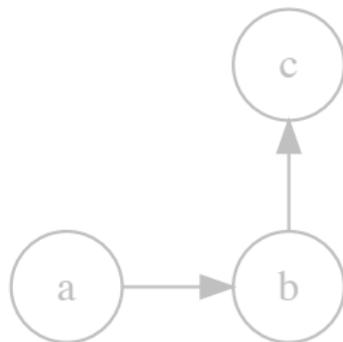
Canonical Example Revisited

Code

```
a ~ Gaussian(0.0, 1.0);  
b ~ Gaussian(a, 1.0);  
c ~ Gaussian(b, 1.0);  
d ~ Gaussian(b, 1.0);  
e ~ Gaussian(d, 1.0);  
stdout.print(c);
```

Checkpoint

assume c



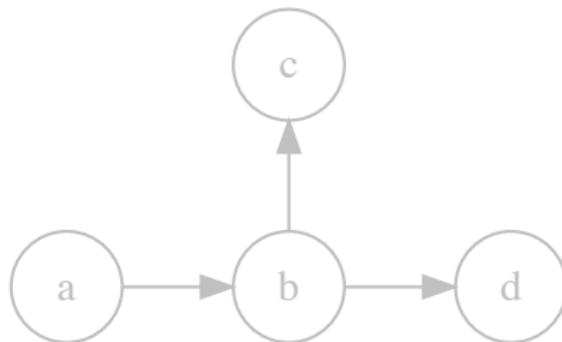
Canonical Example Revisited

Code

```
a ~ Gaussian(0.0, 1.0);  
b ~ Gaussian(a, 1.0);  
c ~ Gaussian(b, 1.0);  
d ~ Gaussian(b, 1.0);  
e ~ Gaussian(d, 1.0);  
stdout.print(c);
```

Checkpoint

assume d



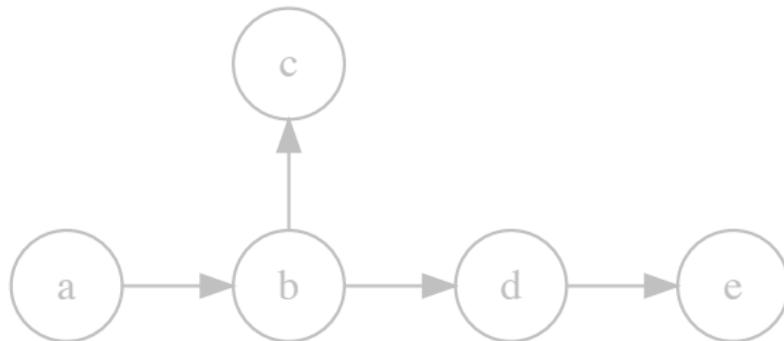
Canonical Example Revisited

Code

```
a ~ Gaussian(0.0, 1.0);  
b ~ Gaussian(a, 1.0);  
c ~ Gaussian(b, 1.0);  
d ~ Gaussian(b, 1.0);  
e ~ Gaussian(d, 1.0);  
stdout.print(c);
```

Checkpoint

observe e



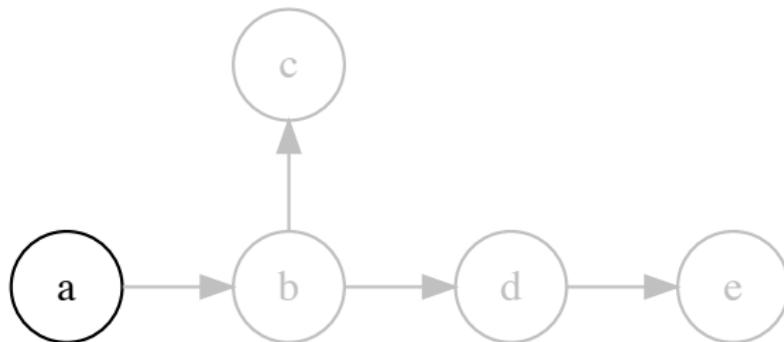
Canonical Example Revisited

Code

```
a ~ Gaussian(0.0, 1.0);  
b ~ Gaussian(a, 1.0);  
c ~ Gaussian(b, 1.0);  
d ~ Gaussian(b, 1.0);  
e ~ Gaussian(d, 1.0);  
stdout.print(c);
```

Checkpoint

observe e



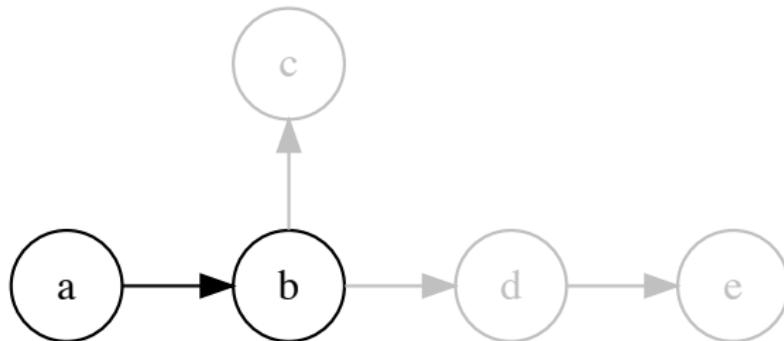
Canonical Example Revisited

Code

```
a ~ Gaussian(0.0, 1.0);  
b ~ Gaussian(a, 1.0);  
c ~ Gaussian(b, 1.0);  
d ~ Gaussian(b, 1.0);  
e ~ Gaussian(d, 1.0);  
stdout.print(c);
```

Checkpoint

observe e



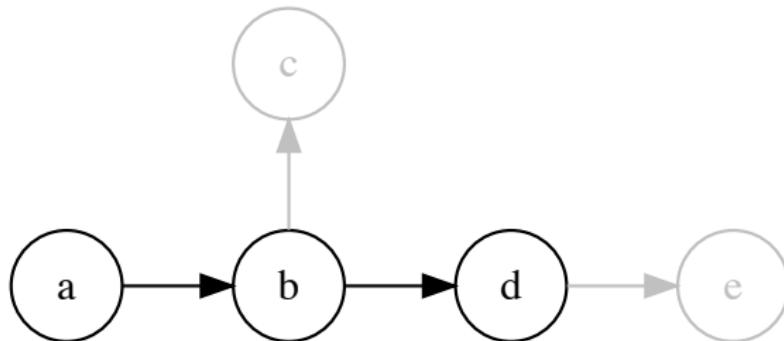
Canonical Example Revisited

Code

```
a ~ Gaussian(0.0, 1.0);  
b ~ Gaussian(a, 1.0);  
c ~ Gaussian(b, 1.0);  
d ~ Gaussian(b, 1.0);  
e ~ Gaussian(d, 1.0);  
stdout.print(c);
```

Checkpoint

observe e



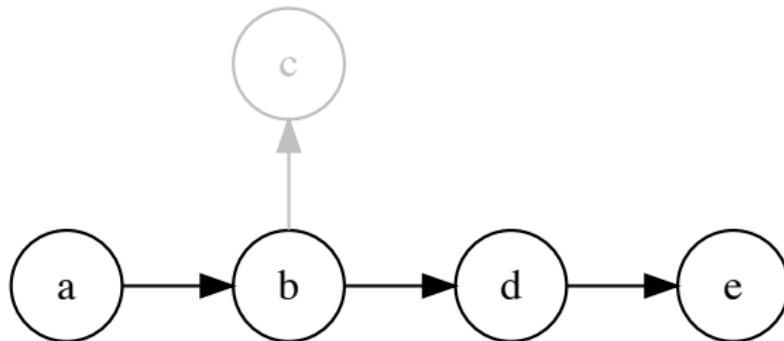
Canonical Example Revisited

Code

```
a ~ Gaussian(0.0, 1.0);  
b ~ Gaussian(a, 1.0);  
c ~ Gaussian(b, 1.0);  
d ~ Gaussian(b, 1.0);  
e ~ Gaussian(d, 1.0);  
stdout.print(c);
```

Checkpoint

observe e



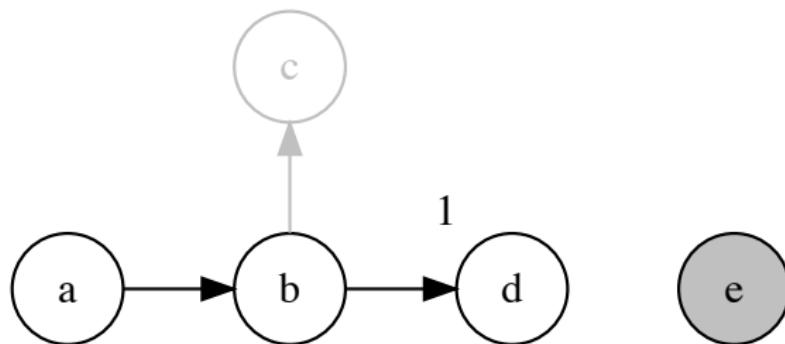
Canonical Example Revisited

Code

```
a ~ Gaussian(0.0, 1.0);  
b ~ Gaussian(a, 1.0);  
c ~ Gaussian(b, 1.0);  
d ~ Gaussian(b, 1.0);  
e ~ Gaussian(d, 1.0);  
stdout.print(c);
```

Checkpoint

observe e



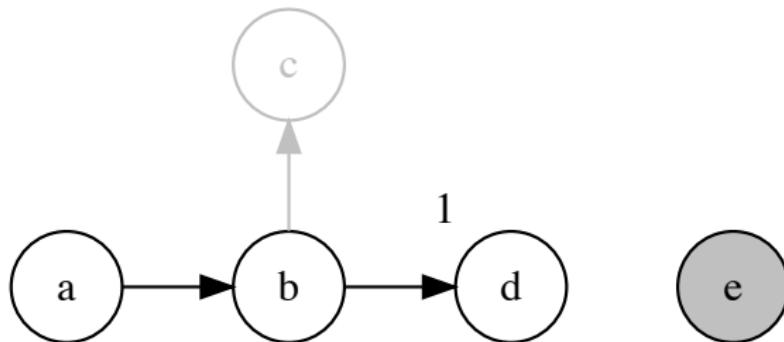
Canonical Example Revisited

Code

```
a ~ Gaussian(0.0, 1.0);  
b ~ Gaussian(a, 1.0);  
c ~ Gaussian(b, 1.0);  
d ~ Gaussian(b, 1.0);  
e ~ Gaussian(d, 1.0);  
stdout.print(c);
```

Checkpoint

value c



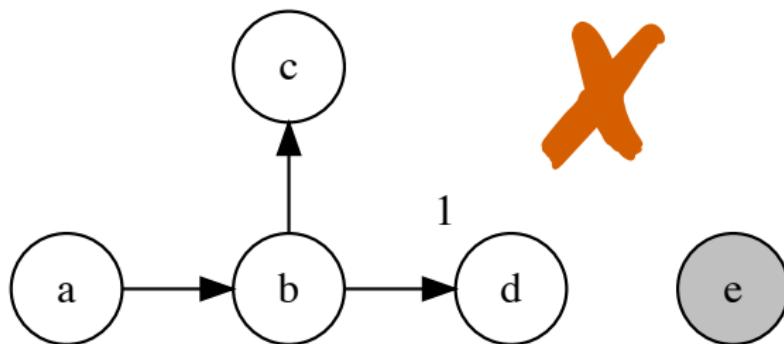
Canonical Example Revisited

Code

```
a ~ Gaussian(0.0, 1.0);  
b ~ Gaussian(a, 1.0);  
c ~ Gaussian(b, 1.0);  
d ~ Gaussian(b, 1.0);  
e ~ Gaussian(d, 1.0);  
stdout.print(c);
```

Checkpoint

value c



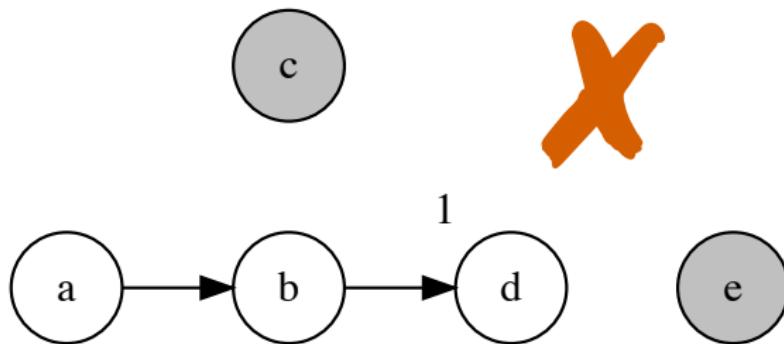
Canonical Example Revisited

Code

```
a ~ Gaussian(0.0, 1.0);  
b ~ Gaussian(a, 1.0);  
c ~ Gaussian(b, 1.0);  
d ~ Gaussian(b, 1.0);  
e ~ Gaussian(d, 1.0);  
stdout.print(c);
```

Checkpoint

value c



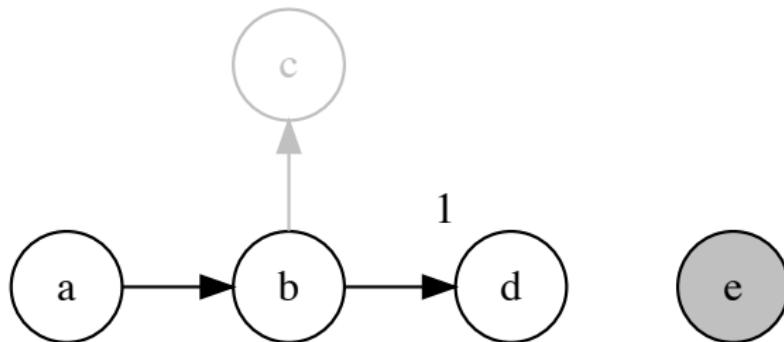
Canonical Example Revisited

Code

```
a ~ Gaussian(0.0, 1.0);  
b ~ Gaussian(a, 1.0);  
c ~ Gaussian(b, 1.0);  
d ~ Gaussian(b, 1.0);  
e ~ Gaussian(d, 1.0);  
stdout.print(c);
```

Checkpoint

value c



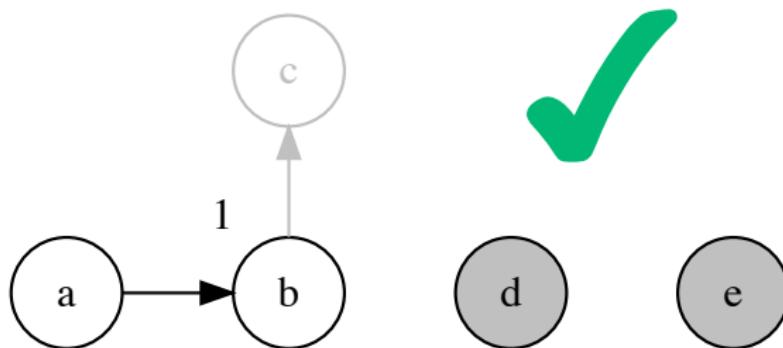
Canonical Example Revisited

Code

```
a ~ Gaussian(0.0, 1.0);  
b ~ Gaussian(a, 1.0);  
c ~ Gaussian(b, 1.0);  
d ~ Gaussian(b, 1.0);  
e ~ Gaussian(d, 1.0);  
stdout.print(c);
```

Checkpoint

value c



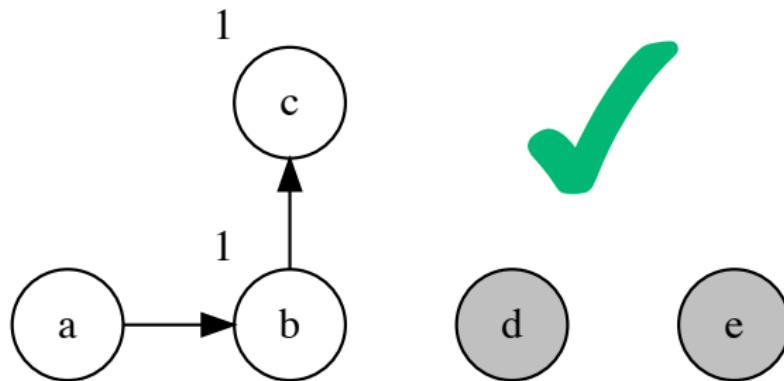
Canonical Example Revisited

Code

```
a ~ Gaussian(0.0, 1.0);  
b ~ Gaussian(a, 1.0);  
c ~ Gaussian(b, 1.0);  
d ~ Gaussian(b, 1.0);  
e ~ Gaussian(d, 1.0);  
stdout.print(c);
```

Checkpoint

value c

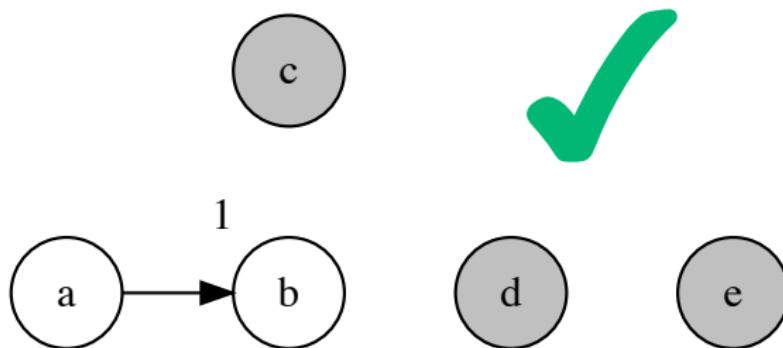


Canonical Example Revisited

Code

```
a ~ Gaussian(0.0, 1.0);  
b ~ Gaussian(a, 1.0);  
c ~ Gaussian(b, 1.0);  
d ~ Gaussian(b, 1.0);  
e ~ Gaussian(d, 1.0);  
stdout.print(c);
```

Checkpoint

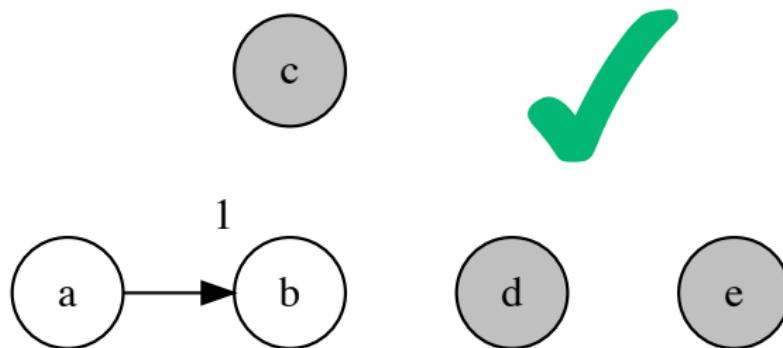


Canonical Example Revisited

Code

```
a ~ Gaussian(0.0, 1.0);  
b ~ Gaussian(a, 1.0);  
c ~ Gaussian(b, 1.0);  
d ~ Gaussian(b, 1.0);  
e ~ Gaussian(d, 1.0);  
stdout.print(c);
```

Checkpoint



Example #1

Code

```
x ~ Gaussian(0.0, 1.0);  
y ~ Gaussian(x, 1.0);  
z ~ Gaussian(y, 1.0);
```

Checkpoint

```
stdout.print(x + "\n");  
stdout.print(y + "\n");  
stdout.print(z + "\n");
```

Example #1

Code

```
x ~ Gaussian(0.0, 1.0);  
y ~ Gaussian(x, 1.0);  
z ~ Gaussian(y, 1.0);  
  
stdout.print(x + "\n");  
stdout.print(y + "\n");  
stdout.print(z + "\n");
```

Checkpoint

assume x



Example #1

Code

```
x ~ Gaussian(0.0, 1.0);  
y ~ Gaussian(x, 1.0);  
z ~ Gaussian(y, 1.0);
```

```
stdout.print(x + "\n");  
stdout.print(y + "\n");  
stdout.print(z + "\n");
```

