UPPSALA UNIVERSITY
Department of Information Technology
Division of Scientific Computing

Exam in Finite element methods II 2013-03-13

- Time: $8^{00} - 13^{00}$
- Tools: Pocket calculator, Beta Mathematics Handbook
- Maximum number of points: 40
- All your answers must be very well argued. Calculations shall be demonstrated in detail. Solutions that are not complete can give points if they include some correct thoughts.
- Use the Cauchy-Schwarz inequality $(v, w)_V \leq \|v\|_V \|w\|_V$ for $v, w \in V$ (Hilbert space) and the Poincare-Friedrichs inequality $\|v\|_{L^2(\Omega)} \leq C\|\nabla v\|_{L^2(\Omega)}$ for $v \in H_0^1(\Omega)$ without proves.

Problem 1

Consider the Neumann problem on weak form: find $u \in V = H^1(\Omega)$ such that,

$$a(u, v) := (A\nabla u, \nabla v) + (Cu, v) = (F, v) := l(v), \quad \forall v \in V,$$

where $\Omega \subset \mathbf{R}^d$ is a bounded domain, $0 < \alpha \leq A(x) \leq \beta$, $0 < \gamma \leq C(x) \leq \delta$, with $\alpha, \beta, \gamma, \delta \in \mathbf{R}$, $F \in L^2(\Omega)$, and $(v, w) = \int_\Omega v \cdot w \, dx$.

a) Use the Riesz Representation Theorem (RRT) to show existence of unique solution $u \in V$ to the weak form (you are not supposed to prove RRT but to show that the assumptions in the statement of the Theorem are fulfilled). [3p.]

b) Bound the solution u in the $H^1(\Omega)$-norm $\|v\|_{H^1(\Omega)} = \left(\|v\|^2_{L^2(\Omega)} + \|\nabla v\|^2_{L^2(\Omega)}\right)^{1/2}$ in terms of the data. [2p.]

c) What assumption in the formulation of RRT does not hold in the case when there is no reaction term (i.e. $C = 0$)? [2p.]

d) Discretize the problem using the finite element method. Present the arising linear system of equations. What can be said about the solvability of the linear system in the case $C = 0$? [3p.]

Problem 2

Let $\{\phi_i\}_{i=1}^3$ be the linear Lagrangian shape functions on the reference triangle K with nodes $(0,0)$, $(0,1)$, and $(1,0)$.

a) Express $\{\phi_i\}_{i=1}^3$ as functions of x and y. [2p.]

b) Let $\{\psi_i\}_{i=1}^6$ be the quadratic Lagrangian shape functions on the reference triangle. Express $\{\psi_i\}_{i=1}^6$ in terms of the linear shape functions $\{\phi_i\}_{i=1}^3$. [3p.]

c) Compute mass matrix entry $m_{22} = \int_K \psi_2 \psi_2 \, dx$, where ψ_2 is the shape function that is equal to one in $(1,0)$. [3p.]

d) Let $f(x) = x^3$. Compute the nodal interpolant of f onto the space of quadratic polynomials on K as a linear combination of $\{\psi_i\}_{i=1}^6$. [2p.]

Problem 3

Let $\Omega \subset \mathbf{R}^d$ and consider the non-linear problem:

$$\begin{aligned} -\nabla \cdot A(u)\nabla u &= f, \quad \text{in } \Omega, \\ u &= 0, \quad \text{in } \partial\Omega, \end{aligned} \tag{1}$$

where $A(u)$ is continuous in u and uniformly bounded $0 < \alpha \leq A(\cdot) \leq \beta$, and $f \in L^2(\Omega)$ independent of u.

a) Construct a fixed point iteration by inserting the k:th iterate of u in A, i.e. $A(u_k)$, and solve for u_{k+1}. Write the arising fixed point iteration on weak form. [2p.]

b) Show that given $u_k \in H_0^1(\Omega)$ there is a unique solution $u_{k+1} \in H_0^1(\Omega)$ to the resulting weak form. You can use the Lax-Milgram Lemma or RRT without proof. Furthermore show that the $H^1(\Omega)$ norm of u_{k+1} can be bounded independent of u_k. [3p.]

c) Schauder's fixed point Theorem states that if $u_k \in B_r = \{v \in H_0^1(\Omega) : \|v\|_{H^1(\Omega)} \leq r\}$ leads to $u_{k+1} \in B_r$ for some $r > 0$ (independent of u_k), there exists at least one weak solution to equation (1). Find such a value r. [2p.]

d) Assume now that $d = 1$. In one spatial dimension the Sobolev imbedding theorem gives $\|v\|_{L^\infty(\Omega)} \leq C\|v\|_{H^1(\Omega)}$ for all $v \in H^1(\Omega)$. Further assume $\|A(u) - A(v)\|_{L^\infty(\Omega)} \leq \kappa\|u - v\|_{L^\infty(\Omega)}$ for all $u, v \in H^1(\Omega)$. Show that if $\kappa > 0$ is sufficiently small, $\{u_k\}$ is a contracting sequence in $H^1(\Omega)$, i.e. $\|u_{k+1} - u_k\|_{H^1(\Omega)} \leq \gamma\|u_k - u_{k-1}\|_{H^1(\Omega)}$ for some $0 \leq \gamma < 1$, independent of k. This result shows existence and uniqueness of a weak solution to equation (1) (under stronger assumptions than in c).
Hint: $(uw, v) \leq \|u\|_{L^\infty(\Omega)}\|v\|_{L^2(\Omega)}\|w\|_{L^2(\Omega)}$ *for all* $v, w \in L^2(\Omega)$ *and* $u \in L^\infty(\Omega)$. [3p.]

Problem 4

Let $\Omega \subset \mathbf{R}^d$ and consider the transport problem:

$$\begin{aligned} -\epsilon\Delta u + b \cdot \nabla u + cu &= f, \quad \text{in } \Omega, \\ u &= 0, \quad \text{in } \partial\Omega, \end{aligned} \tag{2}$$

where $\epsilon > 0$, $b \in \{v \in L^\infty(\Omega) : \nabla \cdot v = 0\}$, $0 \leq c \in L^\infty(\Omega)$ and $f \in L^2(\Omega)$.

a) Derive the weak form of equation (2) with $V = H_0^1(\Omega)$. [2p.]

b) Show that $(b \cdot \nabla u, u) = 0$ if $\nabla \cdot b = 0$ and that the bilinear form (arising in the left hand side of the weak form) is coercive (you may use the full $H^1(\Omega)$ norm or the $H^1(\Omega)$ semi-norm, which are equivalent). How does the coercivity constant depend on ϵ? [4p.]

c) Derive the Galerkin-Least Squares method for equation (2). Use continuous piecewise linear basis functions. [4p.]

Good luck!
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