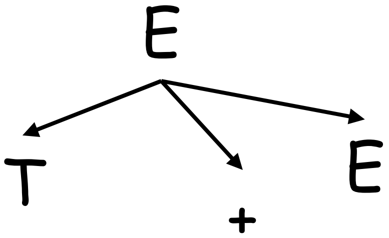
Introduction to Bottom-Up Parsing

Outline

- Review LL parsing
- Shift-reduce parsing
- The LR parsing algorithm
- Constructing LR parsing tables

Top-Down Parsing: Review

- Top-down parsing expands a parse tree from the start symbol to the leaves
  - Always expand the leftmost non-terminal

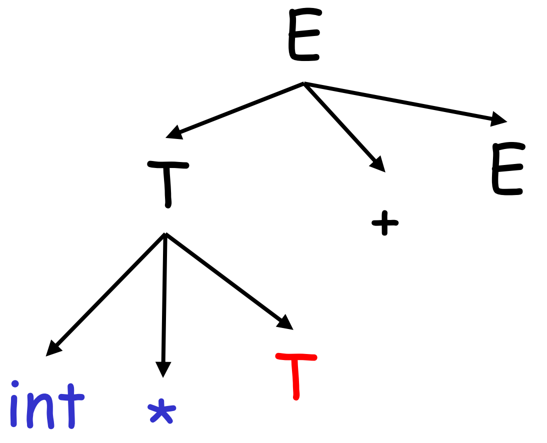


int * int + int

E → T + E | T
T → (E) | int | int * T

Top-Down Parsing: Review

- Top-down parsing expands a parse tree from the start symbol to the leaves
  - Always expand the leftmost non-terminal



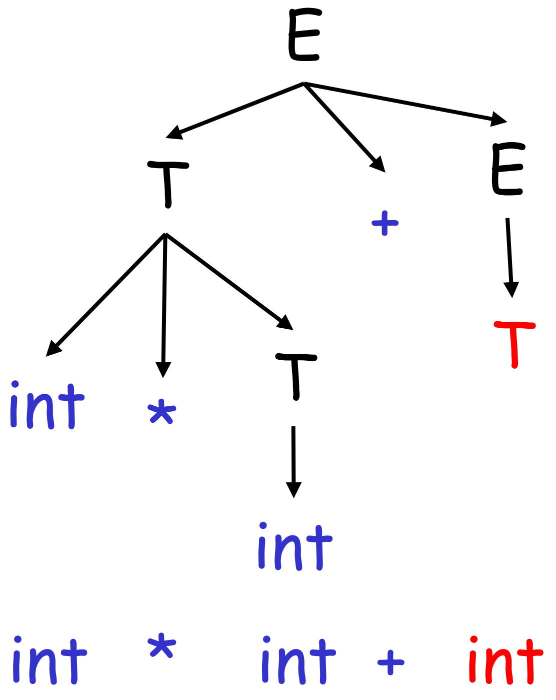
- The leaves at any point form a string βAγ
  - β contains only terminals
  - The input string is βbδ
  - The prefix β matches
  - The next token is b

int * int + int

$E \rightarrow T + E \mid T$
$T \rightarrow (E) \mid int \mid int * T$

Top-Down Parsing: Review

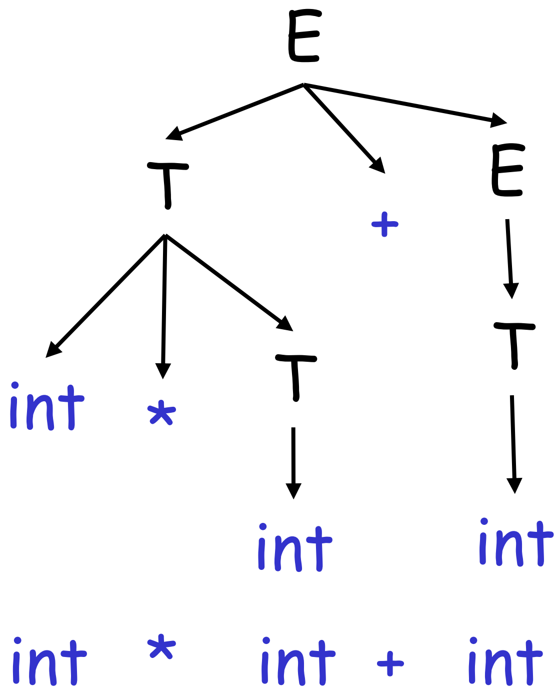
- Top-down parsing expands a parse tree from the start symbol to the leaves
  - Always expand the leftmost non-terminal



- The leaves at any point form a string $\beta A \gamma$
  - β contains only terminals
  - The input string is $\beta b \delta$
  - The prefix β matches
  - The next token is b

Top-Down Parsing: Review

- Top-down parsing expands a parse tree from the start symbol to the leaves
  - Always expand the leftmost non-terminal



- The leaves at any point form a string $\beta A \gamma$
  - β contains only terminals
  - The input string is $\beta b \delta$
  - The prefix β matches
  - The next token is b

Predictive Parsing: Review

- A predictive parser is described by a table
  - For each non-terminal A and for each token b we specify a production $A \to \alpha$
  - When trying to expand A we use $A \to \alpha$ if b is the token that follows next

- Once we have the table
  - The parsing algorithm is simple and fast
  - No backtracking is necessary

Constructing Predictive Parsing Tables

Consider the state $S \rightarrow^* \beta A \gamma$

- With b the next token
- Trying to match $\beta b \delta$

There are two possibilities:

1. Token b belongs to an expansion of A
   - Any $A \rightarrow \alpha$ can be used if b can start a string derived from α
   - We say that $b \in \text{First}(\alpha)$

Or…

Constructing Predictive Parsing Tables (Cont.)