Checkpoint

assume y



Example #1

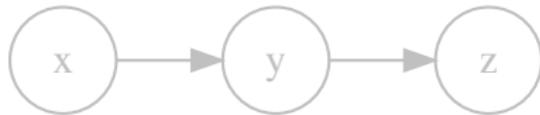
Code

```
x ~ Gaussian(0.0, 1.0);  
y ~ Gaussian(x, 1.0);  
z ~ Gaussian(y, 1.0);
```

Checkpoint

observe z

```
stdout.print(x + "\n");  
stdout.print(y + "\n");  
stdout.print(z + "\n");
```



Example #1

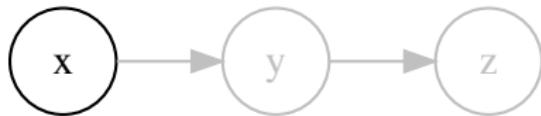
Code

```
x ~ Gaussian(0.0, 1.0);  
y ~ Gaussian(x, 1.0);  
z ~ Gaussian(y, 1.0);
```

Checkpoint

observe z

```
stdout.print(x + "\n");  
stdout.print(y + "\n");  
stdout.print(z + "\n");
```



Example #1

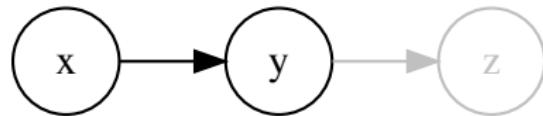
Code

```
x ~ Gaussian(0.0, 1.0);  
y ~ Gaussian(x, 1.0);  
z ~ Gaussian(y, 1.0);
```

Checkpoint

observe z

```
stdout.print(x + "\n");  
stdout.print(y + "\n");  
stdout.print(z + "\n");
```



Example #1

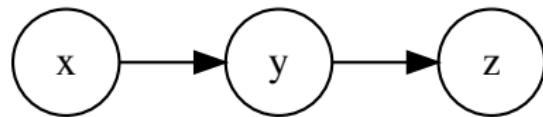
Code

```
x ~ Gaussian(0.0, 1.0);  
y ~ Gaussian(x, 1.0);  
z ~ Gaussian(y, 1.0);
```

Checkpoint

observe z

```
stdout.print(x + "\n");  
stdout.print(y + "\n");  
stdout.print(z + "\n");
```



Example #1

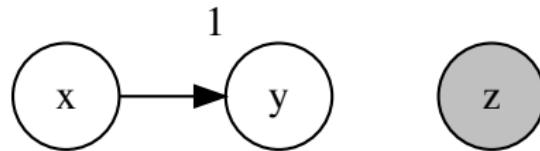
Code

```
x ~ Gaussian(0.0, 1.0);  
y ~ Gaussian(x, 1.0);  
z ~ Gaussian(y, 1.0);
```

Checkpoint

observe z

```
stdout.print(x + "\n");  
stdout.print(y + "\n");  
stdout.print(z + "\n");
```



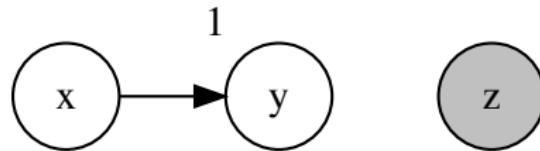
Example #1

Code

```
x ~ Gaussian(0.0, 1.0);  
y ~ Gaussian(x, 1.0);  
z ~ Gaussian(y, 1.0);
```

```
stdout.print(x + "\n");           value x  
stdout.print(y + "\n");  
stdout.print(z + "\n");
```

Checkpoint



Example #1

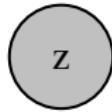
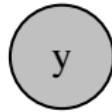
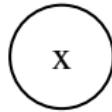
Code

```
x ~ Gaussian(0.0, 1.0);  
y ~ Gaussian(x, 1.0);  
z ~ Gaussian(y, 1.0);
```

```
stdout.print(x + "\n");           value x  
stdout.print(y + "\n");  
stdout.print(z + "\n");
```

Checkpoint

1



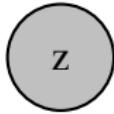
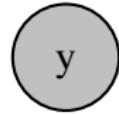
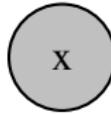
Example #1

Code

```
x ~ Gaussian(0.0, 1.0);  
y ~ Gaussian(x, 1.0);  
z ~ Gaussian(y, 1.0);
```

Checkpoint

```
stdout.print(x + "\n");           value x  
stdout.print(y + "\n");  
stdout.print(z + "\n");
```



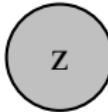
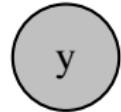
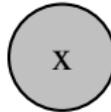
Example #1

Code

```
x ~ Gaussian(0.0, 1.0);  
y ~ Gaussian(x, 1.0);  
z ~ Gaussian(y, 1.0);
```

```
stdout.print(x + "\n");  
stdout.print(y + "\n");  
stdout.print(z + "\n");
```

Checkpoint



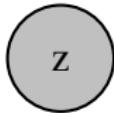
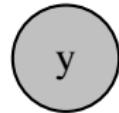
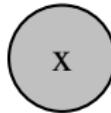
Example #1

Code

```
x ~ Gaussian(0.0, 1.0);  
y ~ Gaussian(x, 1.0);  
z ~ Gaussian(y, 1.0);
```

```
stdout.print(x + "\n");  
stdout.print(y + "\n");  
stdout.print(z + "\n");
```

Checkpoint



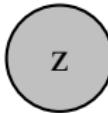
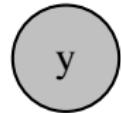
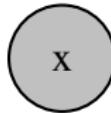
Example #1

Code

```
x ~ Gaussian(0.0, 1.0);  
y ~ Gaussian(x, 1.0);  
z ~ Gaussian(y, 1.0);
```

```
stdout.print(x + "\n");  
stdout.print(y + "\n");  
stdout.print(z + "\n");
```

Checkpoint



Example #2

Code

```
x ~ Gaussian(0.0, 1.0);
for (n in 1..N) {
    y[n] ~ Gaussian(x, 1.0);
}
stdout.print(x);
```

Checkpoint

Example #2

Code

```
x ~ Gaussian(0.0, 1.0);  
for (n in 1..N) {  
    y[n] ~ Gaussian(x, 1.0);  
}  
stdout.print(x);
```

Checkpoint

assume x



Example #2

Code

```
x ~ Gaussian(0.0, 1.0);
for (n in 1..N) {
    y[n] ~ Gaussian(x, 1.0);
}
stdout.print(x);
```

Checkpoint



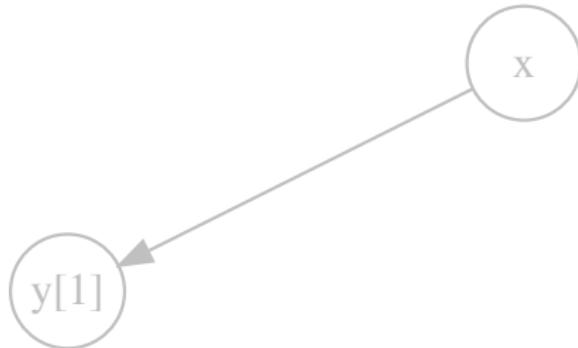
Example #2

Code

```
x ~ Gaussian(0.0, 1.0);  
for (n in 1..N) {  
    y[n] ~ Gaussian(x, 1.0);  
}  
stdout.print(x);
```

Checkpoint

observe $y[n]$



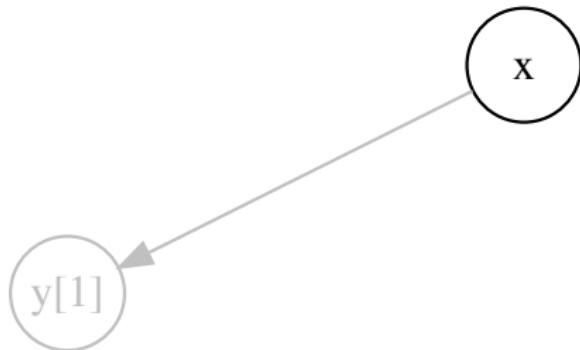
Example #2

Code

```
x ~ Gaussian(0.0, 1.0);  
for (n in 1..N) {  
    y[n] ~ Gaussian(x, 1.0);  
}  
stdout.print(x);
```

Checkpoint

observe $y[n]$



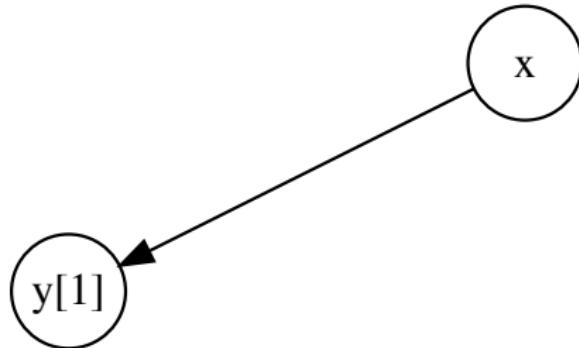
Example #2

Code

```
x ~ Gaussian(0.0, 1.0);
for (n in 1..N) {
    y[n] ~ Gaussian(x, 1.0);
}
stdout.print(x);
```

Checkpoint

observe $y[n]$



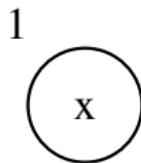
Example #2

Code

```
x ~ Gaussian(0.0, 1.0);  
for (n in 1..N) {  
    y[n] ~ Gaussian(x, 1.0);  
}  
stdout.print(x);
```

Checkpoint

observe $y[n]$



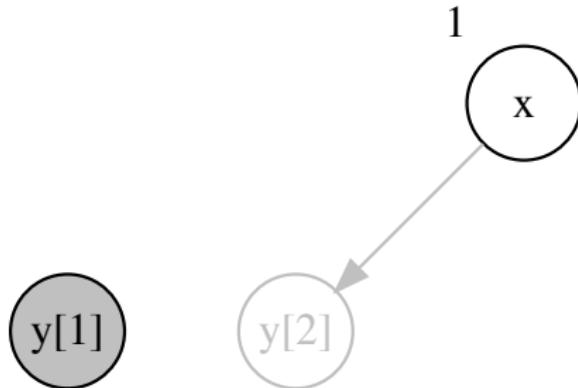
Example #2

Code

```
x ~ Gaussian(0.0, 1.0);  
for (n in 1..N) {  
    y[n] ~ Gaussian(x, 1.0);  
}  
stdout.print(x);
```

Checkpoint

observe $y[n]$



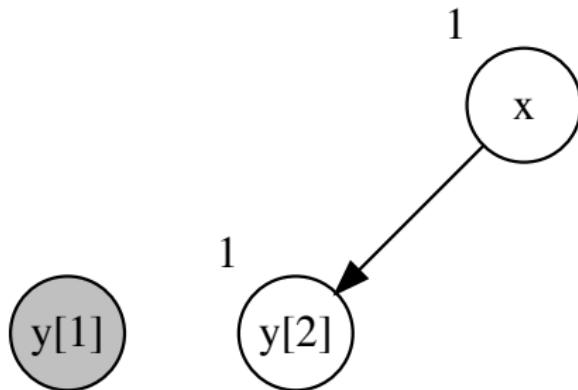
Example #2

Code

```
x ~ Gaussian(0.0, 1.0);  
for (n in 1..N) {  
    y[n] ~ Gaussian(x, 1.0);  
}  
stdout.print(x);
```

Checkpoint

observe $y[n]$



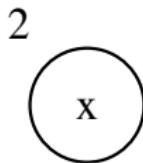
Example #2

Code

```
x ~ Gaussian(0.0, 1.0);  
for (n in 1..N) {  
    y[n] ~ Gaussian(x, 1.0);  
}  
stdout.print(x);
```

Checkpoint

observe $y[n]$



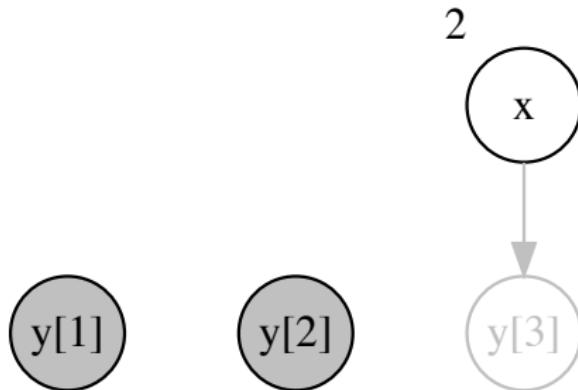
Example #2

Code

```
x ~ Gaussian(0.0, 1.0);
for (n in 1..N) {
    y[n] ~ Gaussian(x, 1.0);
}
stdout.print(x);
```

Checkpoint

observe $y[n]$



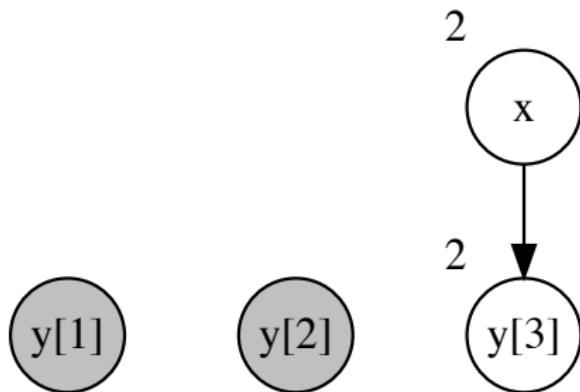
Example #2

Code

```
x ~ Gaussian(0.0, 1.0);  
for (n in 1..N) {  
    y[n] ~ Gaussian(x, 1.0);  
}  
stdout.print(x);
```

Checkpoint

observe $y[n]$



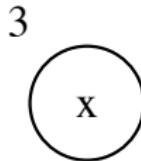
Example #2

Code

```
x ~ Gaussian(0.0, 1.0);  
for (n in 1..N) {  
    y[n] ~ Gaussian(x, 1.0);  
}  
stdout.print(x);
```

Checkpoint

observe $y[n]$



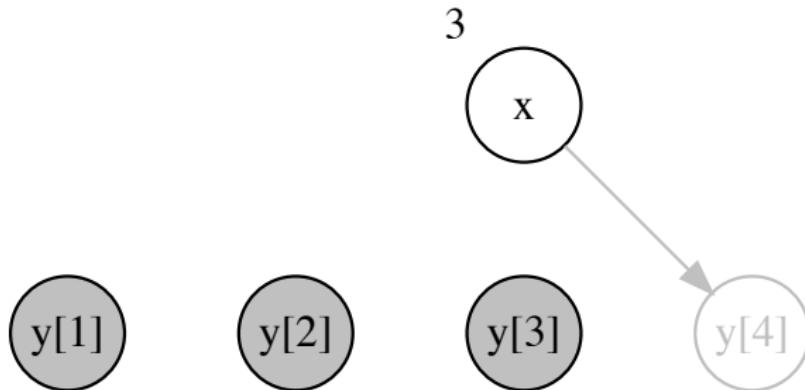
Example #2

Code

```
x ~ Gaussian(0.0, 1.0);  
for (n in 1..N) {  
    y[n] ~ Gaussian(x, 1.0);  
}  
stdout.print(x);
```

Checkpoint

observe $y[n]$



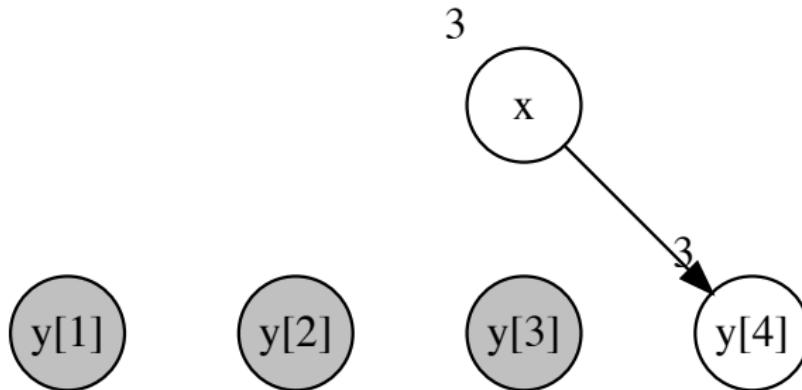
Example #2

Code

```
x ~ Gaussian(0.0, 1.0);  
for (n in 1..N) {  
    y[n] ~ Gaussian(x, 1.0);  
}  
stdout.print(x);
```

Checkpoint

observe $y[n]$



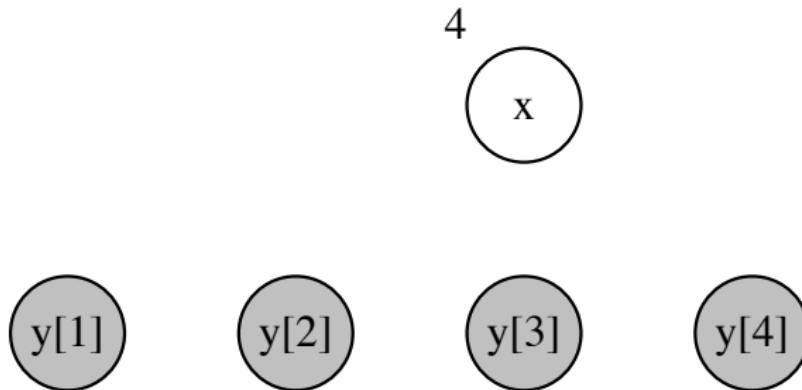
Example #2

Code

```
x ~ Gaussian(0.0, 1.0);  
for (n in 1..N) {  
    y[n] ~ Gaussian(x, 1.0);  
}  
stdout.print(x);
```

Checkpoint

observe $y[n]$



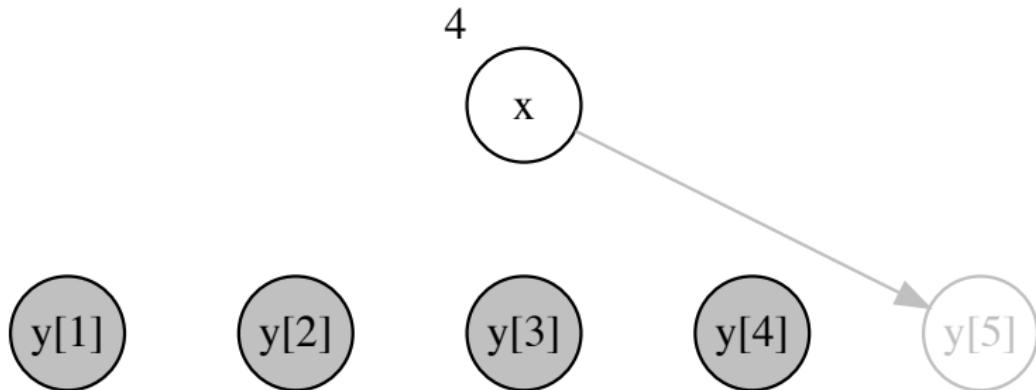
Example #2

Code

```
x ~ Gaussian(0.0, 1.0);  
for (n in 1..N) {  
    y[n] ~ Gaussian(x, 1.0);  
}  
stdout.print(x);
```

