UPPSALA UNIVERSITY
Department of Information Technology
Division of Scientific Computing

Exam in Finite element methods 2010-12-21

- Time: $08^{00} - 13^{00}$.
- Tools: Pocket calculator, Beta Mathematics Handbook.
- This is an exam *without points*; each problem is graded separately with respect to the learning objectives the problem targets.
- All your answers must be well argued and calculations shall be demonstrated in detail. *Solutions that are not complete can still be of value if they include some correct thoughts.*

Question 1

Let $I = (0, 1)$ and let $0 = x_0 < x_1 < \cdots < x_N = 1$ be a mesh of I. Further let $\{\varphi_i\}_{i=0}^N$ be a set of piecewise linear continuous basis functions satisfying $\varphi_i(x_j) = 1$ when $i = j$ and $= 0$ otherwise. You are given a FEM in the form of a linear system of equations $(A + M)\xi = b + d$, where A and M are N-by-N matrices, and ξ, b, and d are vectors of length N. Furthermore, for i, $j = 1 \ldots N$, $A_{ij} = (\varphi_i', \varphi_j')$, $M_{ij} = (\varphi_i, \varphi_j)$, and $b_i = (\varphi_i, f)$ with f some given function. The vector d has a single non-zero element: $d_N = \beta$.

(a) Derive both the variational and the strong formulation for the PDE from the information given.

(b) The mass-matrix M_{ij} can be assembled by using so-called local (element) mass-matrices. Describe briefly how this works.

(c) Suppose that $\beta = 0$. Derive an *a priori* bound for $\|U_x\|$, where $U = \sum_{j=1}^N \varphi_j \xi_j$ is the finite element solution. *Hint:* after using the standard energy-approach you might find it helpful to consider the inequality $ab \le (a^2 + b^2)/2$ for a, b real.

Question 2

The heat equation in 2D is given by $u_t = \kappa \Delta u$ and is posed in some smooth domain Ω with boundary conditions $u = 0$ on $\partial\Omega$ and some given initial data $u = u_0$ for $t = 0$.

(a) Formulate in continuous time a finite element method using standard linear basis functions on a discretization $\mathcal{K}$ of Ω. Then use the backward Euler method for discretizing time and formulate the equations that need to be solved in each step.

(b) Assume that at some time T, the error e has been estimated to satisfy $\|e\| \approx 10^{-1}$ in the $L^2(\Omega)$-norm. Given that the discretization is parameterized by $k := \max_i t_i - t_{i-1}$ and $h := \max_K h_K$, estimate how much smaller (h, k) needs to be in order to reach an error $\approx 10^{-3}$.

(c) For the analytical solution u, prove decay in the $L^2(\Omega)$-norm so that $\|u\| \le \|u_0\|$. Prove that the same result holds true for the fully discrete solution U_n from (a).

Question 3

Consider the mesh in Figure 1.

(a) State the (triangle) T-matrix for the mesh and also indicate the sparsity pattern of the associated mass-matrix.

(b) Refine triangle 1-2-4 uniformly once and triangle 4-7-6 by splitting the edge 4-7. Draw the resulting mesh and also explain the meaning of the term ‘hanging node’ — preferably by giving an example.

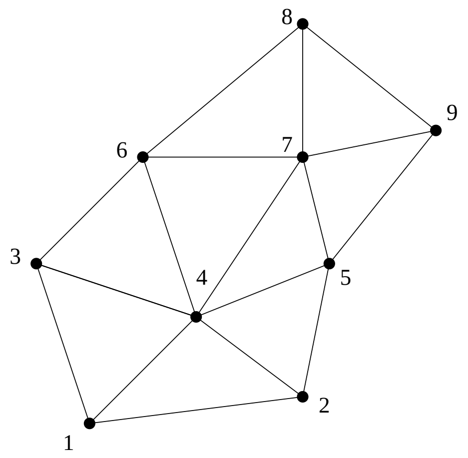

Figure 1: Sample 2D triangulation with the node numbering indicated.

(c) Write a mini-essay where you reflect over how finite element software might be used in the industry. Approximately 5–8 sentences will typically be enough.

(d) List a few techniques by which the accuracy obtained by a finite element software can be judged by the end-user.

Question 4

Consider the PDE $-\varepsilon u_{xx} = f$ on $I = (0, 1)$ with homogeneous Dirichlet boundary conditions and ε a positive constant.

(a) Derive the variational and finite element formulations (including the discrete set of equations to be solved). Use the same mesh and basis functions as defined in Question 1.

(b) State and prove a version of *Galerkin orthogonality* for this problem. Subsequently derive a *best approximation result* and explain the label "best approximation".

(c) Derive the *a posteriori* estimate $\varepsilon^2\|e_x\|^2 \le \text{const.} \times \sum_{i=1}^{N-1} R_i(U)^2$, with $R_i(U) = h_i\|f + \varepsilon U_{xx}\|_{L^2(I_i)}$ in terms of $I_i = [x_{i-1}, x_i]$, $h_i = x_i - x_{i-1}$, and where e is the difference between the true solution u and the FEM solution U. Use without proof whatever interpolation estimates you believe you need, for example $\|g - \pi g\|_{L^2(I_i)} \le Ch_i\|g_x\|_{L^2(I_i)}$ for π a linear interpolant on $\{x_{i-1}, x_i\}$.

(d) Explain in a couple of sentences how *a posteriori* error estimates such as the one above can be used to implement adaptivity.

Question 5

A simple version of the 2D wave equation is $u_{tt} = \epsilon\Delta u + f$ for $t > 0$ in some domain Ω together with homogeneous Neumann conditions $n \cdot \nabla u = 0$ on $\partial\Omega$. Assume also that suitable initial data $u = u_0$ and $u_t = v_0$ for $t = 0$ are available.

(a) Derive a semi-discrete (time-continuous) finite element method using standard linear basis functions on a discretization $\mathcal{K}$ of Ω. Then discretize time by using the trapezoidal rule (Crank-Nicolson method) and formulate the equations that need to be solved in each step.

(b) For $f = 0$, prove that $\|u_t(T)\|^2 + \epsilon\|\nabla u(T)\|^2 = \|v_0\|^2 + \epsilon\|\nabla u_0\|^2$ in the $L^2(\Omega)$-norm. Also interpret the result.

Good luck!
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