2. Token b does not belong to an expansion of A
   - The expansion of A is empty and b belongs to an expansion of γ
   - Means that b can appear after A in a derivation of the form $S \to^* \beta A b \omega$
   - We say that $b \in \text{Follow}(A)$ in this case

   - What productions can we use in this case?
     - Any $A \to \alpha$ can be used if α can expand to ε
     - We say that $\varepsilon \in \text{First}(A)$ in this case

Computing First Sets

Definition

$$\text{First}(X) = \{\, b \mid X \to^* b\alpha \,\} \cup \{\, \varepsilon \mid X \to^* \varepsilon \,\}$$

Algorithm sketch

1. $\text{First}(b) = \{\, b \,\}$
2. $\varepsilon \in \text{First}(X)$ if $X \to \varepsilon$ is a production
3. $\varepsilon \in \text{First}(X)$ if $X \to A_1 \ldots A_n$
   and $\varepsilon \in \text{First}(A_i)$ for $1 \le i \le n$
4. $\text{First}(\alpha) \subseteq \text{First}(X)$ if $X \to A_1 \ldots A_n\, \alpha$
   and $\varepsilon \in \text{First}(A_i)$ for $1 \le i \le n$

First Sets: Example

- Recall the grammar

  E → T X
  T → (E) | int Y

  X → + E | ε
  Y → * T | ε

- First sets

  First(( ) = { (}
  First( )) = {) }
  First(int) = { int }
  First(+) = { + }
  First(*) = { * }

  First(T) = { int, (}
  First(E) = { int, (}
  First(X) = { +, ε }
  First(Y) = { *, ε }

Computing Follow Sets

- Definition

  $$\text{Follow}(X) = \{ b \mid S \rightarrow^* \beta\, X\, b\, \delta \}$$

- Intuition
  - If $X \rightarrow A\ B$ then $\text{First}(B) \subseteq \text{Follow}(A)$ and $\text{Follow}(X) \subseteq \text{Follow}(B)$
  - Also if $B \rightarrow^* \varepsilon$ then $\text{Follow}(X) \subseteq \text{Follow}(A)$
  - If S is the start symbol then $\$ \in \text{Follow}(S)$

Computing Follow Sets (Cont.)

Algorithm sketch

1. $\$ \in \text{Follow}(S)$
2. $\text{First}(\beta) - \{\varepsilon\} \subseteq \text{Follow}(X)$
   - For each production $A \to \alpha\ X\ \beta$
3. $\text{Follow}(A) \subseteq \text{Follow}(X)$
   - For each production $A \to \alpha\ X\ \beta$ where $\varepsilon \in \text{First}(\beta)$

Follow Sets: Example

First(T) = { int, (}
First(E) = { int, (}
First(X) = { +, ε }
First(Y) = { *, ε }

- Recall the grammar

$E \to T\ X$

$T \to (\ E\ )\ |\ \text{int}\ Y$

$X \to +\ E\ |\ \varepsilon$

$Y \to *\ T\ |\ \varepsilon$

- Follow sets

Follow(+) = { int, (}
Follow(( ) = { int, (}
Follow(X) = { $,) }
Follow( )) = { +,) , $ }
Follow(int) = { *, +,) , $ }

Follow(*) = { int, (}
Follow(E) = {), $ }
Follow(T) = { +,) , $ }
Follow(Y) = { +,) , $ }

Constructing LL(1) Parsing Tables

- Construct a parsing table T for CFG G

- For each production $A \to \alpha$ in G do:
  - For each terminal $b \in \text{First}(\alpha)$ do

    $T[A, b] = \alpha$
  - If $\varepsilon \in \text{First}(\alpha)$, for each $b \in \text{Follow}(A)$ do

    $T[A, b] = \alpha$
  - If $\varepsilon \in \text{First}(\alpha)$ and $\$ \in \text{Follow}(A)$ do

    $T[A, \$] = \alpha$

Constructing LL(1) Tables: Example

- Recall the grammar

$E \rightarrow T\,X \qquad\qquad X \rightarrow +\,E \mid \varepsilon$

$T \rightarrow (\,E\,) \mid \text{int}\;Y \qquad\qquad Y \rightarrow *\;T \mid \varepsilon$

- Where in the line of Y do we put $Y \rightarrow * \; T$?
  - In the lines of First(*T) = { * }

- Where in the line of Y do we put $Y \rightarrow \varepsilon$?
  - In the lines of Follow(Y) = { $, +,) }

Notes on LL(1) Parsing Tables

- If any entry is multiply defined then G is not LL(1)
  - If G is ambiguous
  - If G is left recursive
  - If G is not left-factored
  - And in other cases as well

- For some grammars there is a simple parsing strategy: *Predictive parsing*
- Most programming language grammars are not LL(1)
- Thus, we need more powerful parsing strategies

Bottom Up Parsing

Bottom-Up Parsing

- Bottom-up parsing is more general than top-down parsing
  - And just as efficient
  - Builds on ideas in top-down parsing
  - Preferred method in practice

- Also called LR parsing
  - L means that tokens are read left-to-right
  - R means that it constructs a rightmost derivation !