Checkpoint

observe $y[n]$



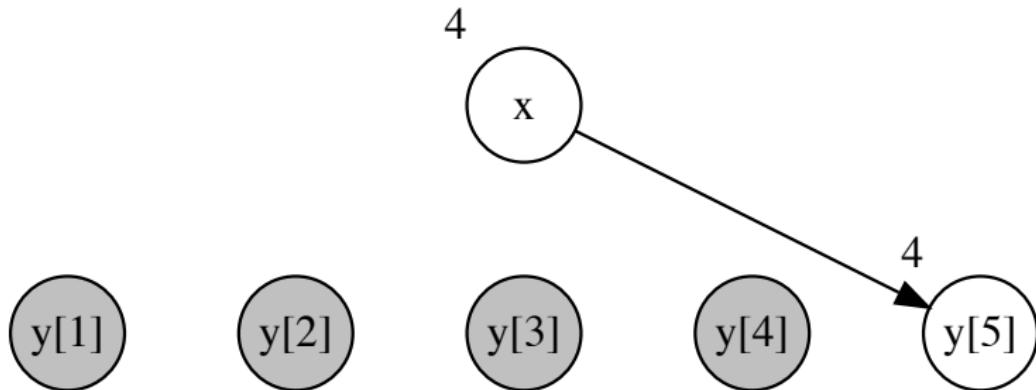
Example #2

Code

```
x ~ Gaussian(0.0, 1.0);  
for (n in 1..N) {  
    y[n] ~ Gaussian(x, 1.0);  
}  
stdout.print(x);
```

Checkpoint

observe $y[n]$



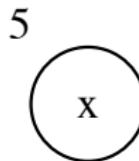
Example #2

Code

```
x ~ Gaussian(0.0, 1.0);  
for (n in 1..N) {  
    y[n] ~ Gaussian(x, 1.0);  
}  
stdout.print(x);
```

Checkpoint

observe $y[n]$



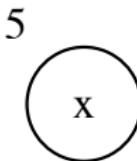
Example #2

Code

```
x ~ Gaussian(0.0, 1.0);  
for (n in 1..N) {  
    y[n] ~ Gaussian(x, 1.0);  
}  
  
stdout.print(x);
```

Checkpoint

value x

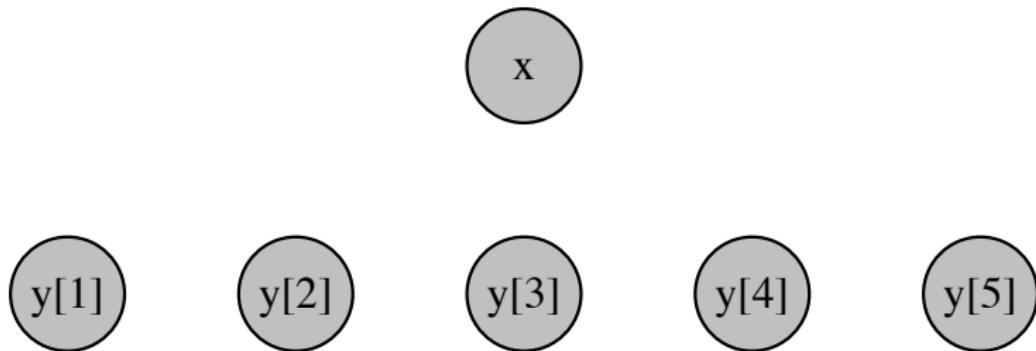


Example #2

Code

```
x ~ Gaussian(0.0, 1.0);  
for (n in 1..N) {  
    y[n] ~ Gaussian(x, 1.0);  
}  
stdout.print(x);
```

Checkpoint

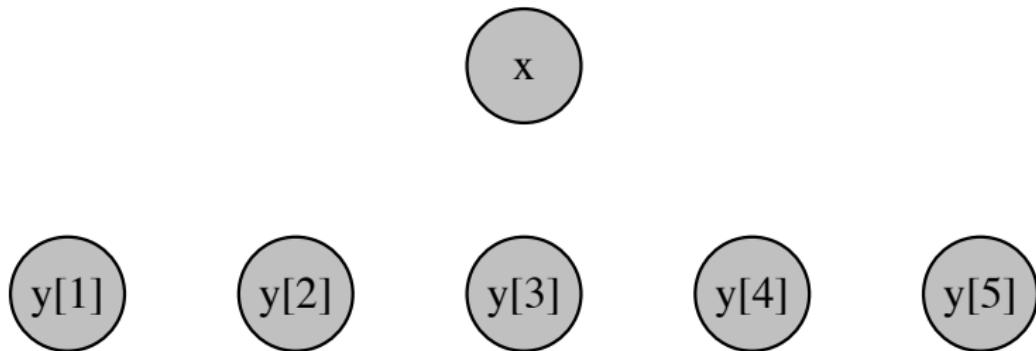


Example #2

Code

```
x ~ Gaussian(0.0, 1.0);  
for (n in 1..N) {  
    y[n] ~ Gaussian(x, 1.0);  
}  
stdout.print(x);
```

Checkpoint



Example #3

Code

```
ρ ~ Bernoulli(0.5);  
if (ρ) {  
    x ~ Gaussian(0.0, 1.0);  
} else {  
    x <- 0.0;  
}  
y ~ Gaussian(x, 1.0);
```

Checkpoint

Example #3

Code

```
 $\varrho \sim \text{Bernoulli}(0.5);$ 
if ( $\varrho$ ) {
     $x \sim \text{Gaussian}(0.0, 1.0);$ 
} else {
     $x \leftarrow 0.0;$ 
}
 $y \sim \text{Gaussian}(x, 1.0);$ 
```

Checkpoint

assume ϱ



Example #3

Code

```
 $\varrho \sim \text{Bernoulli}(0.5);$ 
if ( $\varrho$ ) {
     $x \sim \text{Gaussian}(0.0, 1.0);$ 
} else {
     $x <- 0.0;$ 
}
 $y \sim \text{Gaussian}(x, 1.0);$ 
```

Checkpoint

value ϱ



Example #3

Code

```
 $\varrho \sim \text{Bernoulli}(0.5);$ 
if ( $\varrho$ ) {
     $x \sim \text{Gaussian}(0.0, 1.0);$ 
} else {
     $x <- 0.0;$ 
}
 $y \sim \text{Gaussian}(x, 1.0);$ 
```

Checkpoint

value ϱ



Example #3

Code

```
ρ ~ Bernoulli(0.5);  
if (ρ) {  
    x ~ Gaussian(0.0, 1.0);  
} else {  
    x <- 0.0;  
}  
y ~ Gaussian(x, 1.0);
```

Checkpoint



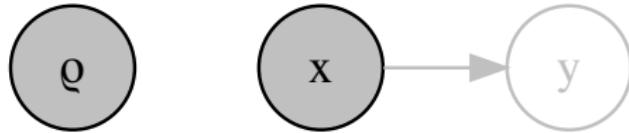
Example #3

Code

```
 $\varrho \sim \text{Bernoulli}(0.5);$ 
if ( $\varrho$ ) {
     $x \sim \text{Gaussian}(0.0, 1.0);$ 
} else {
     $x <- 0.0;$ 
}
 $y \sim \text{Gaussian}(x, 1.0);$ 
```

Checkpoint

assume y

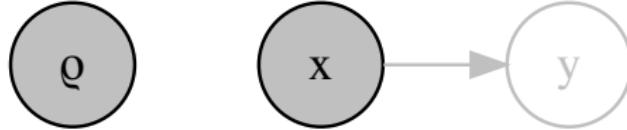


Example #3

Code

```
 $\varrho \sim \text{Bernoulli}(0.5);$ 
 $\text{if } (\varrho) \{$ 
     $x \sim \text{Gaussian}(0.0, 1.0);$ 
 $\} \text{ else } \{$ 
     $x <- 0.0;$ 
 $\}$ 
 $y \sim \text{Gaussian}(x, 1.0);$ 
```

Checkpoint



Example #3

Code

```
ρ ~ Bernoulli(0.5);  
if (ρ) {  
    x ~ Gaussian(0.0, 1.0);  
} else {  
    x <- 0.0;  
}  
y ~ Gaussian(x, 1.0);
```

Checkpoint

Example #3

Code

```
 $\varrho \sim \text{Bernoulli}(0.5);$ 
if ( $\varrho$ ) {
     $x \sim \text{Gaussian}(0.0, 1.0);$ 
} else {
     $x \leftarrow 0.0;$ 
}
 $y \sim \text{Gaussian}(x, 1.0);$ 
```

Checkpoint

assume ϱ



Example #3

Code

```
 $\varrho \sim \text{Bernoulli}(0.5);$ 
if ( $\varrho$ ) {
     $x \sim \text{Gaussian}(0.0, 1.0);$ 
} else {
     $x <- 0.0;$ 
}
 $y \sim \text{Gaussian}(x, 1.0);$ 
```

Checkpoint

value ϱ



Example #3

Code

```
 $\varrho \sim \text{Bernoulli}(0.5);$ 
if ( $\varrho$ ) {
     $x \sim \text{Gaussian}(0.0, 1.0);$ 
} else {
     $x <- 0.0;$ 
}
 $y \sim \text{Gaussian}(x, 1.0);$ 
```

Checkpoint

value ϱ



Example #3

Code

```
 $\varrho \sim \text{Bernoulli}(0.5);$ 
if ( $\varrho$ ) {
     $x \sim \text{Gaussian}(0.0, 1.0);$ 
} else {
     $x \leftarrow 0.0;$ 
}
 $y \sim \text{Gaussian}(x, 1.0);$ 
```

Checkpoint

assume x



Example #3

Code

```
 $\varrho \sim \text{Bernoulli}(0.5);$ 
if ( $\varrho$ ) {
     $x \sim \text{Gaussian}(0.0, 1.0);$ 
} else {
     $x <- 0.0;$ 
}
 $y \sim \text{Gaussian}(x, 1.0);$ 
```

Checkpoint

assume y



Example #3

Code

```
 $\varrho \sim \text{Bernoulli}(0.5);$ 
 $\text{if } (\varrho) \{$ 
     $x \sim \text{Gaussian}(0.0, 1.0);$ 
 $\} \text{ else } \{$ 
     $x <- 0.0;$ 
 $\}$ 
 $y \sim \text{Gaussian}(x, 1.0);$ 
```

Checkpoint



Example #4

Code

```
x[1] ~ Gaussian(0.0, 1.0);
y[1] ~ Gaussian(x[1], 1.0);
for (t in 2..T) {
    x[t] ~ Gaussian(a*x[t - 1], 1.0);
    y[t] ~ Gaussian(x[t], 1.0);
}
stdout.print(x[1]);
```

Checkpoint

Example #4

Code

```
x[1] ~ Gaussian(0.0, 1.0);
y[1] ~ Gaussian(x[1], 1.0);
for (t in 2..T) {
    x[t] ~ Gaussian(a*x[t - 1], 1.0);
    y[t] ~ Gaussian(x[t], 1.0);
}
stdout.print(x[1]);
```

Checkpoint

assume $x[1]$

x[1]

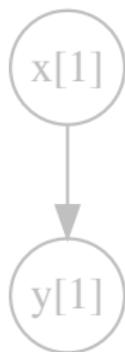
Example #4

Code

```
x[1] ~ Gaussian(0.0, 1.0);
y[1] ~ Gaussian(x[1], 1.0);
for (t in 2..T) {
    x[t] ~ Gaussian(a*x[t - 1], 1.0);
    y[t] ~ Gaussian(x[t], 1.0);
}
stdout.print(x[1]);
```

Checkpoint

observe y[1]



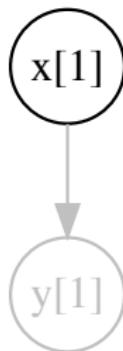
Example #4

Code

```
x[1] ~ Gaussian(0.0, 1.0);
y[1] ~ Gaussian(x[1], 1.0);
for (t in 2..T) {
    x[t] ~ Gaussian(a*x[t - 1], 1.0);
    y[t] ~ Gaussian(x[t], 1.0);
}
stdout.print(x[1]);
```

Checkpoint

observe y[1]



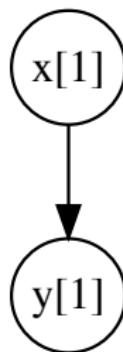
Example #4

Code

```
x[1] ~ Gaussian(0.0, 1.0);
y[1] ~ Gaussian(x[1], 1.0);
for (t in 2..T) {
    x[t] ~ Gaussian(a*x[t - 1], 1.0);
    y[t] ~ Gaussian(x[t], 1.0);
}
stdout.print(x[1]);
```

Checkpoint

observe y[1]



Example #4

Code

```
x[1] ~ Gaussian(0.0, 1.0);
y[1] ~ Gaussian(x[1], 1.0);
for (t in 2..T) {
    x[t] ~ Gaussian(a*x[t - 1], 1.0);
    y[t] ~ Gaussian(x[t], 1.0);
}
stdout.print(x[1]);
```

Checkpoint

observe y[1]

1



y[1]

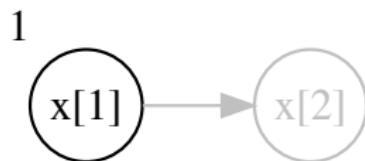
Example #4

Code

```
x[1] ~ Gaussian(0.0, 1.0);
y[1] ~ Gaussian(x[1], 1.0);
for (t in 2..T) {
    x[t] ~ Gaussian(a*x[t - 1], 1.0);
    y[t] ~ Gaussian(x[t], 1.0);
}
stdout.print(x[1]);
```

Checkpoint

assume x[t]



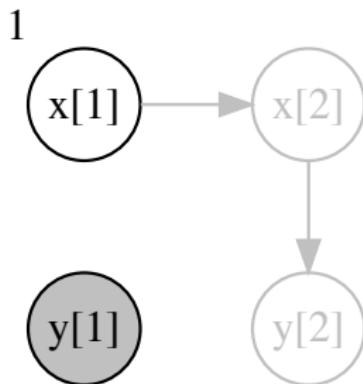
Example #4

Code

```
x[1] ~ Gaussian(0.0, 1.0);
y[1] ~ Gaussian(x[1], 1.0);
for (t in 2..T) {
    x[t] ~ Gaussian(a*x[t - 1], 1.0);
    y[t] ~ Gaussian(x[t], 1.0);
}
stdout.print(x[1]);
```

Checkpoint

observe $y[t]$



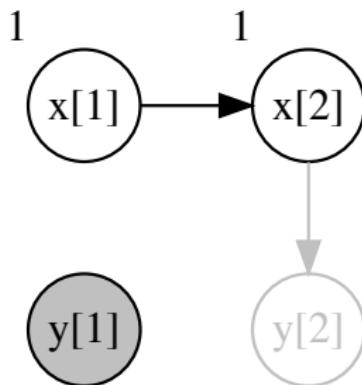
Example #4

Code

```
x[1] ~ Gaussian(0.0, 1.0);
y[1] ~ Gaussian(x[1], 1.0);
for (t in 2..T) {
    x[t] ~ Gaussian(a*x[t - 1], 1.0);
    y[t] ~ Gaussian(x[t], 1.0);
}
stdout.print(x[1]);
```

Checkpoint

observe $y[t]$



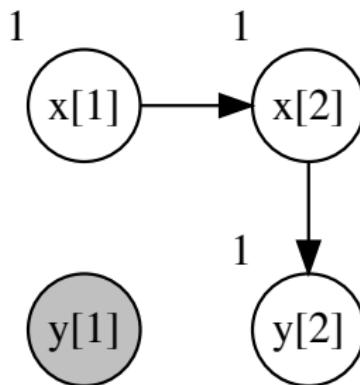
Example #4

Code

```
x[1] ~ Gaussian(0.0, 1.0);
y[1] ~ Gaussian(x[1], 1.0);
for (t in 2..T) {
    x[t] ~ Gaussian(a*x[t - 1], 1.0);
    y[t] ~ Gaussian(x[t], 1.0);
}
stdout.print(x[1]);
```

Checkpoint

observe $y[t]$



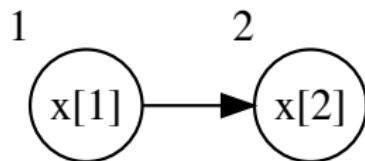
Example #4

Code

```
x[1] ~ Gaussian(0.0, 1.0);
y[1] ~ Gaussian(x[1], 1.0);
for (t in 2..T) {
    x[t] ~ Gaussian(a*x[t - 1], 1.0);
    y[t] ~ Gaussian(x[t], 1.0);
}
stdout.print(x[1]);
```

Checkpoint

observe y[t]



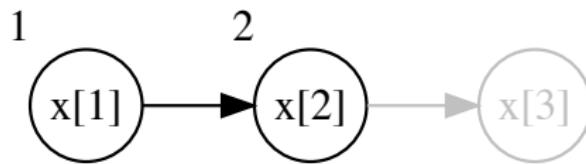
Example #4

Code

```
x[1] ~ Gaussian(0.0, 1.0);
y[1] ~ Gaussian(x[1], 1.0);
for (t in 2..T) {
    x[t] ~ Gaussian(a*x[t - 1], 1.0);
    y[t] ~ Gaussian(x[t], 1.0);
}
stdout.print(x[1]);
```

Checkpoint

assume x[t]



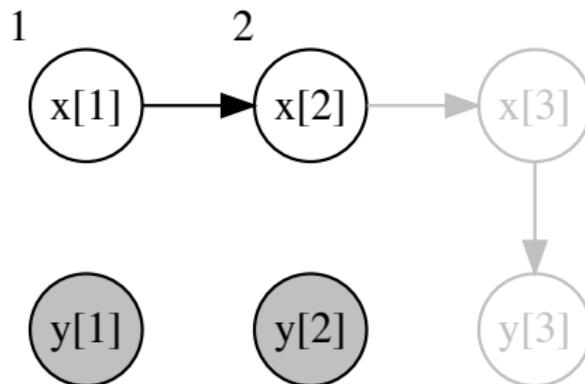
Example #4

Code

```
x[1] ~ Gaussian(0.0, 1.0);
y[1] ~ Gaussian(x[1], 1.0);
for (t in 2..T) {
    x[t] ~ Gaussian(a*x[t - 1], 1.0);
    y[t] ~ Gaussian(x[t], 1.0);
}
stdout.print(x[1]);
```

Checkpoint

observe $y[t]$



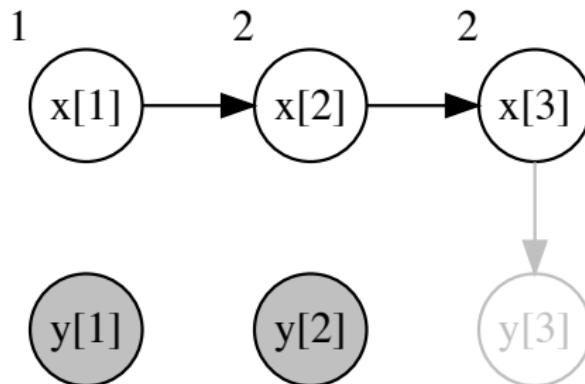
Example #4

Code

```
x[1] ~ Gaussian(0.0, 1.0);
y[1] ~ Gaussian(x[1], 1.0);
for (t in 2..T) {
    x[t] ~ Gaussian(a*x[t - 1], 1.0);
    y[t] ~ Gaussian(x[t], 1.0);
}
stdout.print(x[1]);
```

Checkpoint

observe $y[t]$



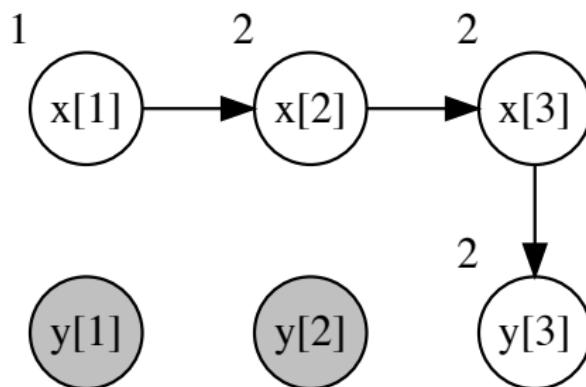
Example #4

Code

```
x[1] ~ Gaussian(0.0, 1.0);
y[1] ~ Gaussian(x[1], 1.0);
for (t in 2..T) {
    x[t] ~ Gaussian(a*x[t - 1], 1.0);
    y[t] ~ Gaussian(x[t], 1.0);
}
stdout.print(x[1]);
```

Checkpoint

observe $y[t]$



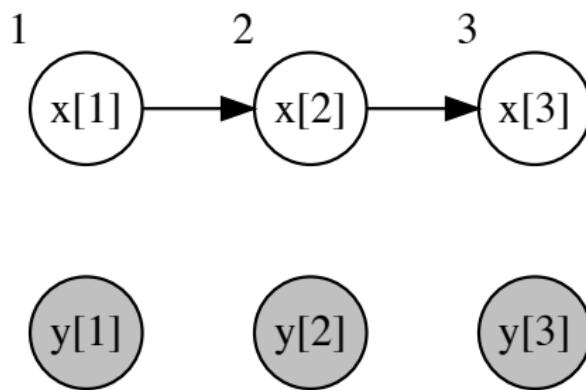
Example #4

Code

```
x[1] ~ Gaussian(0.0, 1.0);
y[1] ~ Gaussian(x[1], 1.0);
for (t in 2..T) {
    x[t] ~ Gaussian(a*x[t - 1], 1.0);
    y[t] ~ Gaussian(x[t], 1.0);
}
stdout.print(x[1]);
```

Checkpoint

observe $y[t]$



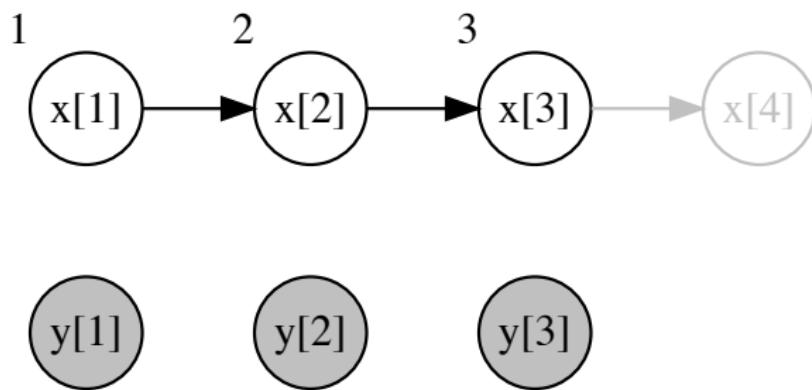
Example #4

Code

```
x[1] ~ Gaussian(0.0, 1.0);
y[1] ~ Gaussian(x[1], 1.0);
for (t in 2..T) {
    x[t] ~ Gaussian(a*x[t - 1], 1.0);
    y[t] ~ Gaussian(x[t], 1.0);
}
stdout.print(x[1]);
```

Checkpoint

assume x[t]



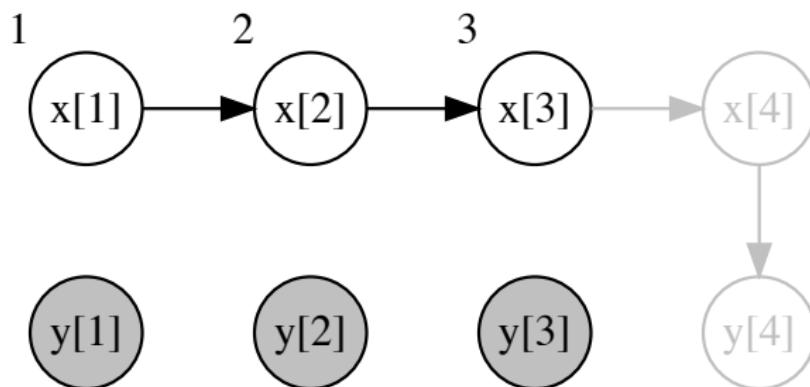
Example #4

Code

```
x[1] ~ Gaussian(0.0, 1.0);
y[1] ~ Gaussian(x[1], 1.0);
for (t in 2..T) {
    x[t] ~ Gaussian(a*x[t - 1], 1.0);
    y[t] ~ Gaussian(x[t], 1.0);
}
stdout.print(x[1]);
```

Checkpoint

observe $y[t]$



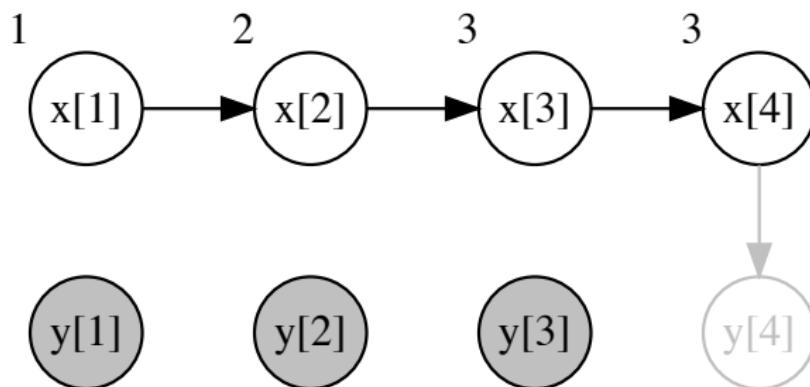
Example #4

Code

```
x[1] ~ Gaussian(0.0, 1.0);
y[1] ~ Gaussian(x[1], 1.0);
for (t in 2..T) {
    x[t] ~ Gaussian(a*x[t - 1], 1.0);
    y[t] ~ Gaussian(x[t], 1.0);
}
stdout.print(x[1]);
```

Checkpoint

observe $y[t]$



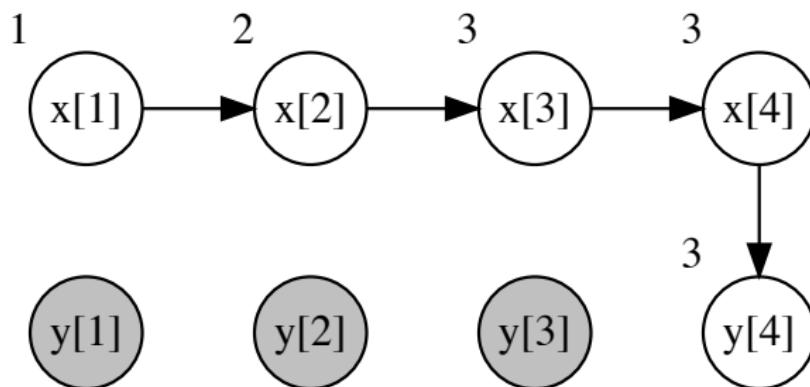
Example #4

Code

```
x[1] ~ Gaussian(0.0, 1.0);
y[1] ~ Gaussian(x[1], 1.0);
for (t in 2..T) {
    x[t] ~ Gaussian(a*x[t - 1], 1.0);
    y[t] ~ Gaussian(x[t], 1.0);
}
stdout.print(x[1]);
```

Checkpoint

observe $y[t]$



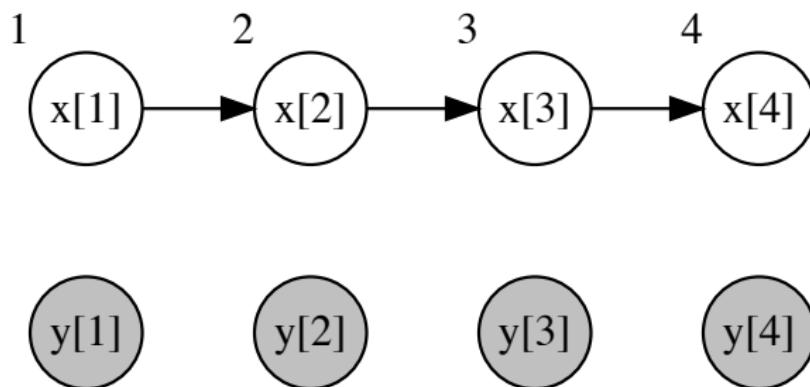
Example #4

Code

```
x[1] ~ Gaussian(0.0, 1.0);
y[1] ~ Gaussian(x[1], 1.0);
for (t in 2..T) {
    x[t] ~ Gaussian(a*x[t - 1], 1.0);
    y[t] ~ Gaussian(x[t], 1.0);
}
stdout.print(x[1]);
```

Checkpoint

observe $y[t]$



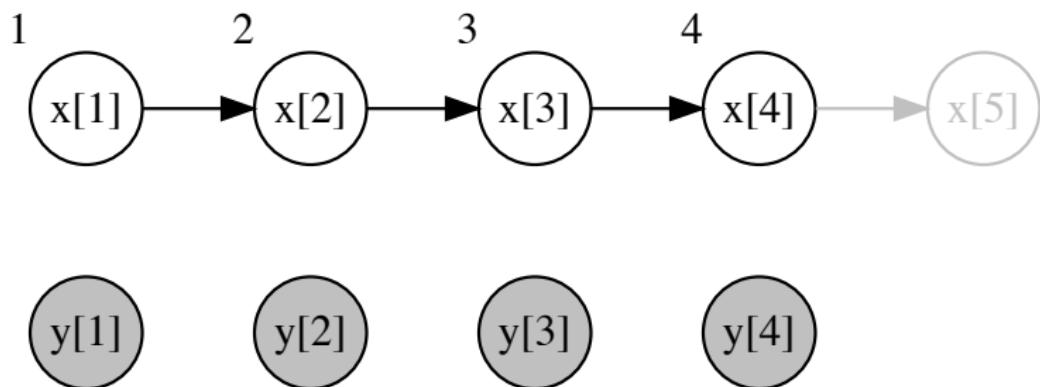
Example #4

Code

```
x[1] ~ Gaussian(0.0, 1.0);
y[1] ~ Gaussian(x[1], 1.0);
for (t in 2..T) {
    x[t] ~ Gaussian(a*x[t - 1], 1.0);
    y[t] ~ Gaussian(x[t], 1.0);
}
stdout.print(x[1]);
```

Checkpoint

assume x[t]



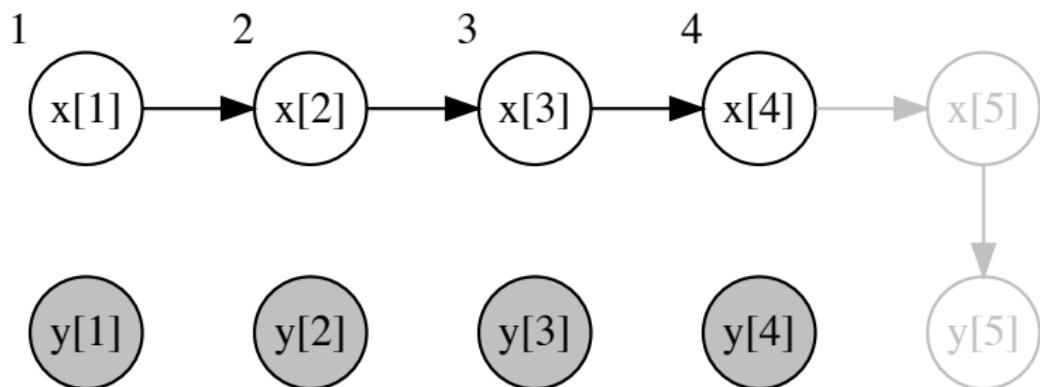
Example #4

Code

```
x[1] ~ Gaussian(0.0, 1.0);
y[1] ~ Gaussian(x[1], 1.0);
for (t in 2..T) {
    x[t] ~ Gaussian(a*x[t - 1], 1.0);
    y[t] ~ Gaussian(x[t], 1.0);
}
stdout.print(x[1]);
```

Checkpoint

observe $y[t]$



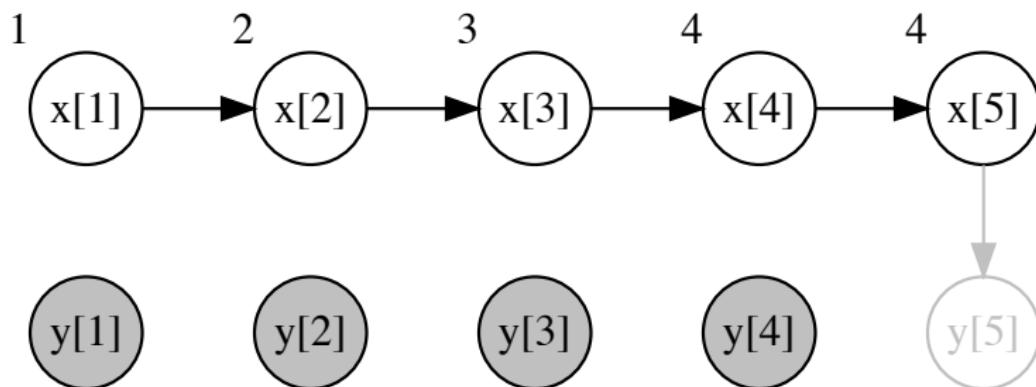
Example #4

Code

```
x[1] ~ Gaussian(0.0, 1.0);
y[1] ~ Gaussian(x[1], 1.0);
for (t in 2..T) {
    x[t] ~ Gaussian(a*x[t - 1], 1.0);
    y[t] ~ Gaussian(x[t], 1.0);
}
stdout.print(x[1]);
```

Checkpoint

observe $y[t]$



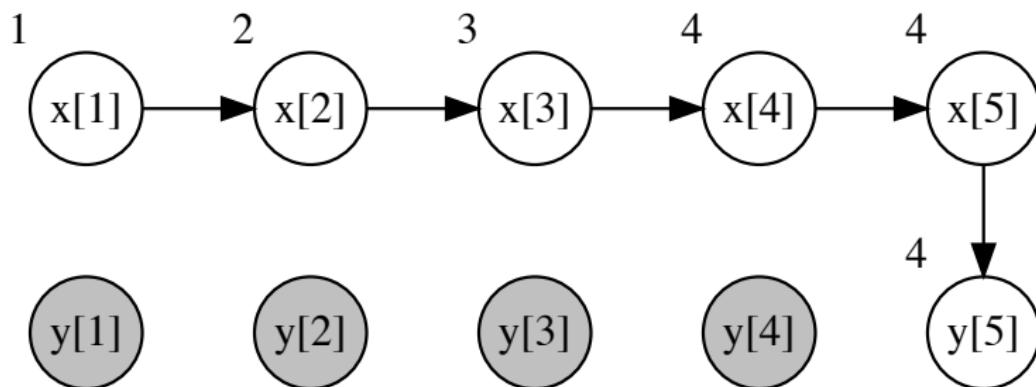
Example #4

Code

```
x[1] ~ Gaussian(0.0, 1.0);
y[1] ~ Gaussian(x[1], 1.0);
for (t in 2..T) {
    x[t] ~ Gaussian(a*x[t - 1], 1.0);
    y[t] ~ Gaussian(x[t], 1.0);
}
stdout.print(x[1]);
```

Checkpoint

observe $y[t]$



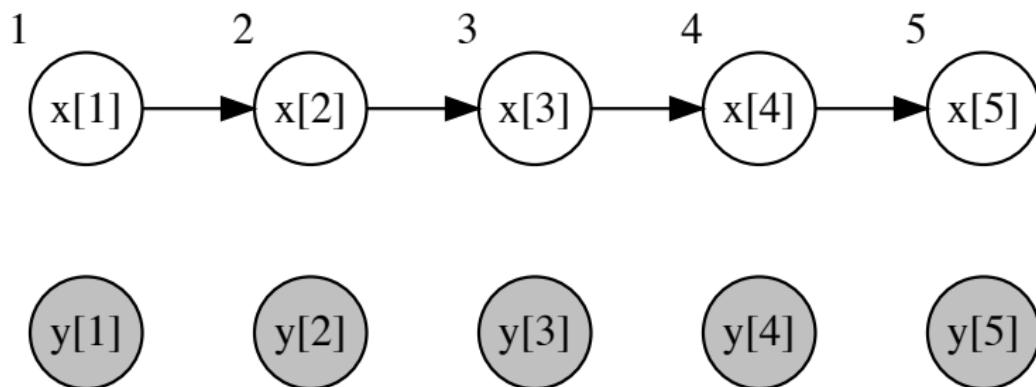
Example #4

Code

```
x[1] ~ Gaussian(0.0, 1.0);
y[1] ~ Gaussian(x[1], 1.0);
for (t in 2..T) {
    x[t] ~ Gaussian(a*x[t - 1], 1.0);
    y[t] ~ Gaussian(x[t], 1.0);
}
stdout.print(x[1]);
```

Checkpoint

observe $y[t]$



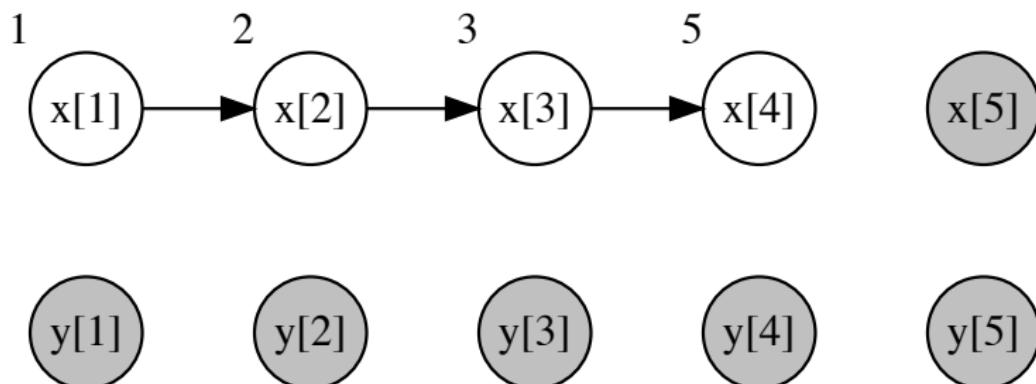
Example #4: Kalman Filter

Code

```
x[1] ~ Gaussian(0.0, 1.0);  
y[1] ~ Gaussian(x[1], 1.0);  
for (t in 2..T) {  
    x[t] ~ Gaussian(a*x[t - 1], 1.0);  
    y[t] ~ Gaussian(x[t], 1.0);  
}  
stdout.print(x[1]);
```

Checkpoint

value x[1]



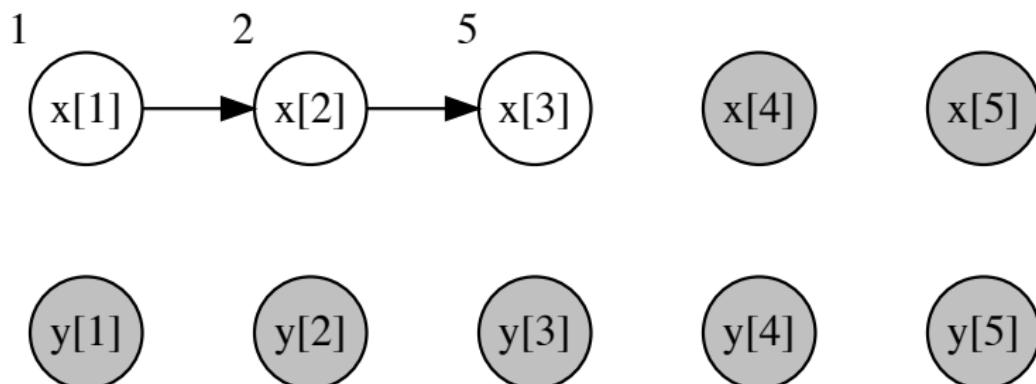
Example #4: Kalman Filter

Code

```
x[1] ~ Gaussian(0.0, 1.0);  
y[1] ~ Gaussian(x[1], 1.0);  
for (t in 2..T) {  
    x[t] ~ Gaussian(a*x[t - 1], 1.0);  
    y[t] ~ Gaussian(x[t], 1.0);  
}  
stdout.print(x[1]);
```

Checkpoint

value x[1]