An Introductory Example

- LR parsers don't need left-factored grammars and can also handle left-recursive grammars

- Consider the following grammar:

  E → E + (E) | int

  - Why is this not LL(1)?

- Consider the string: int + (int) + (int)

The Idea

- LR parsing *reduces* a string to the start symbol by inverting productions:

```
str w input string of terminals
repeat
    – Identify $\beta$ in str such that $A \rightarrow \beta$ is a production
      (i.e., str = $\alpha\ \beta\ \gamma$)
    – Replace $\beta$ by $A$ in str (i.e., str w = $\alpha\ A\ \gamma$)
until str = S  (the start symbol)
  OR all possibilities are exhausted
```

A Bottom-up Parse in Detail (1)

E → E + (E) | int

int + (int) + (int)

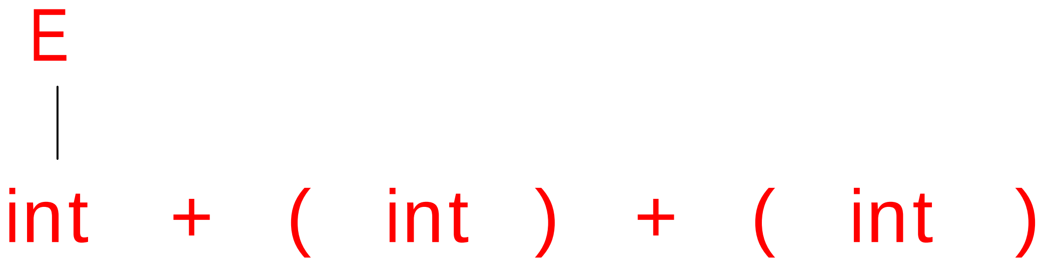
int + (int) + (int)

A Bottom-up Parse in Detail (2)

E → E + (E) | int

int + (int) + (int)

E + (int) + (int)



E

int + (int) + (int)

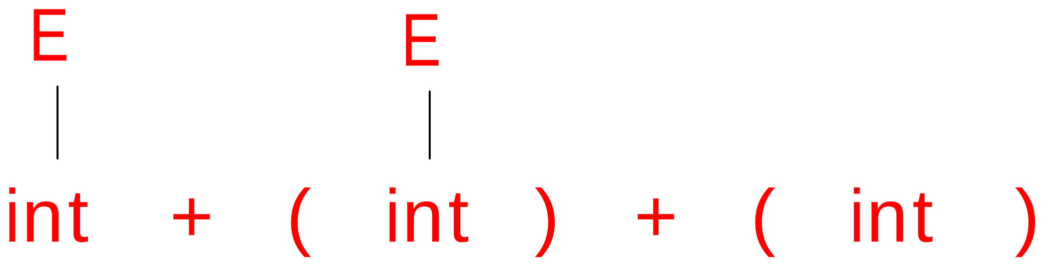
A Bottom-up Parse in Detail (3)

E → E + (E) | int

int + (int) + (int)

E + (int) + (int)

E + (E) + (int)



A Bottom-up Parse in Detail (4)

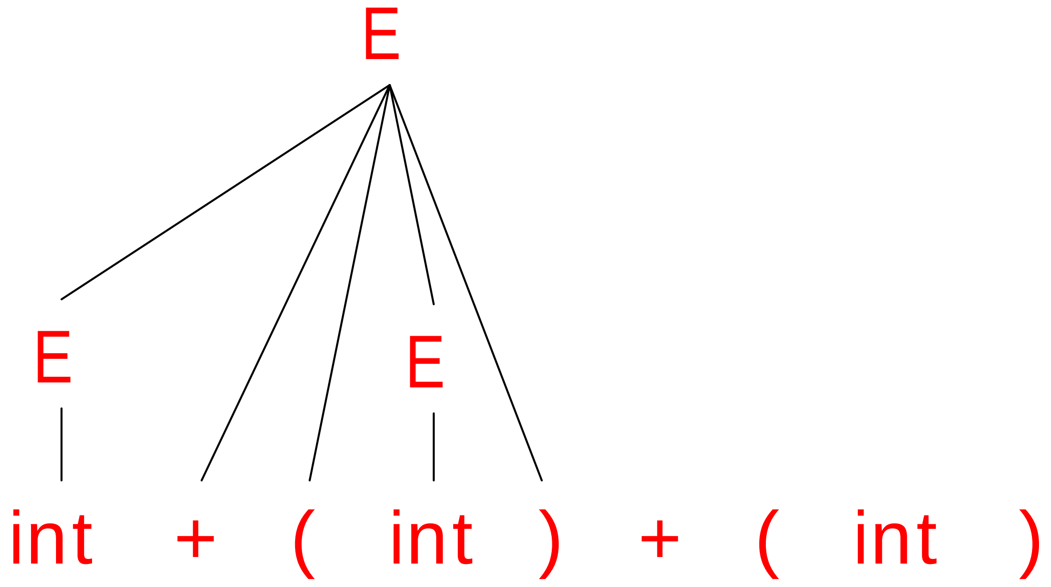
E → E + (E) | int

int + (int) + (int)

E + (int) + (int)

E + (E) + (int)

E + (int)