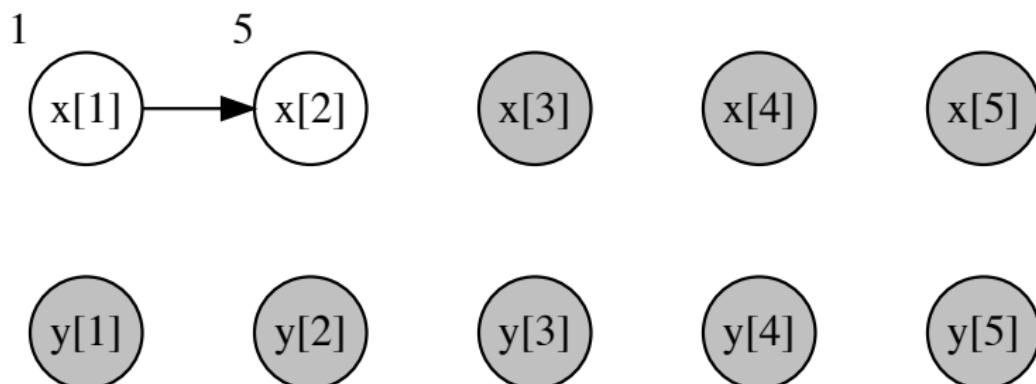
Example #4: Kalman Filter

Code

```
x[1] ~ Gaussian(0.0, 1.0);  
y[1] ~ Gaussian(x[1], 1.0);  
for (t in 2..T) {  
    x[t] ~ Gaussian(a*x[t - 1], 1.0);  
    y[t] ~ Gaussian(x[t], 1.0);  
}  
stdout.print(x[1]);
```

Checkpoint

value x[1]



Example #4: Kalman Filter

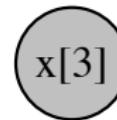
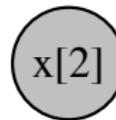
Code

```
x[1] ~ Gaussian(0.0, 1.0);  
y[1] ~ Gaussian(x[1], 1.0);  
for (t in 2..T) {  
    x[t] ~ Gaussian(a*x[t - 1], 1.0);  
    y[t] ~ Gaussian(x[t], 1.0);  
}  
stdout.print(x[1]);
```

Checkpoint

value x[1]

5



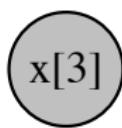
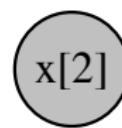
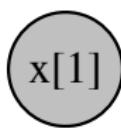
Example #4: Kalman Filter

Code

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x[1] ~ Gaussian(0.0, 1.0);  
y[1] ~ Gaussian(x[1], 1.0);  
for (t in 2..T) {  
    x[t] ~ Gaussian(a*x[t - 1], 1.0);  
    y[t] ~ Gaussian(x[t], 1.0);  
}  
stdout.print(x[1]);
```

Checkpoint

value x[1]



Example #4: Kalman Filter

Code

```
x[1] ~ Gaussian(0.0, 1.0);  
y[1] ~ Gaussian(x[1], 1.0);  
for (t in 2..T) {  
    x[t] ~ Gaussian(a*x[t - 1], 1.0);  
    y[t] ~ Gaussian(x[t], 1.0);  
}  
stdout.print(x[1]);
```

Checkpoint

x[1]

x[2]

x[3]

x[4]

x[5]

y[1]

y[2]

y[3]

y[4]

y[5]

Example #5

Code

```
x_n[1] ~ Gaussian(vector(0.0, 1), identity(1, 1));
x_l[1] ~ Gaussian(vector(0.0, 3), identity(3, 3));
y_n[1] ~ Gaussian(g(x_n[1]),  $\Sigma_{y_n}$ );
y_l[1] ~ Gaussian(C*x_l[1],  $\Sigma_{y_l}$ );
for (t in 2..T) {
    x_n[t] ~ Gaussian(f(x_n[t-1]) + B*x_l[t-1],  $\Sigma_{x_n}$ );
    x_l[t] ~ Gaussian(A*x_l[t-1],  $\Sigma_{x_l}$ );
    y_n[t] ~ Gaussian(g(x_n[t]),  $\Sigma_{y_n}$ );
    y_l[t] ~ Gaussian(C*x_l[t],  $\Sigma_{y_l}$ );
}
```

Example #5

Example #5

x_n[1]

Example #5

x_n[1]

x_l[1]

Example #5

x_n[1]

x_l[1]

Example #5



Example #5

x_n[1]

x_l[1]

y_n[1]

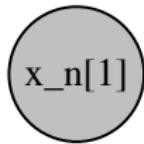
Example #5

x_n[1]

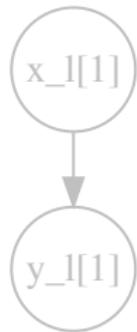
x_l[1]

y_n[1]

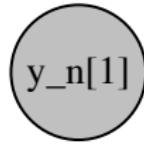
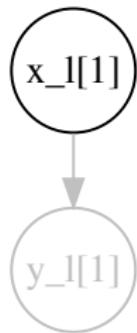
Example #5



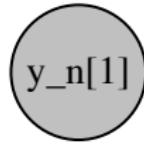
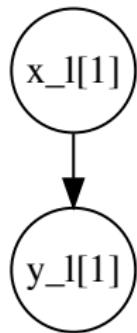
Example #5



Example #5



Example #5



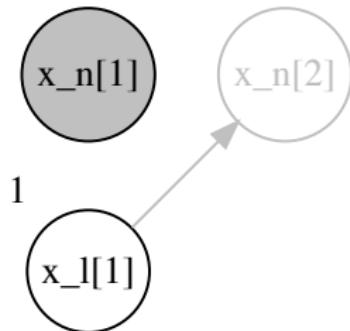
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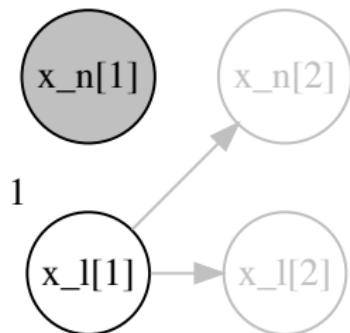
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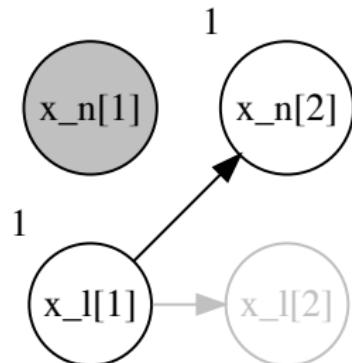
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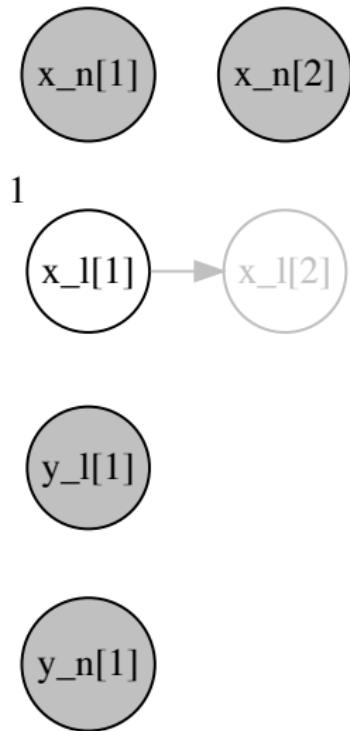
Example #5



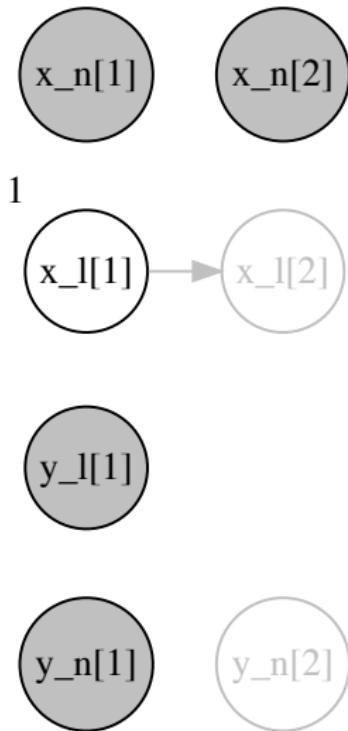
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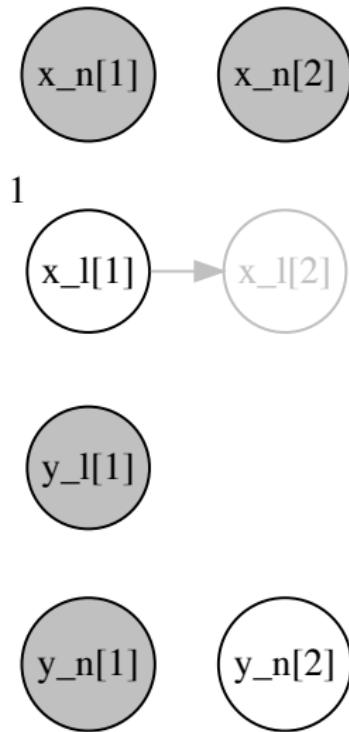
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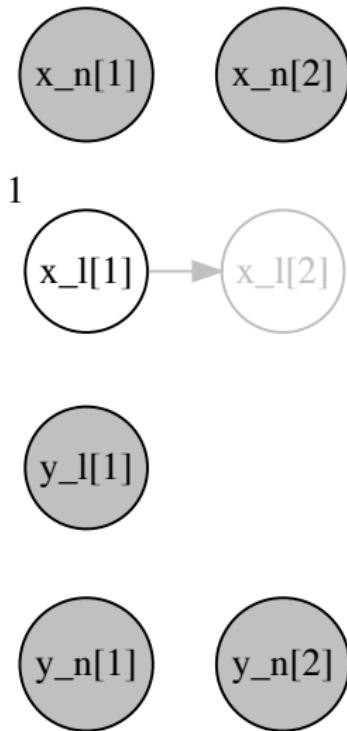
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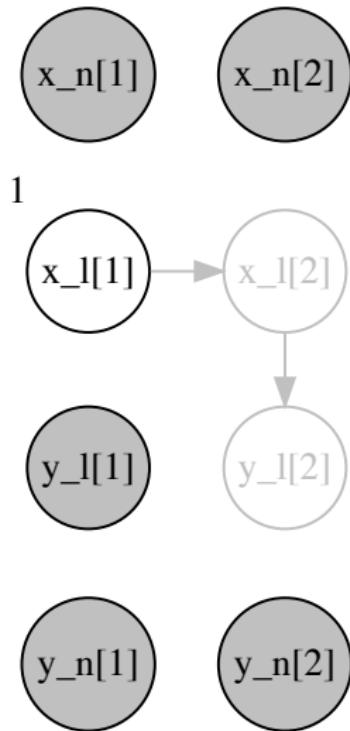
Example #5



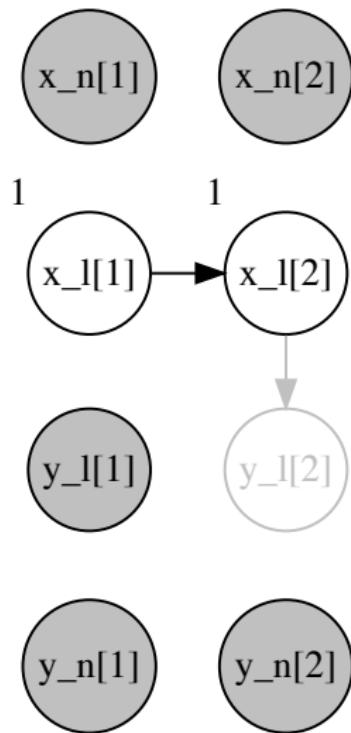
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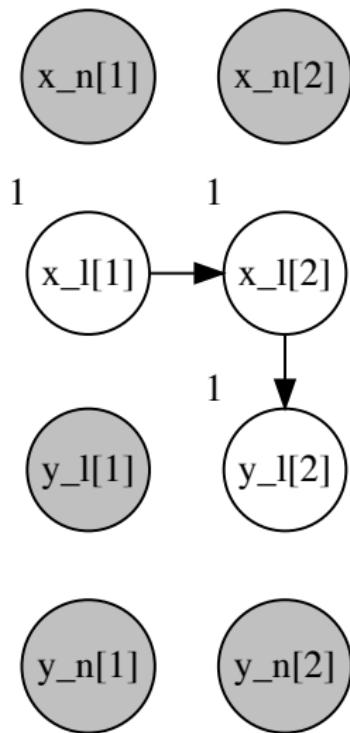
Example #5



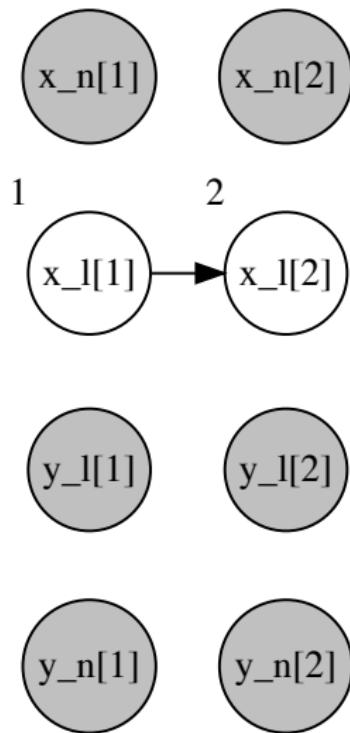
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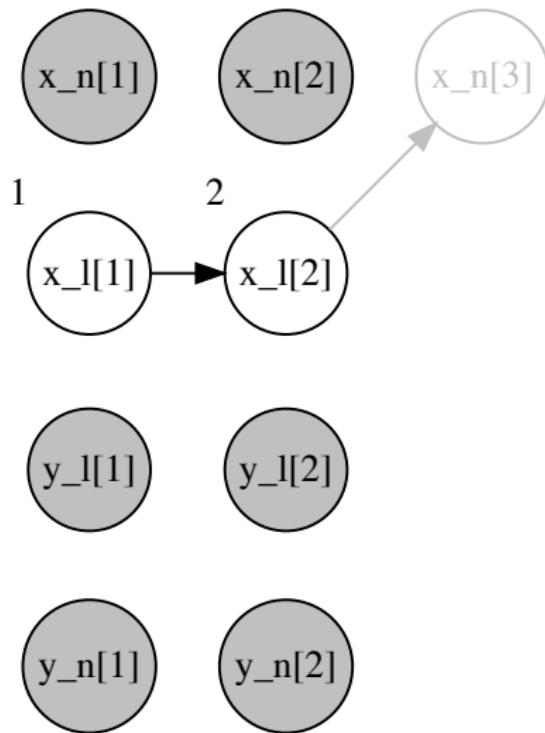
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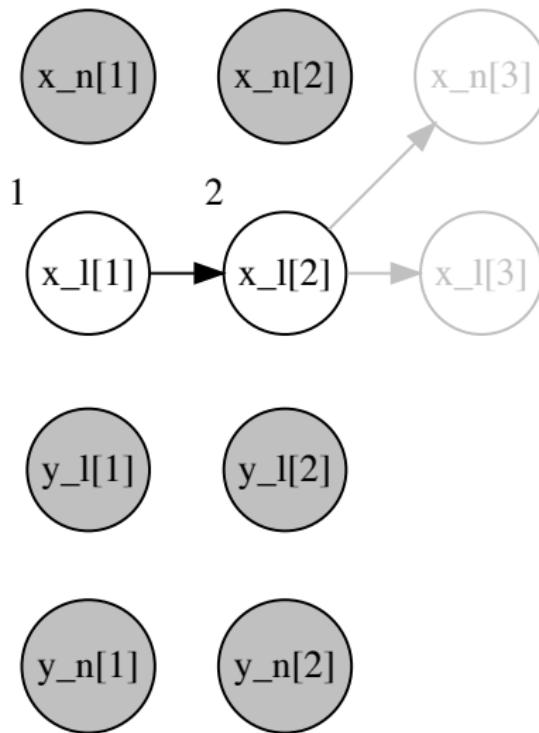
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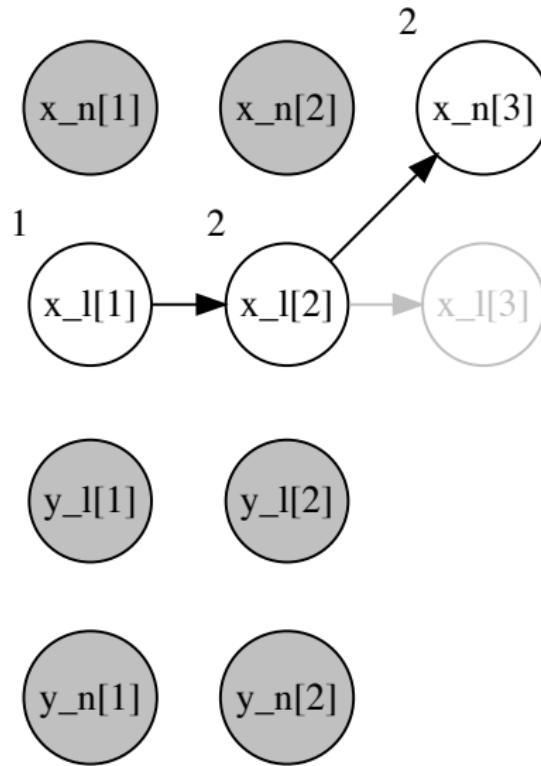
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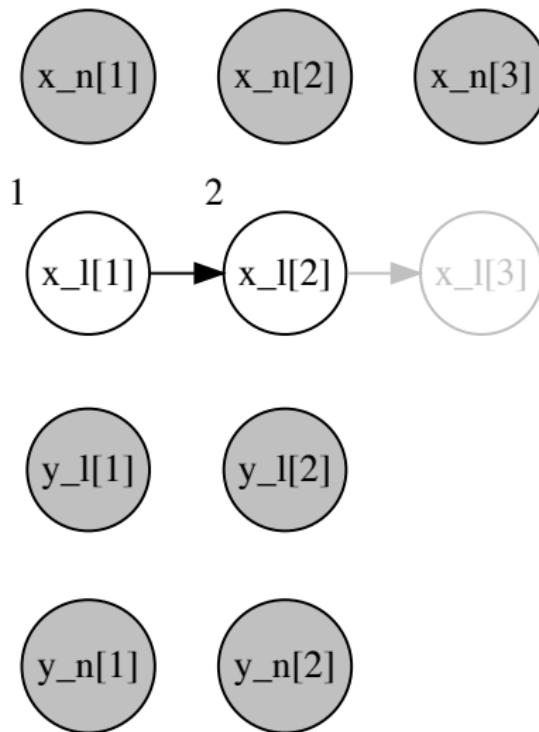
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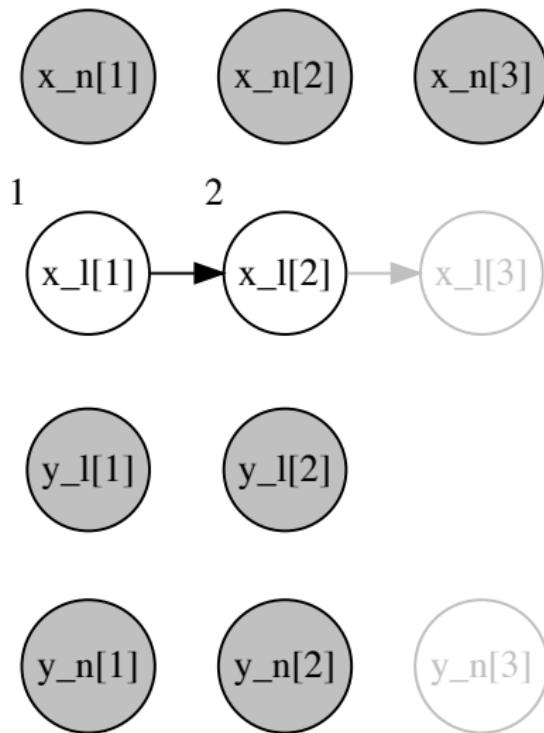