A Bottom-up Parse in Detail (5)

E → E + (E) | int

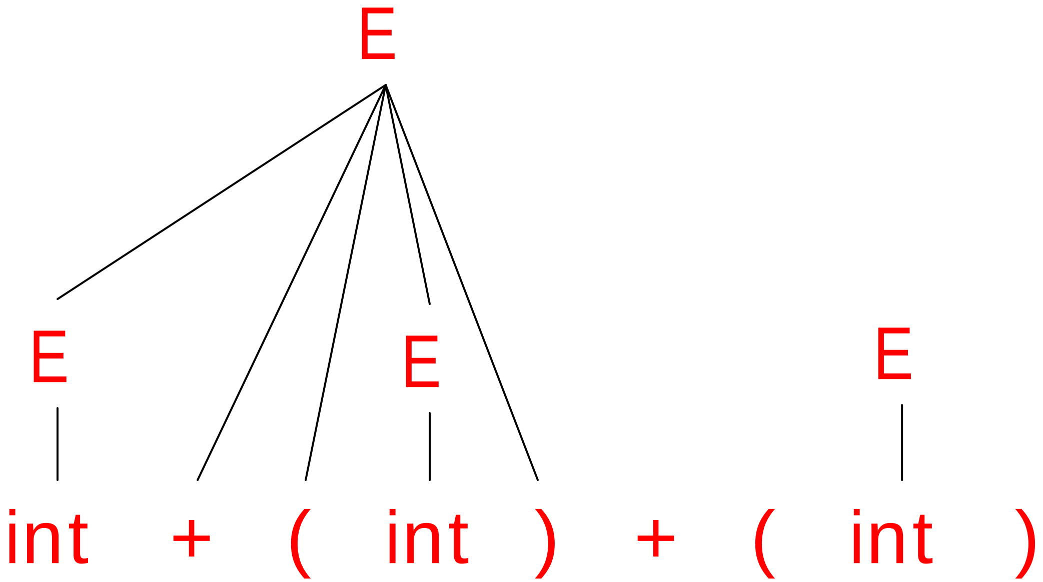
int + (int) + (int)

E + (int) + (int)

E + (E) + (int)

E + (int)

E + (E)

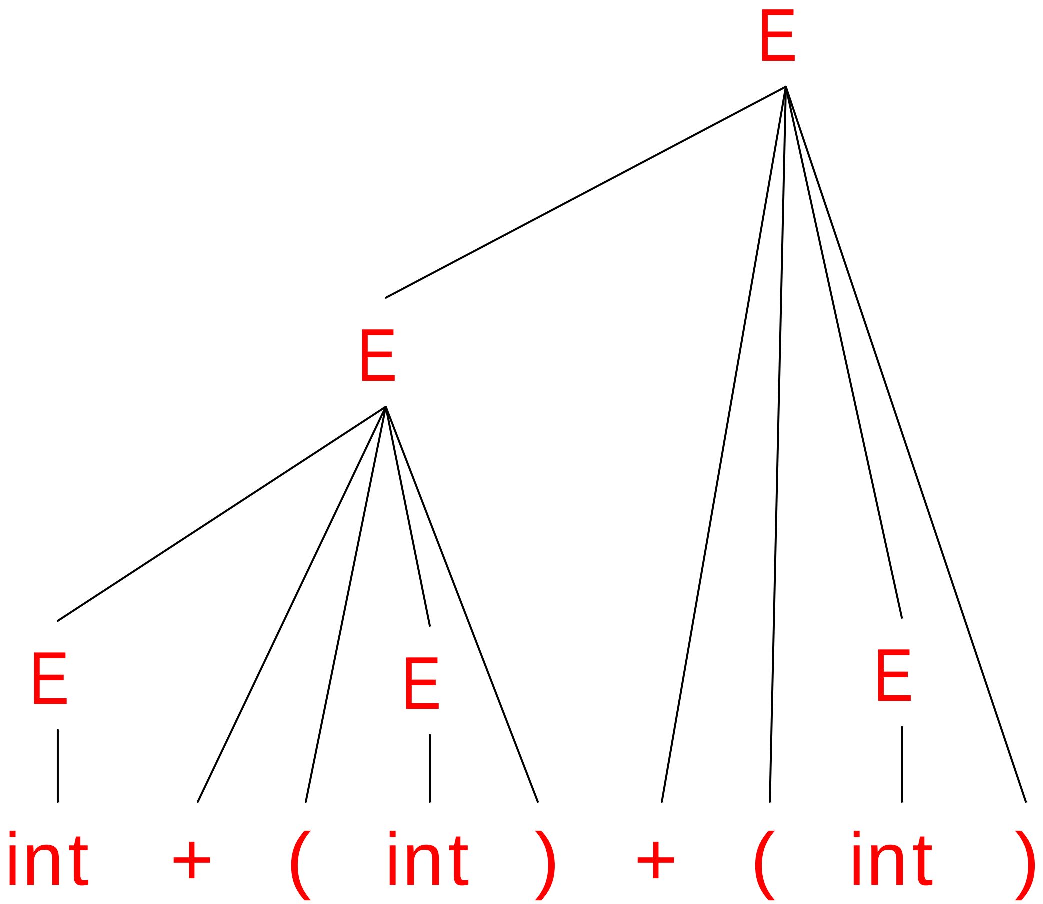


A Bottom-up Parse in Detail (6)

$E \rightarrow E + (E) \mid \text{int}$

int + (int) + (int)
E + (int) + (int)
E + (E) + (int)
E + (int)
E + (E)
E

A rightmost derivation in reverse



Important Fact #1 about Bottom-up Parsing

An LR parser traces a rightmost derivation in reverse

Where Do Reductions Happen

Fact #1 has an interesting consequence:

- Let $\alpha\beta\gamma$ be a step of a bottom-up parse
- Assume the next reduction is by using $A \rightarrow \beta$
- Then γ is a string of terminals

Why?

Because $\alpha A\gamma \rightarrow \alpha\beta\gamma$ is a step in a right-most derivation

Notation

- Idea: Split string into two substrings
  - Right substring is as yet unexamined by parsing (a string of terminals)
  - Left substring has terminals and non-terminals

- The dividing point is marked by a |
  - The | is not part of the string

- Initially, all input is unexamined: $|x_1x_2 \ldots x_n$

Shift-Reduce Parsing

Bottom-up parsing uses only two kinds of actions:

Shift

Reduce

Shift

Shift: Move | one place to the right

- Shifts a terminal to the left string

$$E + (\,|\ \mathrm{int}\ ) \Rightarrow E + (\mathrm{int}\ |\ )$$

In general:

$$ABC\ |\ xyz \Rightarrow ABCx\ |\ yz$$

Reduce

Reduce: Apply an inverse production at the right end of the left string

- If $E \rightarrow E + (E)$ is a production, then

$$E + (\underline{E + (E)} \mid) \Rightarrow E + (\underline{E} \mid)$$

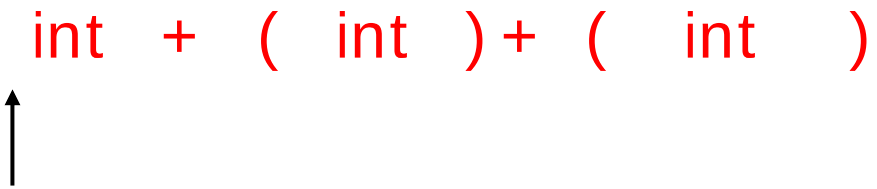
In general, given $A \rightarrow xy$, then:

$$Cbxy \mid ijk \Rightarrow CbA \mid ijk$$

Shift-Reduce Example

E → E + (E) | int

| int + (int) + (int)$ shift

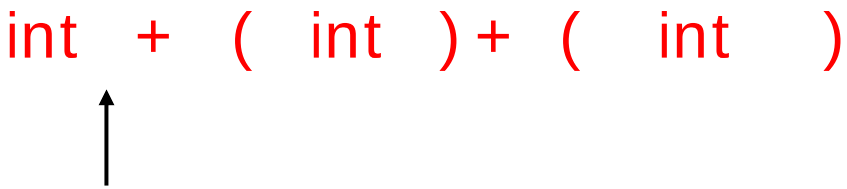


Shift-Reduce Example

E → E + (E) | int

| int + (int) + (int)$ shift

int | + (int) + (int)$ reduce E → int

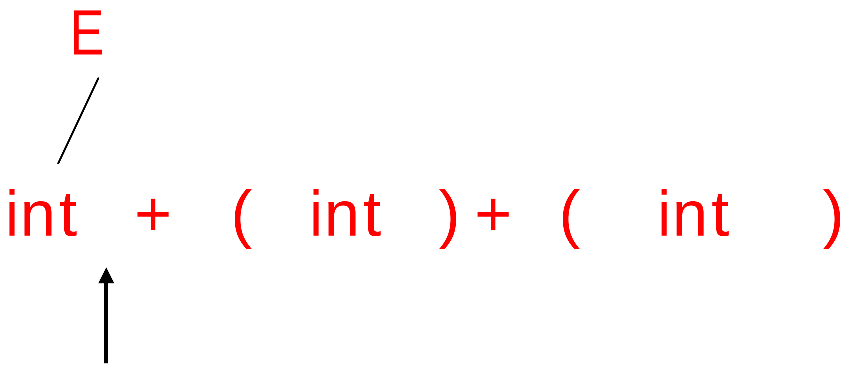


int + (int) + (int)

Shift-Reduce Example

E → E + (E) | int

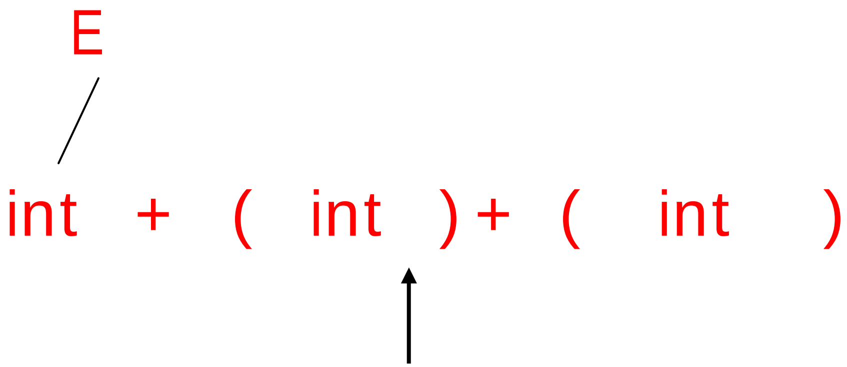
| int + (int) + (int)$	shift
int | + (int) + (int)$	reduce E → int
E | + (int) + (int)$	shift 3 times