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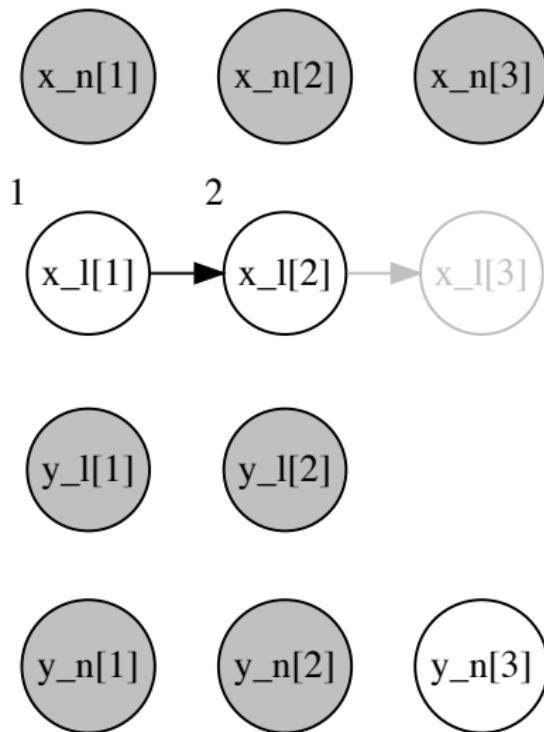
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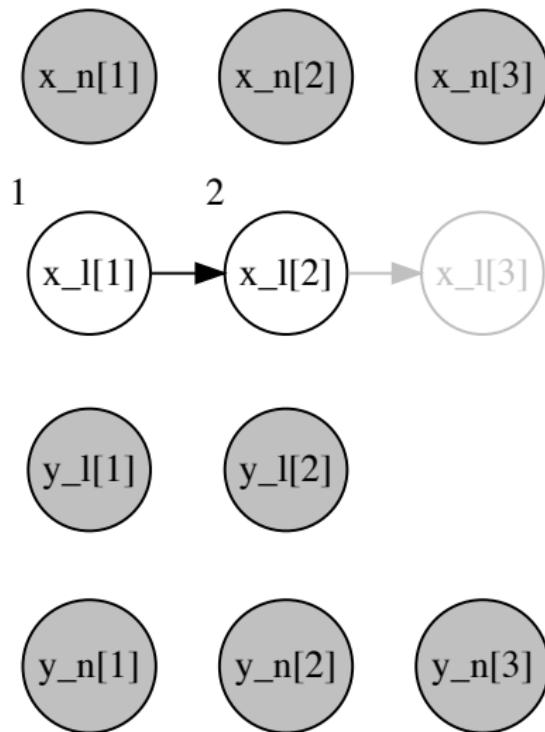
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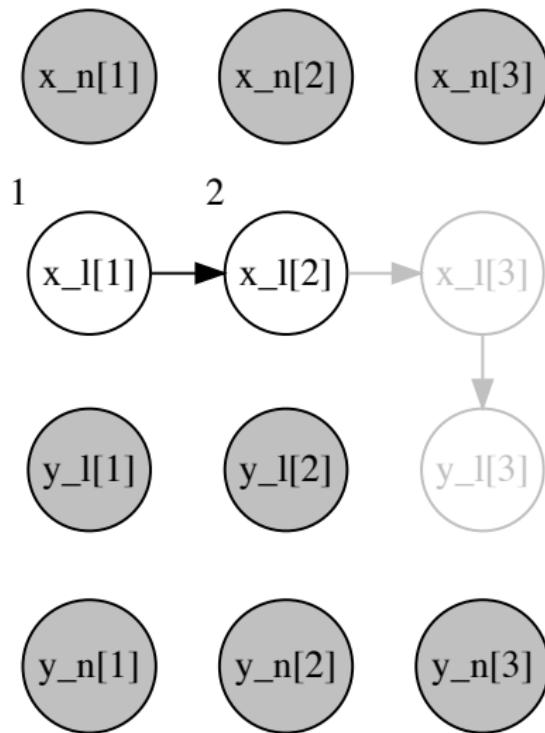
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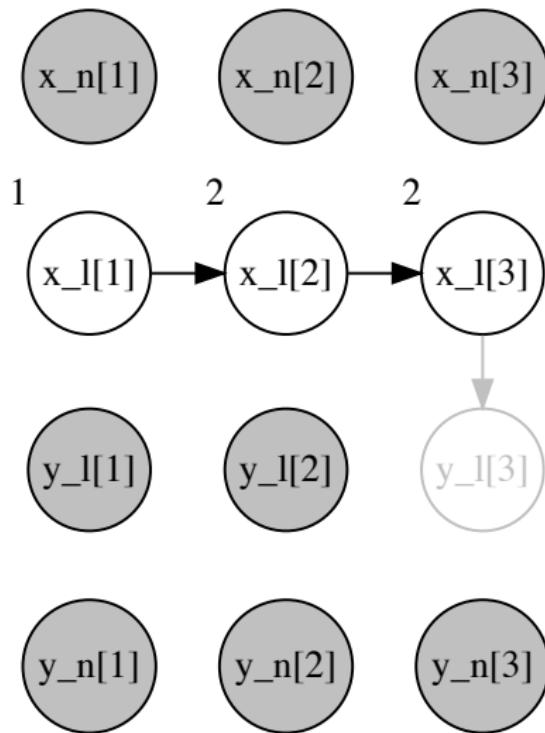
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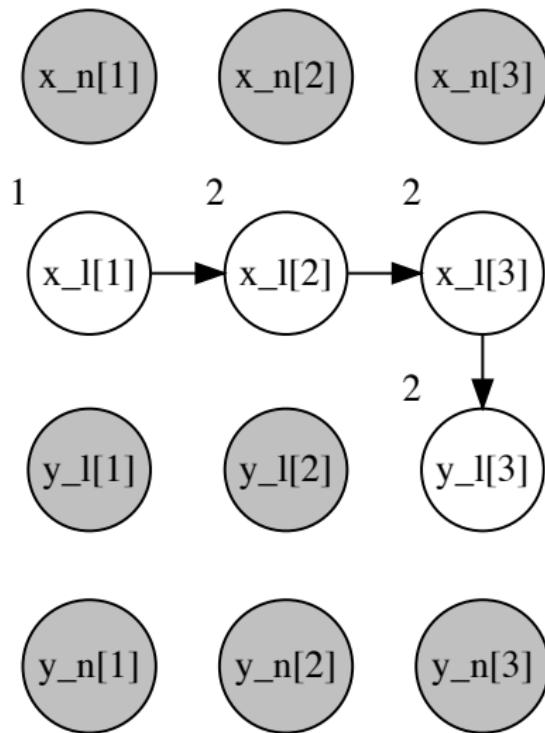
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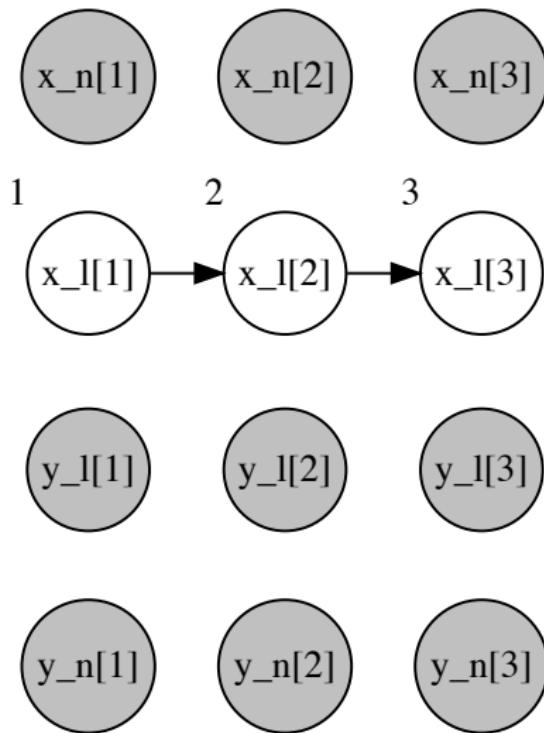
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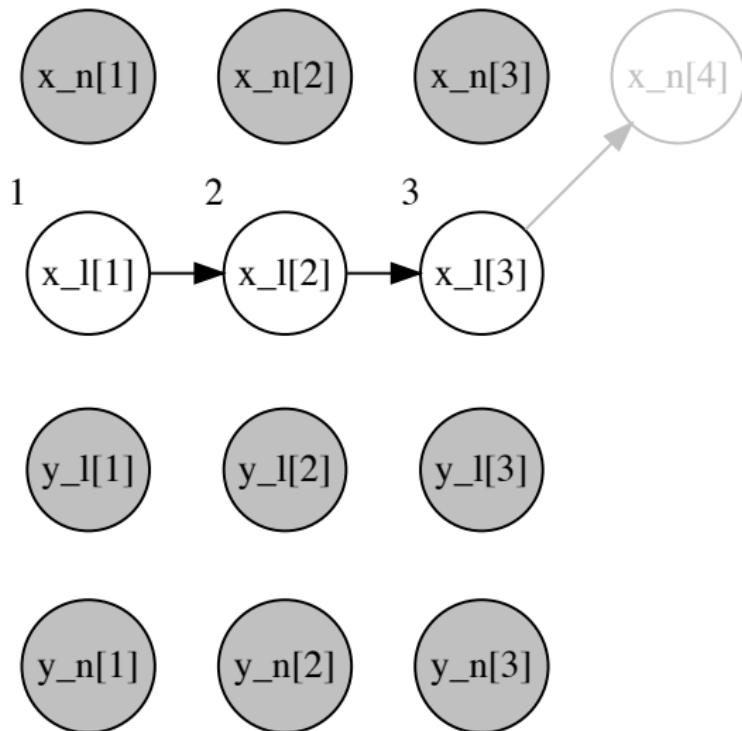
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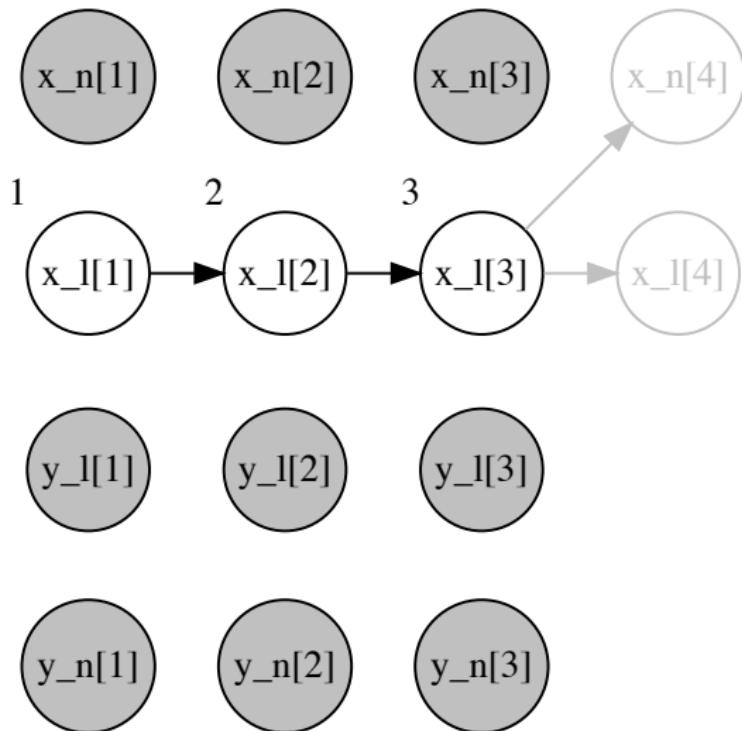
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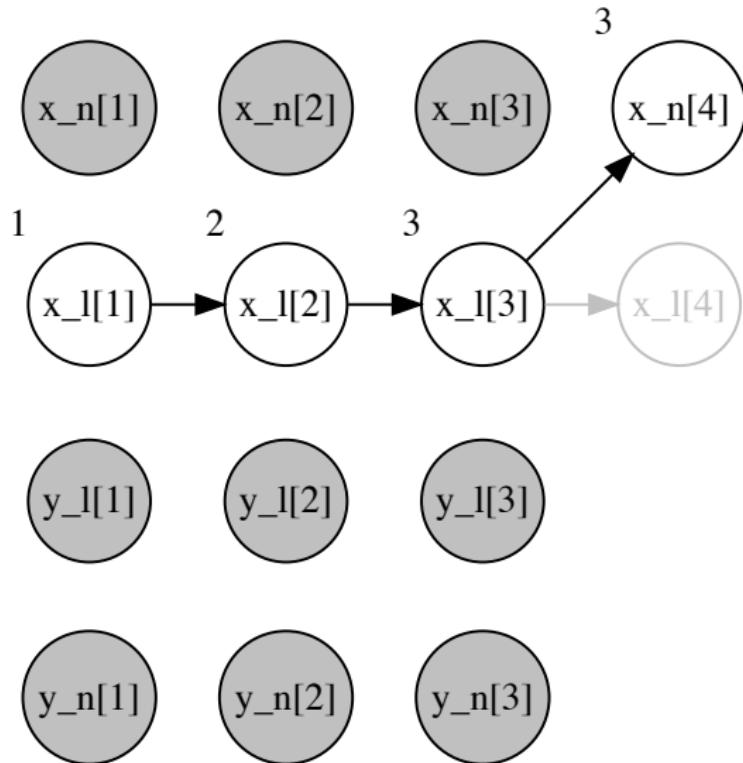
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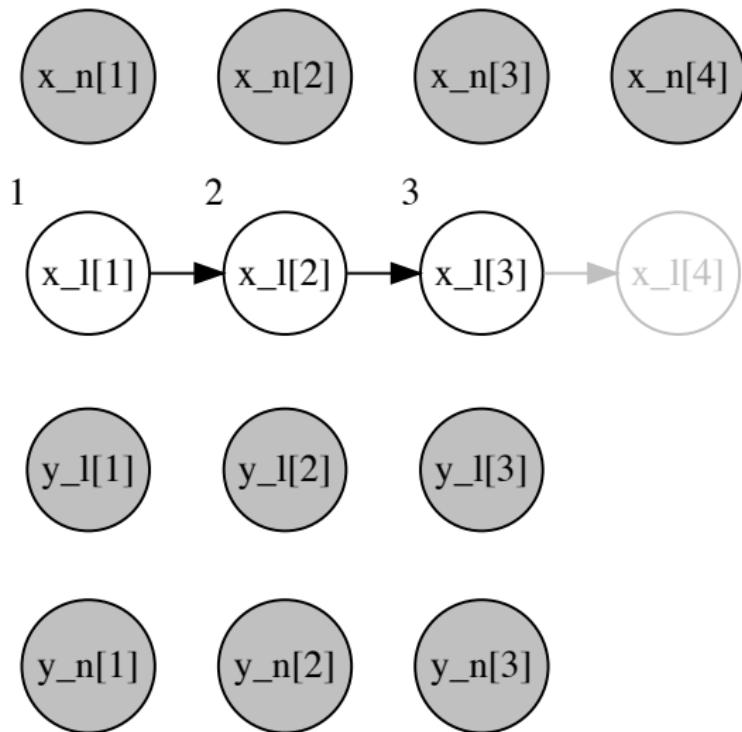
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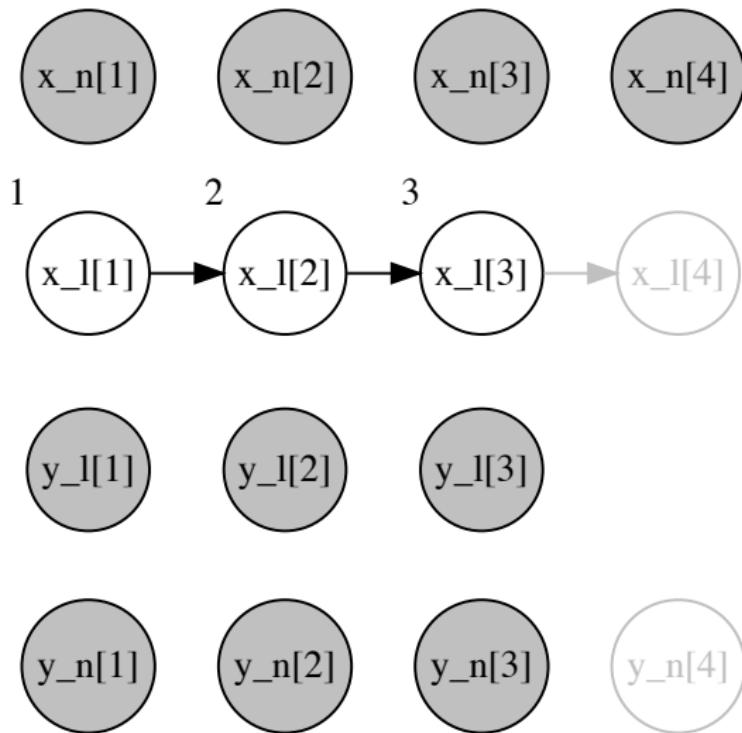
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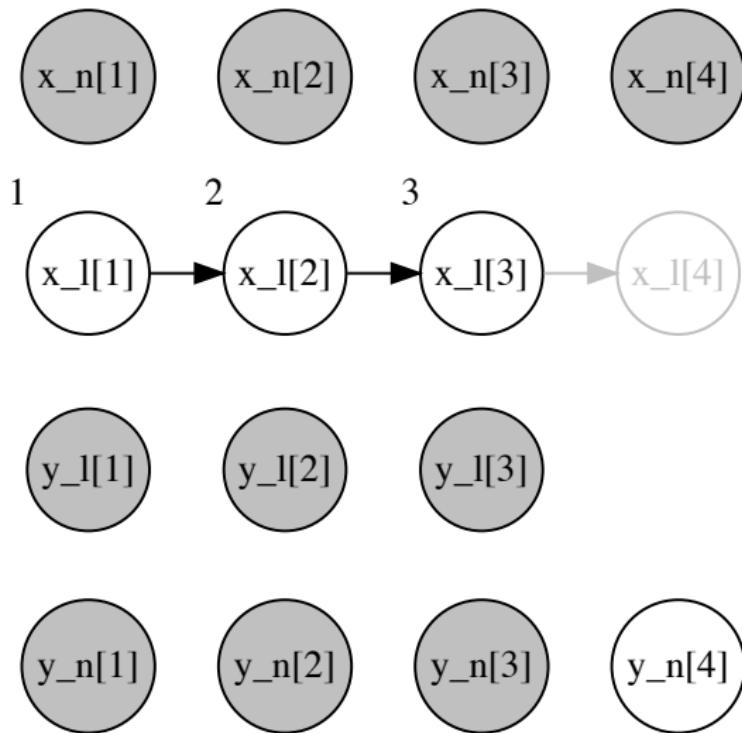
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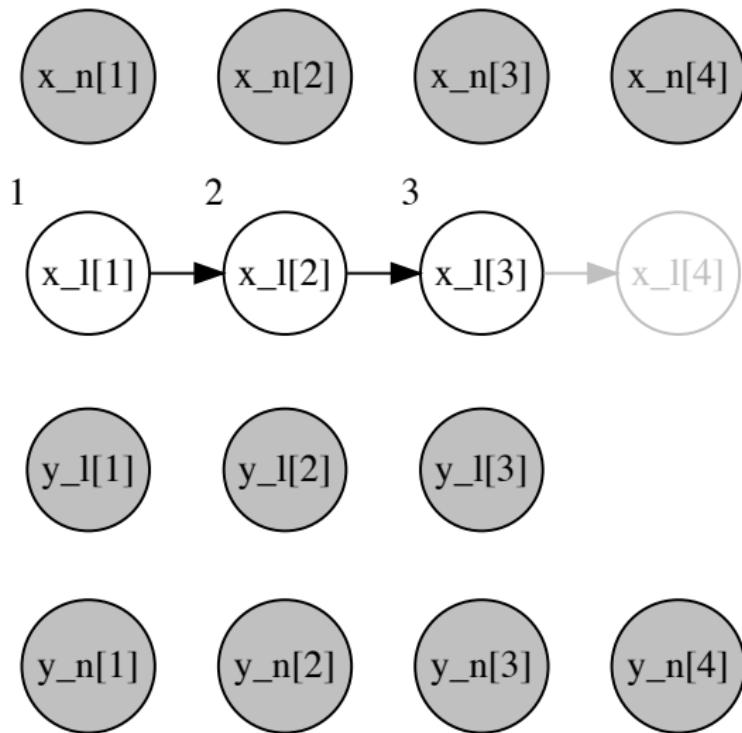
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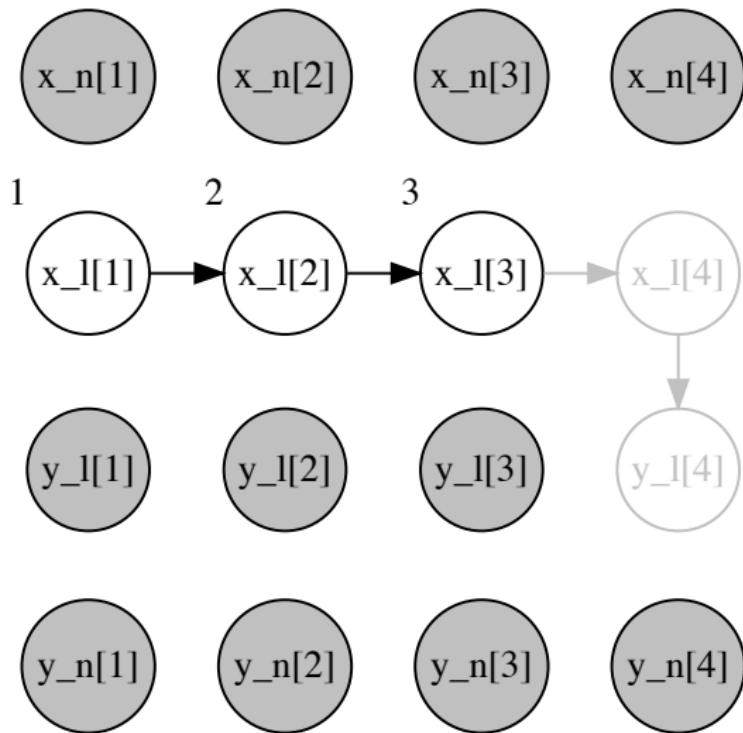
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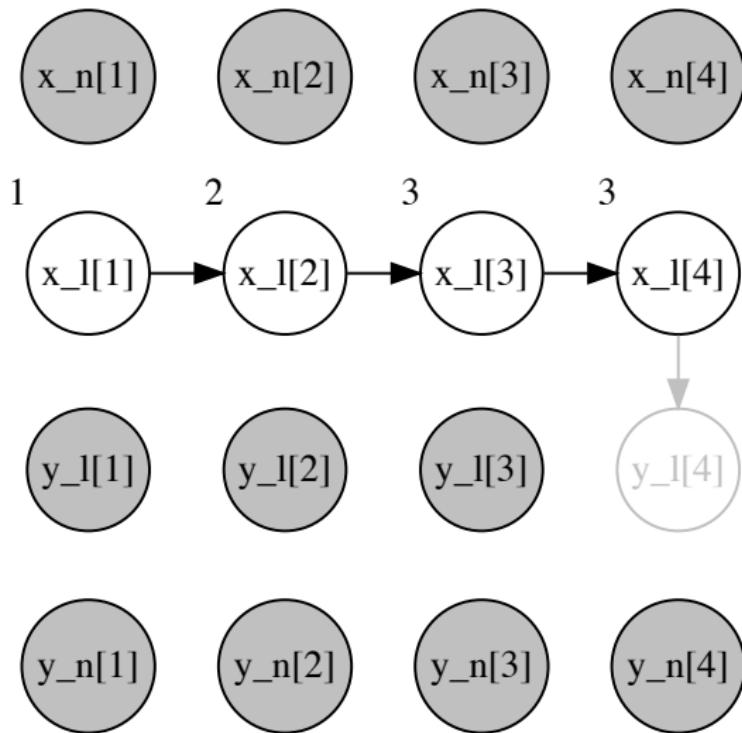
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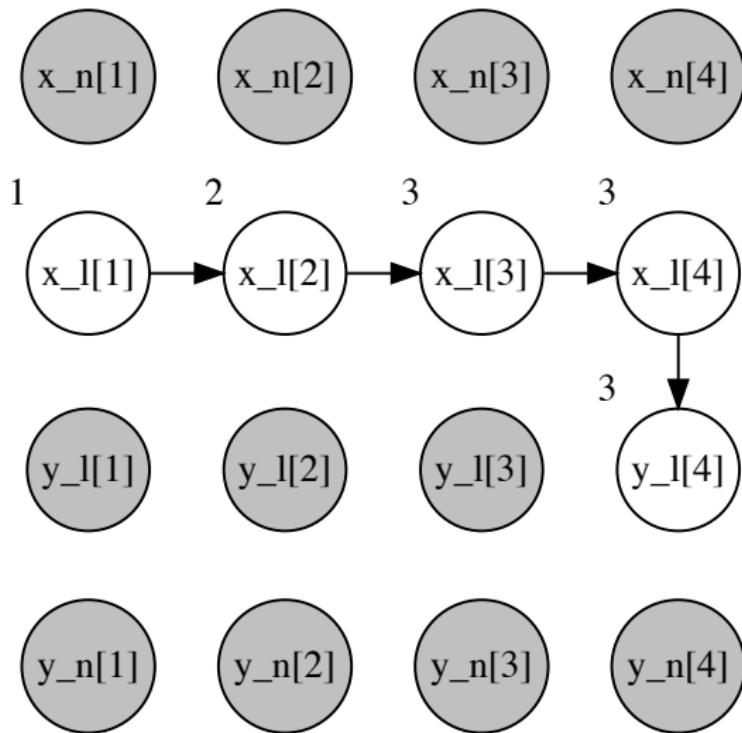
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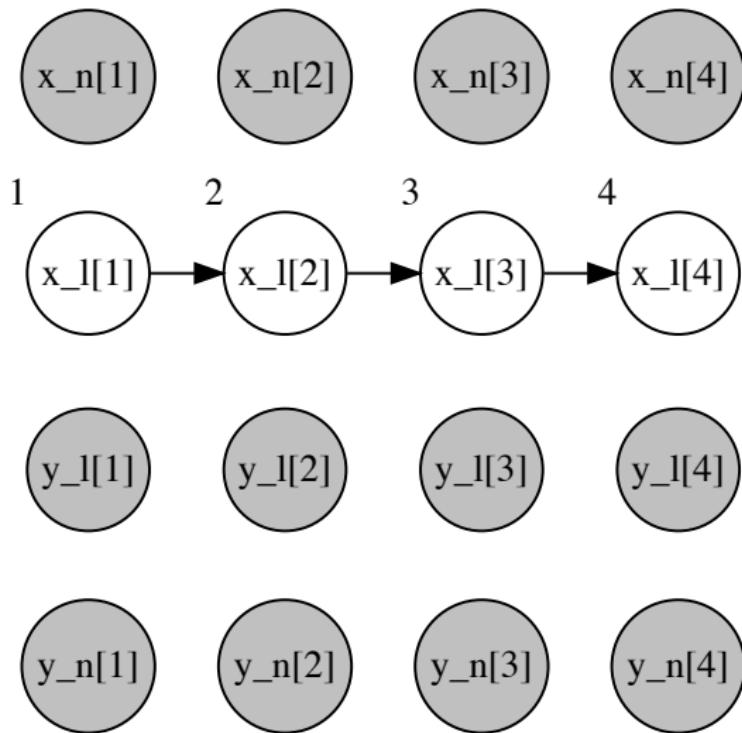
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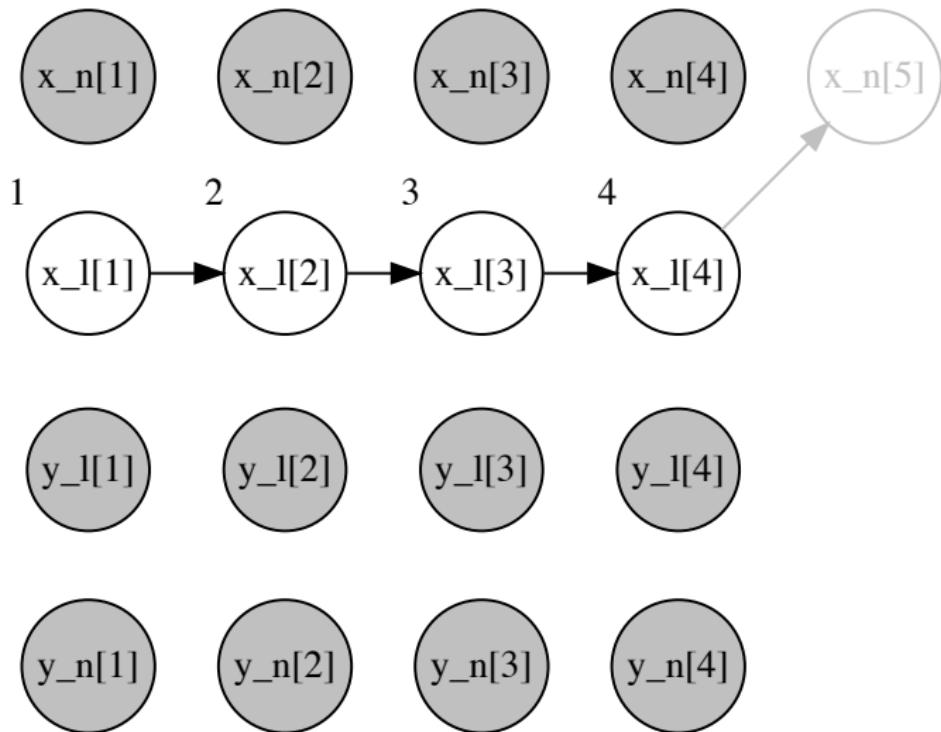
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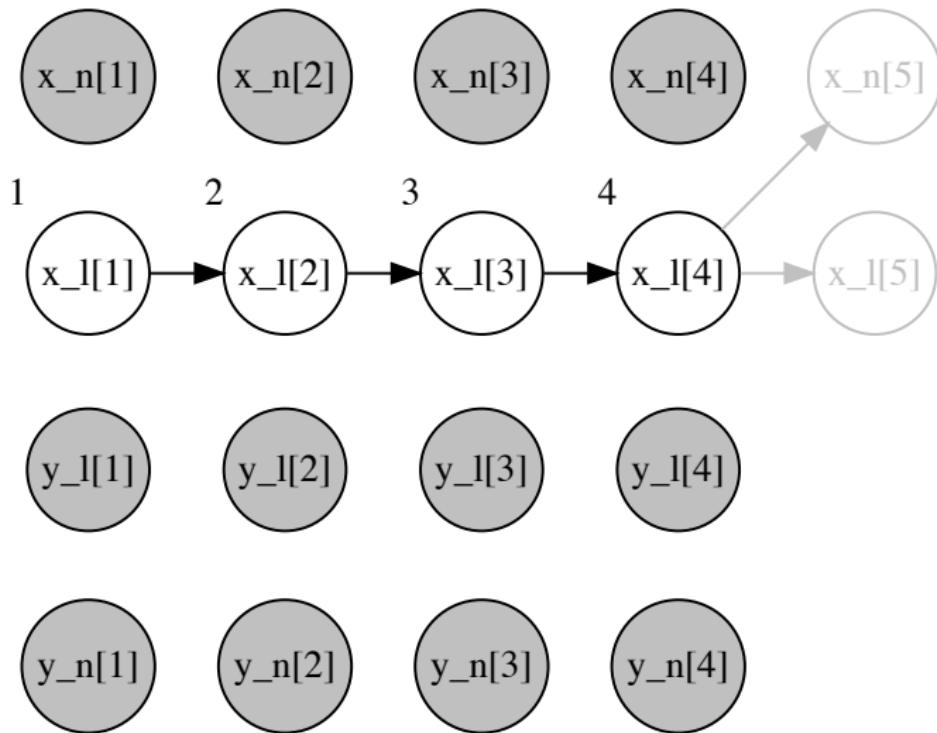
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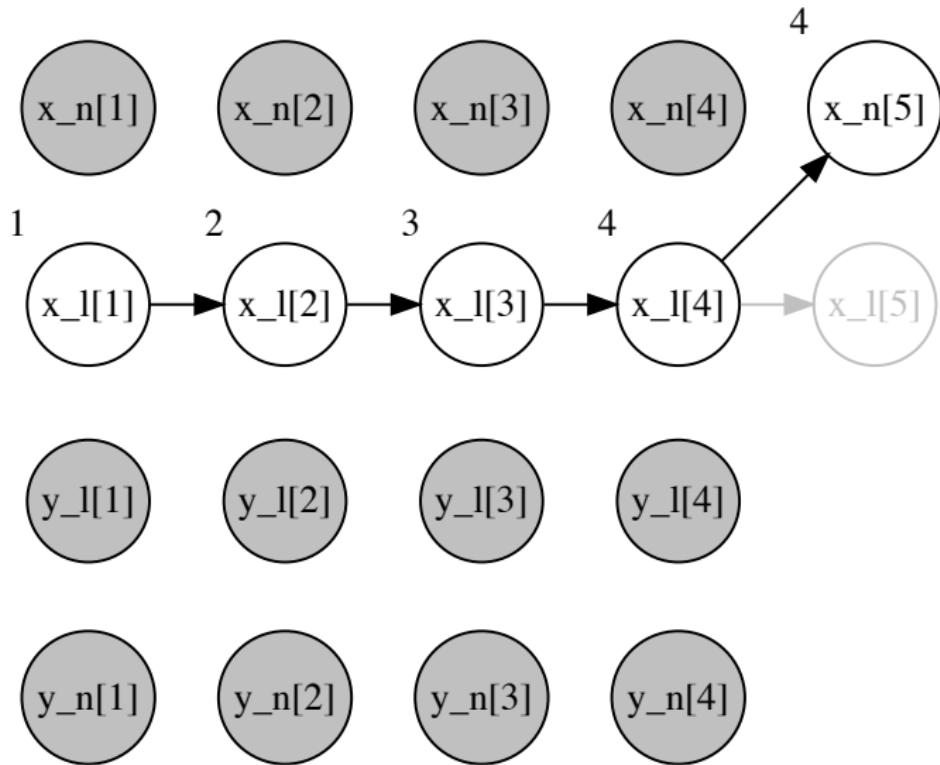
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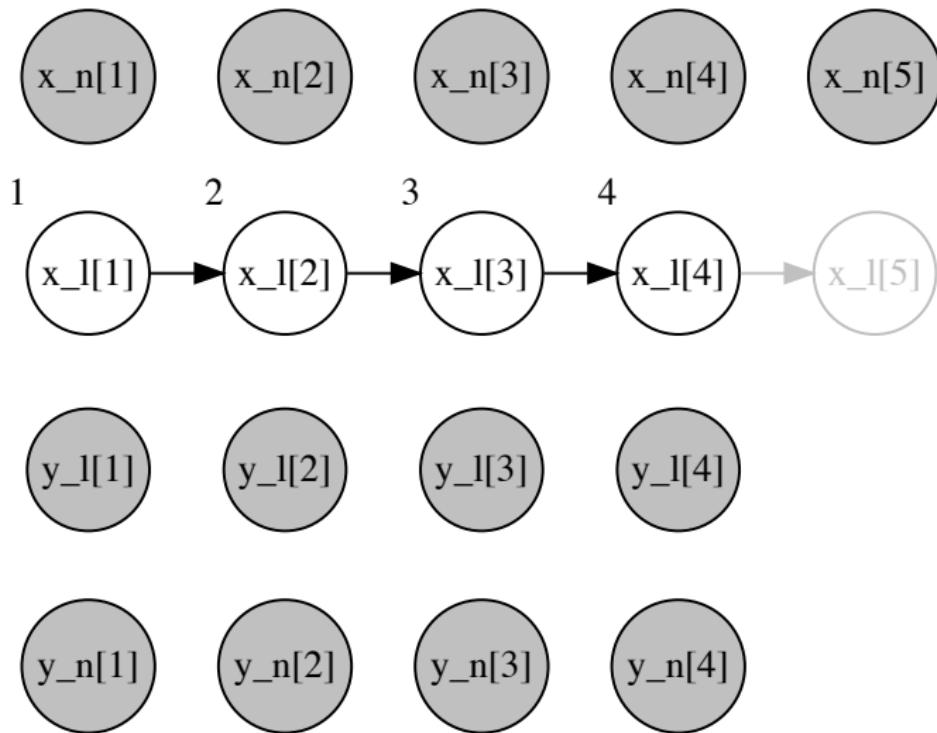
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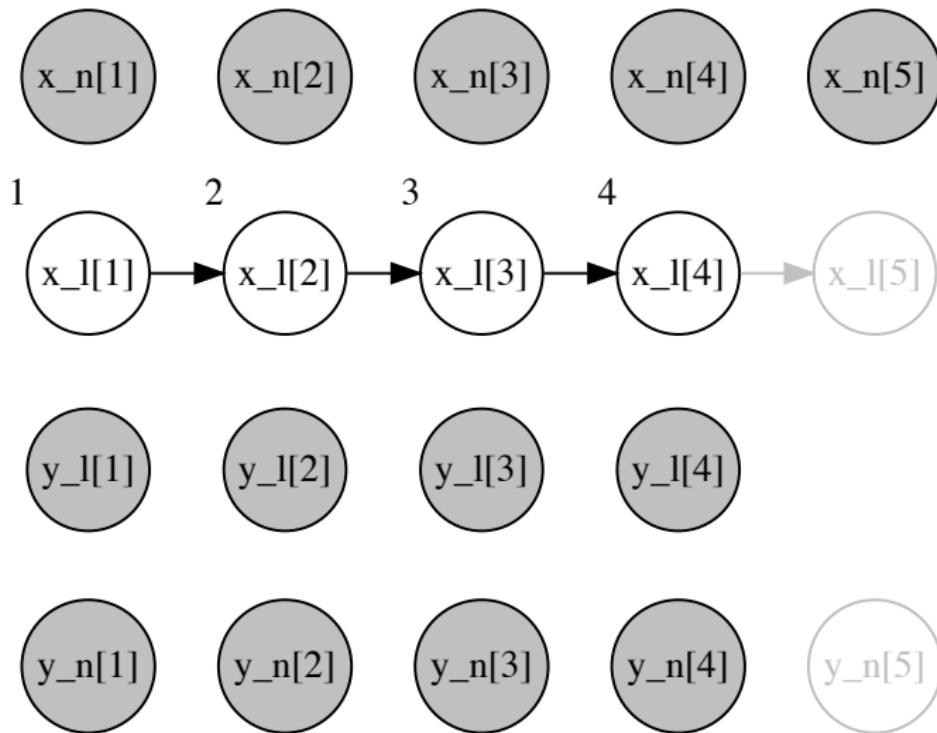
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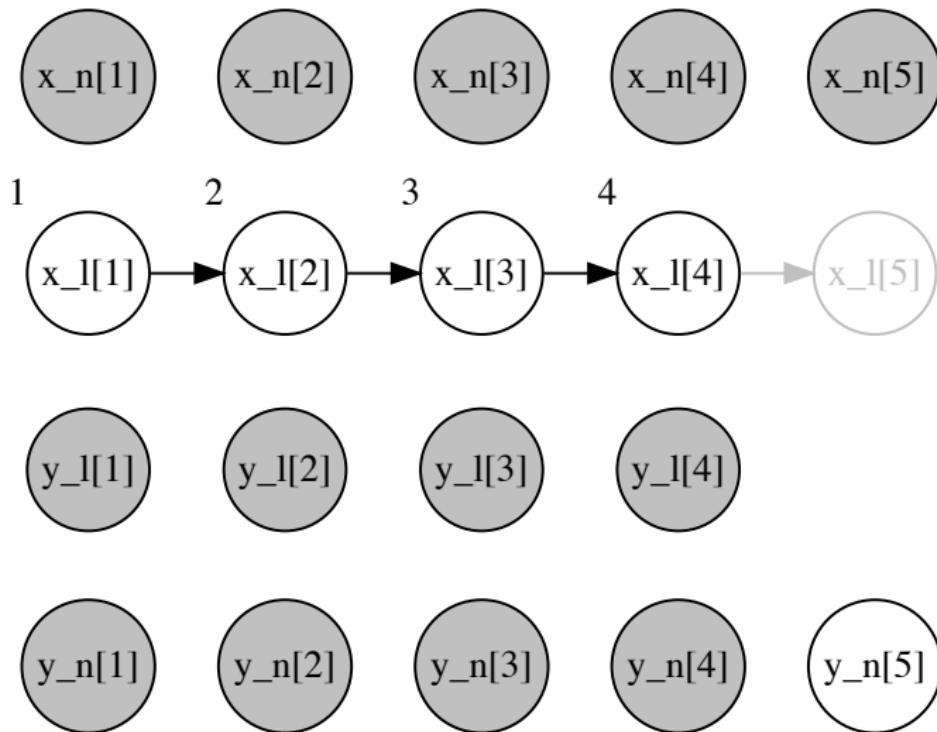
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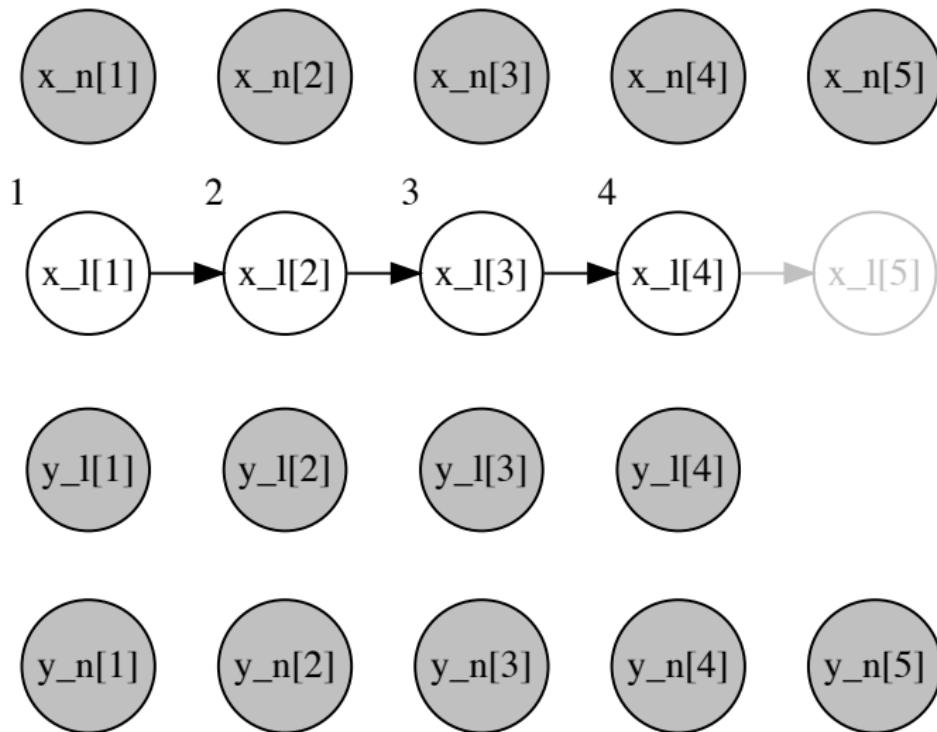