Shift-Reduce Example

E → E + (E) | int

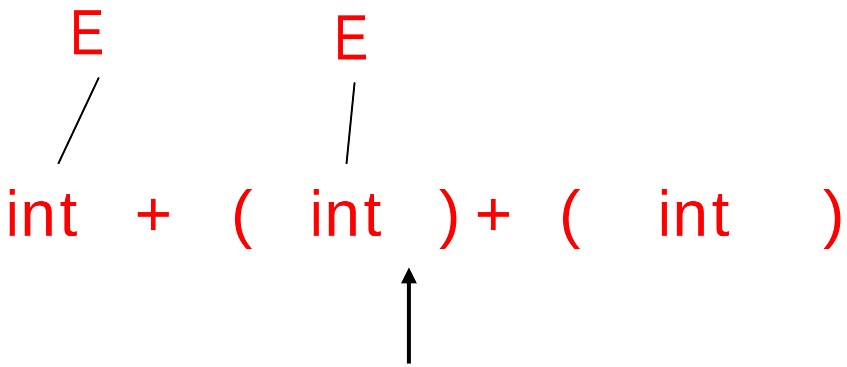
I int + (int) + (int)$	shift
int I + (int) + (int)$	reduce E → int
E I + (int) + (int)$	shift 3 times
E + (int I) + (int)$	reduce E → int



Shift-Reduce Example

E → E + (E) | int

| int + (int) + (int)$	shift
int | + (int) + (int)$	reduce E → int
E | + (int) + (int)$	shift 3 times
E + (int |) + (int)$	reduce E → int
E + (E |) + (int)$	shift



Shift-Reduce Example

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| int + (int) + (int)$	shift
int | + (int) + (int)$	reduce E → int
E | + (int) + (int)$	shift 3 times
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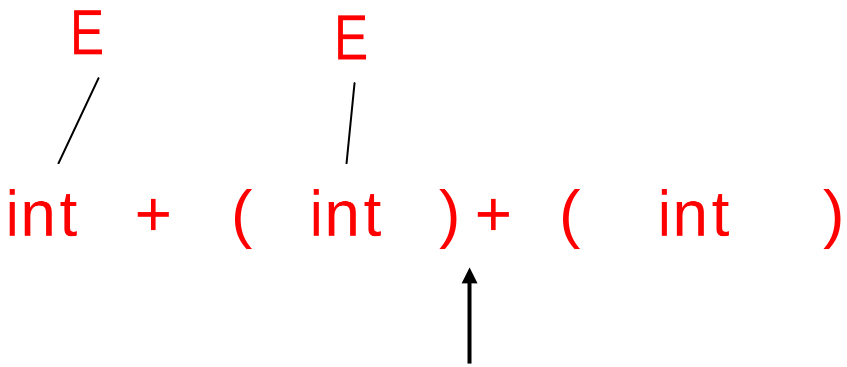
E + (E |) + (int)$	shift
E + (E) | + (int)$	reduce E → E + (E)



Shift-Reduce Example

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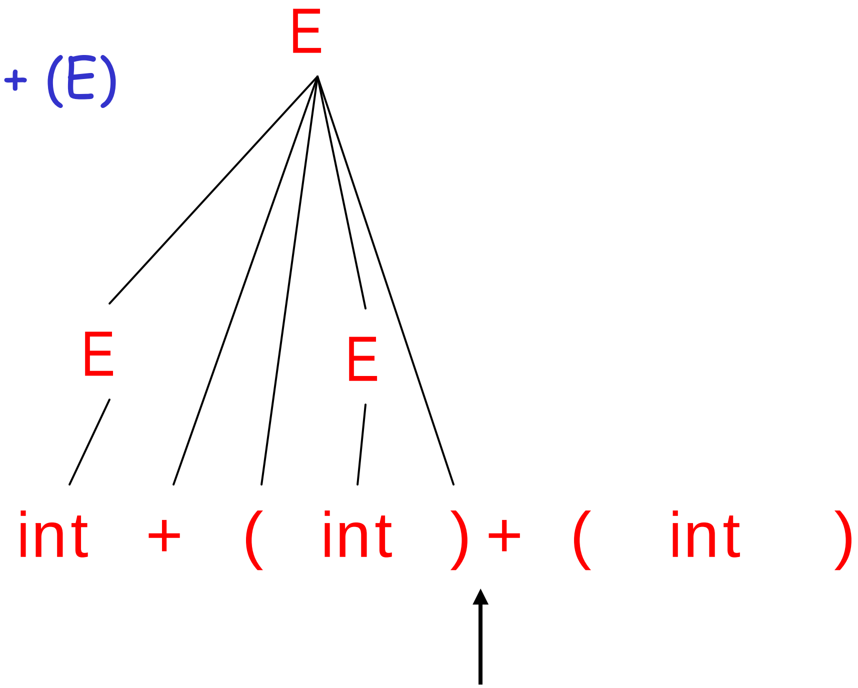
E + (E |) + (int)$	shift
E + (E) | + (int)$	reduce E → E + (E)
E | + (int)$	shift 3 times



Shift-Reduce Example

E → E + (E) | int

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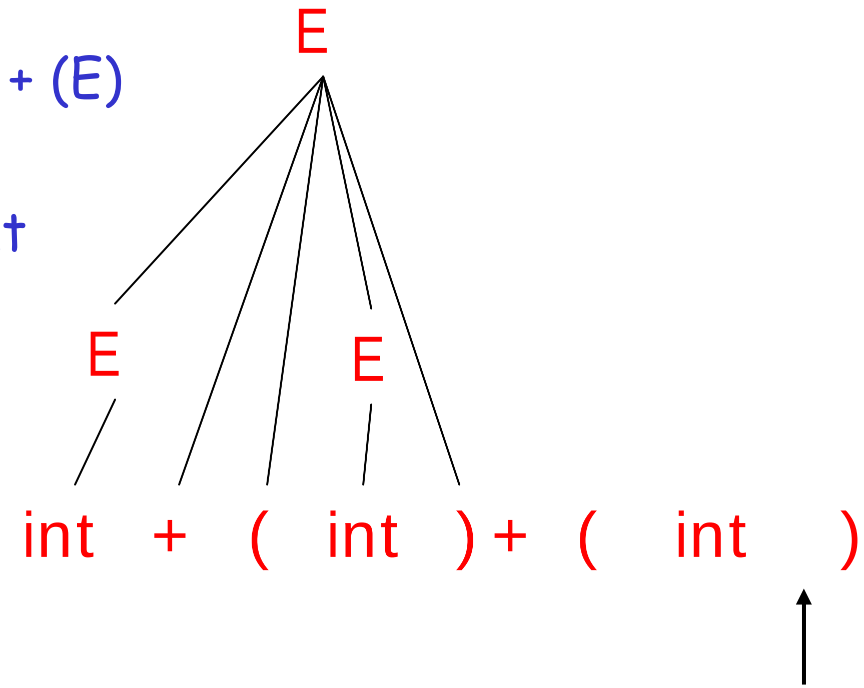
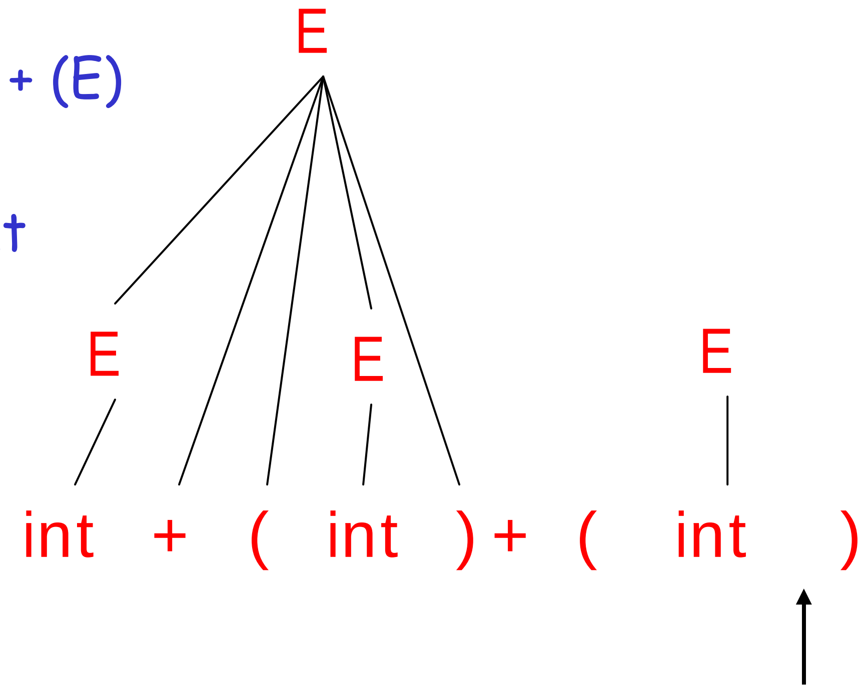
int | + (int) + (int)$	reduce E → int
E | + (int) + (int)$	shift 3 times
E + (int |) + (int)$	reduce E → int
E + (E |) + (int)$	shift
E + (E) | + (int)$	reduce E → E + (E)
E | + (int)$	shift 3 times
E + (int |)$	reduce E → int