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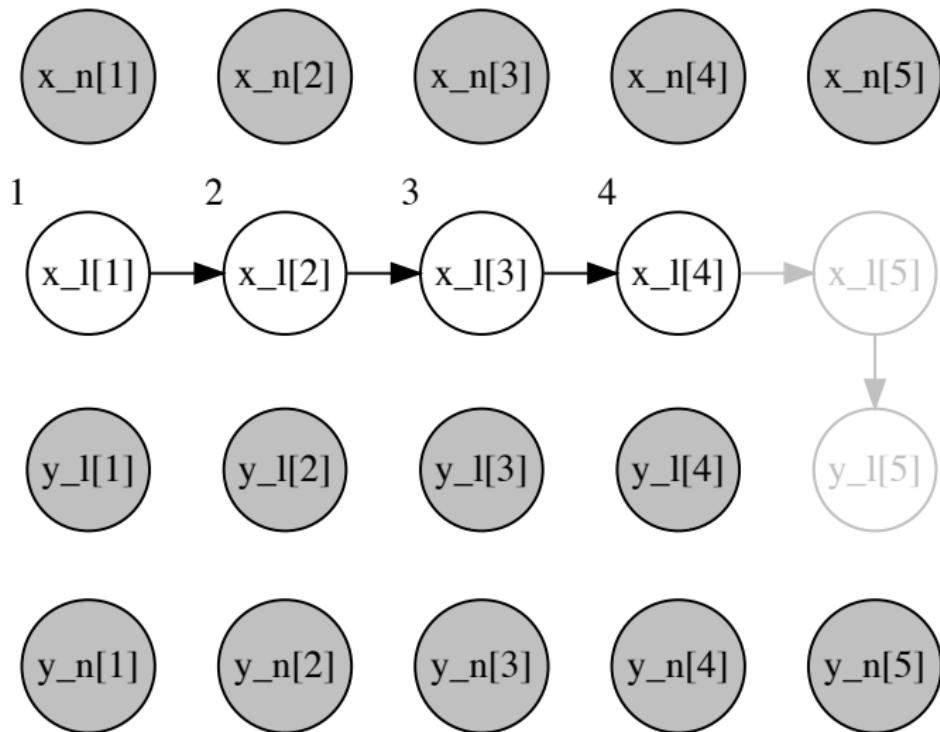
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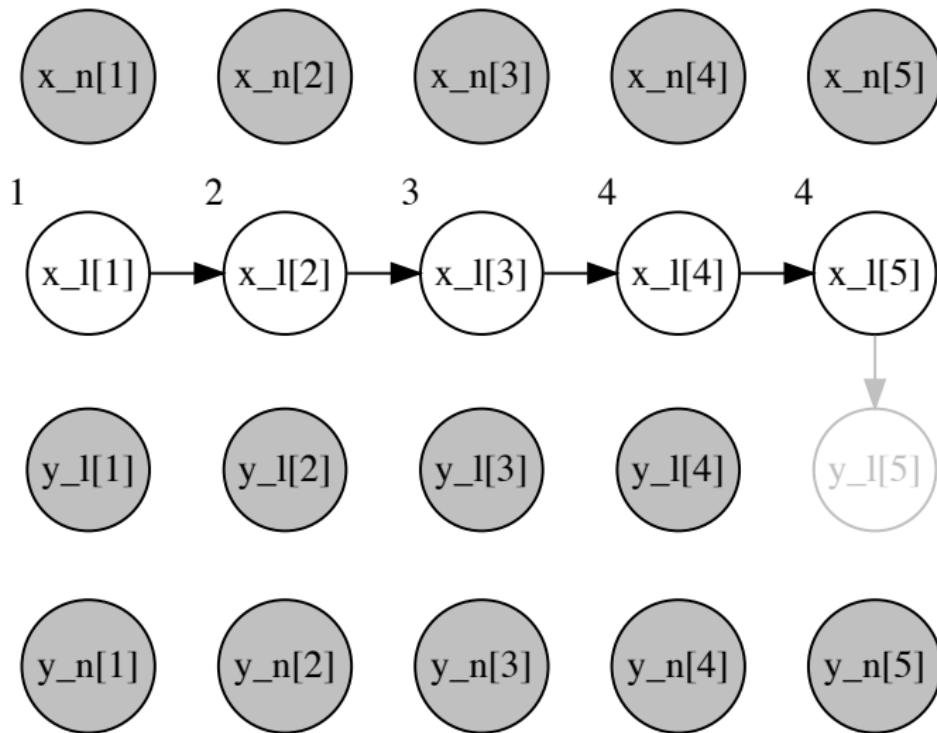
Example #5



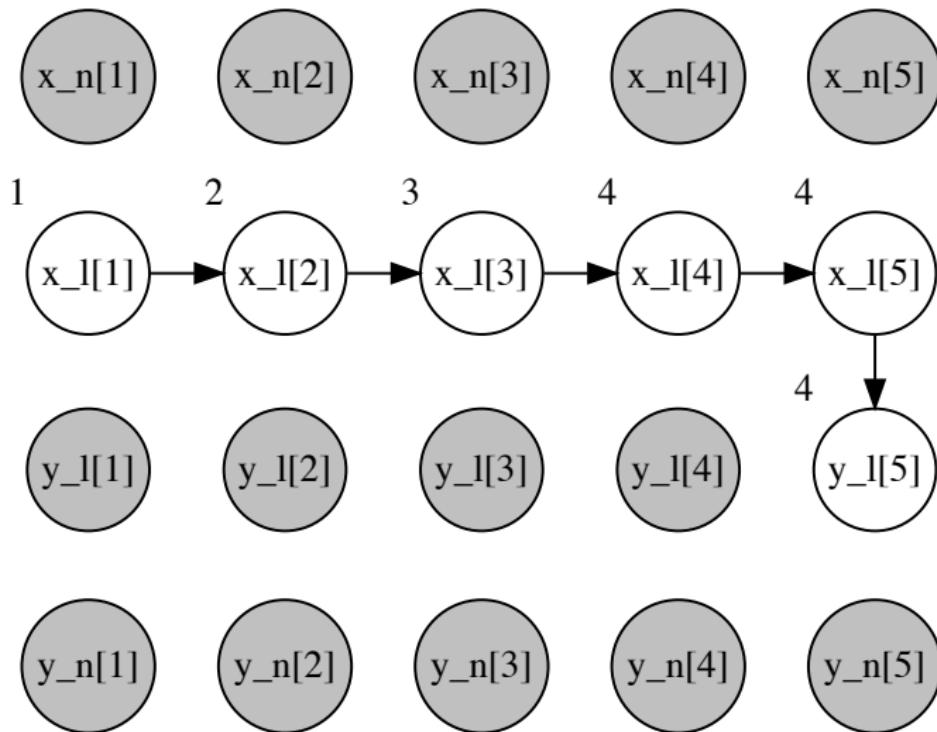
Example #5



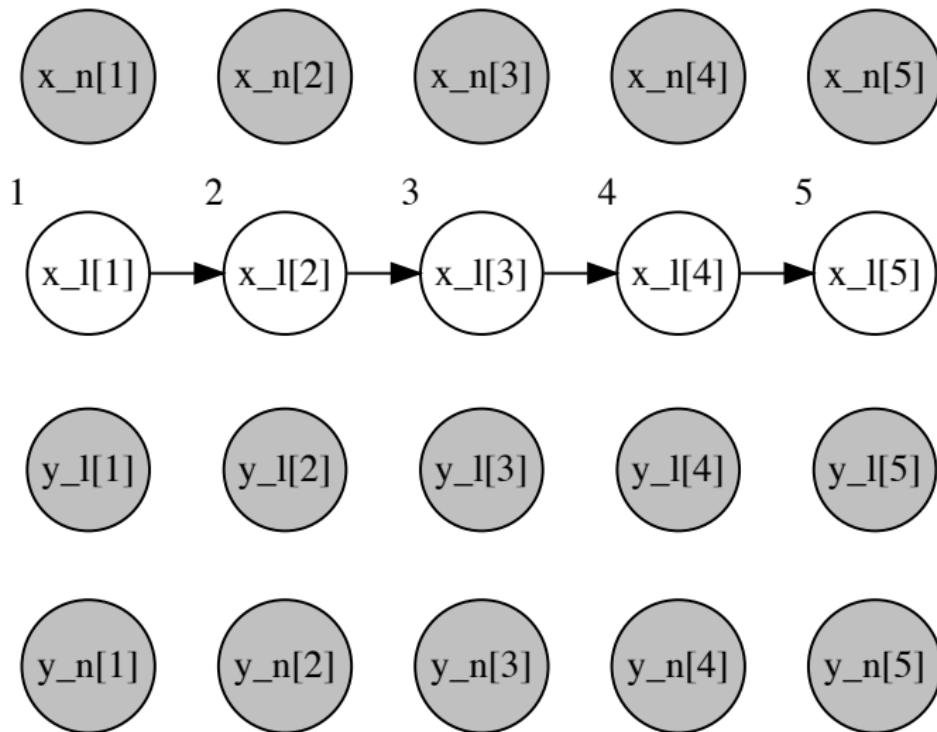
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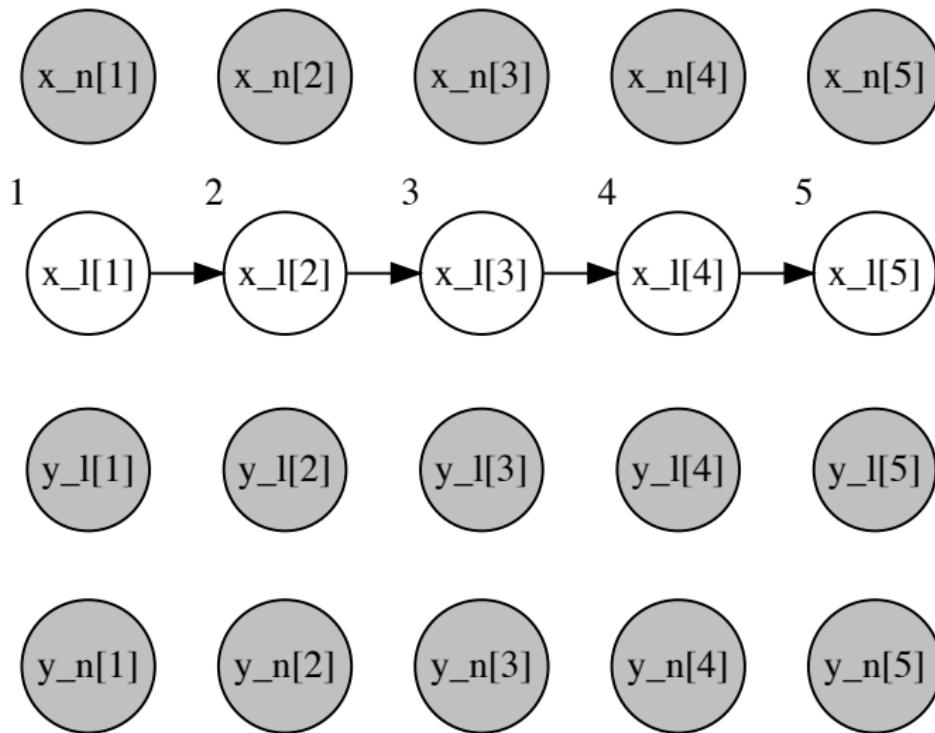
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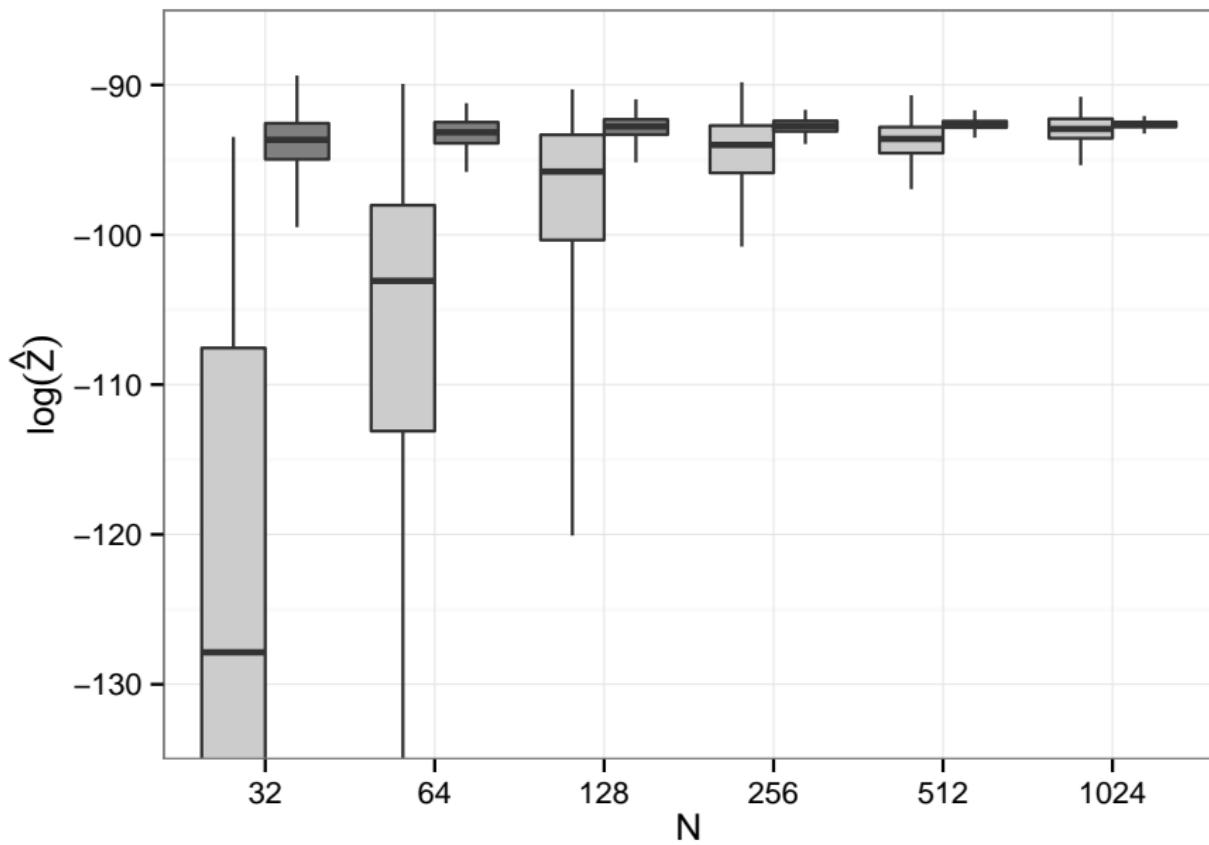
Example #5



Example #5: Rao–Blackwellized Particle Filter



Example #5: Rao–Blackwellized Particle Filter



Limitation #1

The graph kept alongside the running probabilistic program must be a forest of disjoint trees.

- ▶ For more general graphs (e.g. with multivariate normals), multiple nodes can be grouped into “supernodes”, where the supernodes form trees (c.f. the junction tree algorithm, belief propagation on factor graphs).
- ▶ Or, one or more analytical relationships can simply be ignored, so that some variables are sampled eagerly.
- ▶ But delayed sampling still produces correctly-weighted importance samples, it just misses an opportunity for variance reduction.

Limitation #2

Delayed sampling arbitrarily changes the order of random variables that are sampled, and their interleaving amongst random variables that are observed, but does not reorder random variables that are observed. This means that some analytical relationships are not visible.

- ▶ Reordering of observations is a possible extension, but requires more complex graph operations.
- ▶ The order of observations given by the programmer is likely to be a reasonable one.
- ▶ Again, delayed sampling still produces correctly-weighted importance samples, it just misses an opportunity for variance reduction.

Summary

- ▶ Delayed sampling can automate variance reduction techniques such as Rao–Blackwellization, locally-optimal proposals, and variable elimination.
- ▶ Very little modification is required to program code.
- ▶ It has been implemented in Birch, a new probabilistic programming language under development at Uppsala University, as well as in Anglican.
- ▶ The paper is on arXiv: <http://arxiv.org/abs/1708.07787>