Shift-Reduce Example

E → E + (E) | int

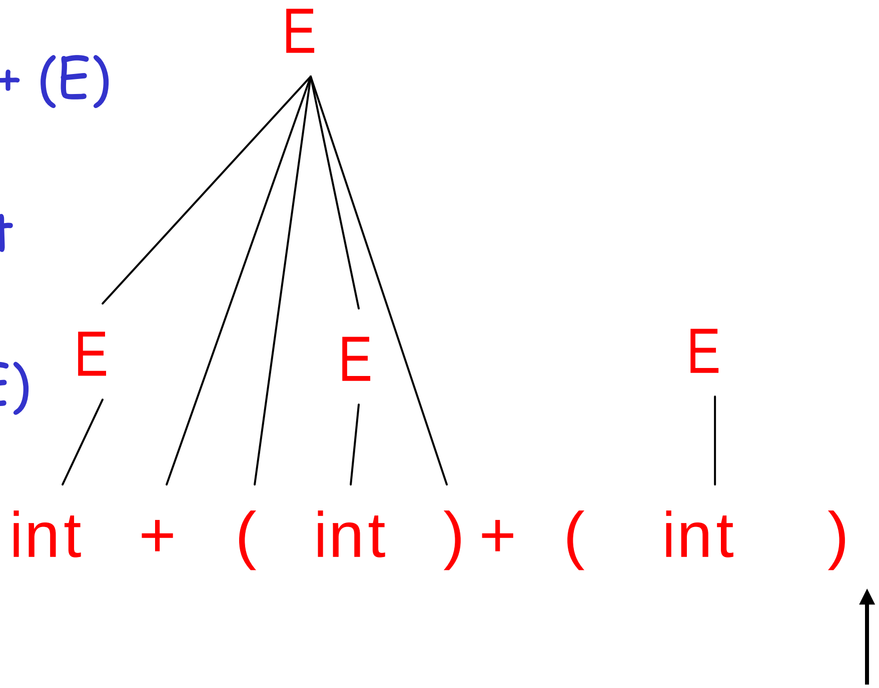
| int + (int) + (int)$	shift
int | + (int) + (int)$	reduce E → int
E | + (int) + (int)$	shift 3 times
E + (int |) + (int)$	reduce E → int
E + (E |) + (int)$	shift
E + (E) | + (int)$	reduce E → E + (E)
E | + (int)$	shift 3 times
E + (int |)$	reduce E → int
E + (E |)$	shift



Shift-Reduce Example

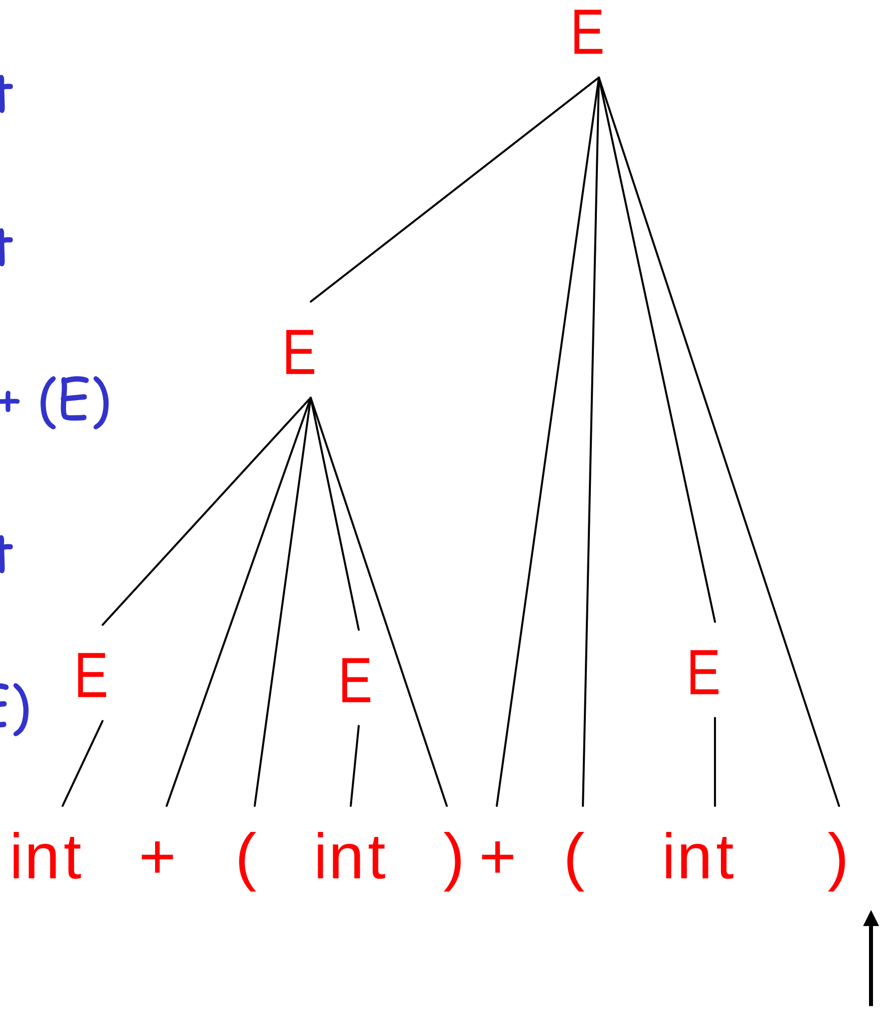
E → E + (E) | int

| int + (int) + (int)$	shift
int | + (int) + (int)$	reduce E → int
E | + (int) + (int)$	shift 3 times
E + (int |) + (int)$	reduce E → int
E + (E |) + (int)$	shift
E + (E) | + (int)$	reduce E → E + (E)
E | + (int)$	shift 3 times
E + (int |)$	reduce E → int
E + (E |)$	shift
E + (E) | $	reduce E → E + (E)



Shift-Reduce Example

| int + (int) + (int)$	shift
int | + (int) + (int)$	reduce E → int
E | + (int) + (int)$	shift 3 times
E + (int |) + (int)$	reduce E → int
E + (E |) + (int)$	shift
E + (E) | + (int)$	reduce E → E + (E)
E | + (int)$	shift 3 times
E + (int |)$	reduce E → int
E + (E |)$	shift
E + (E) | $	reduce E → E + (E)
E | $	accept



The Stack

- Left string can be implemented by a stack
  - Top of the stack is the |

- Shift pushes a terminal on the stack

- Reduce pops 0 or more symbols off of the stack (production RHS) and pushes a non-terminal on the stack (production LHS)

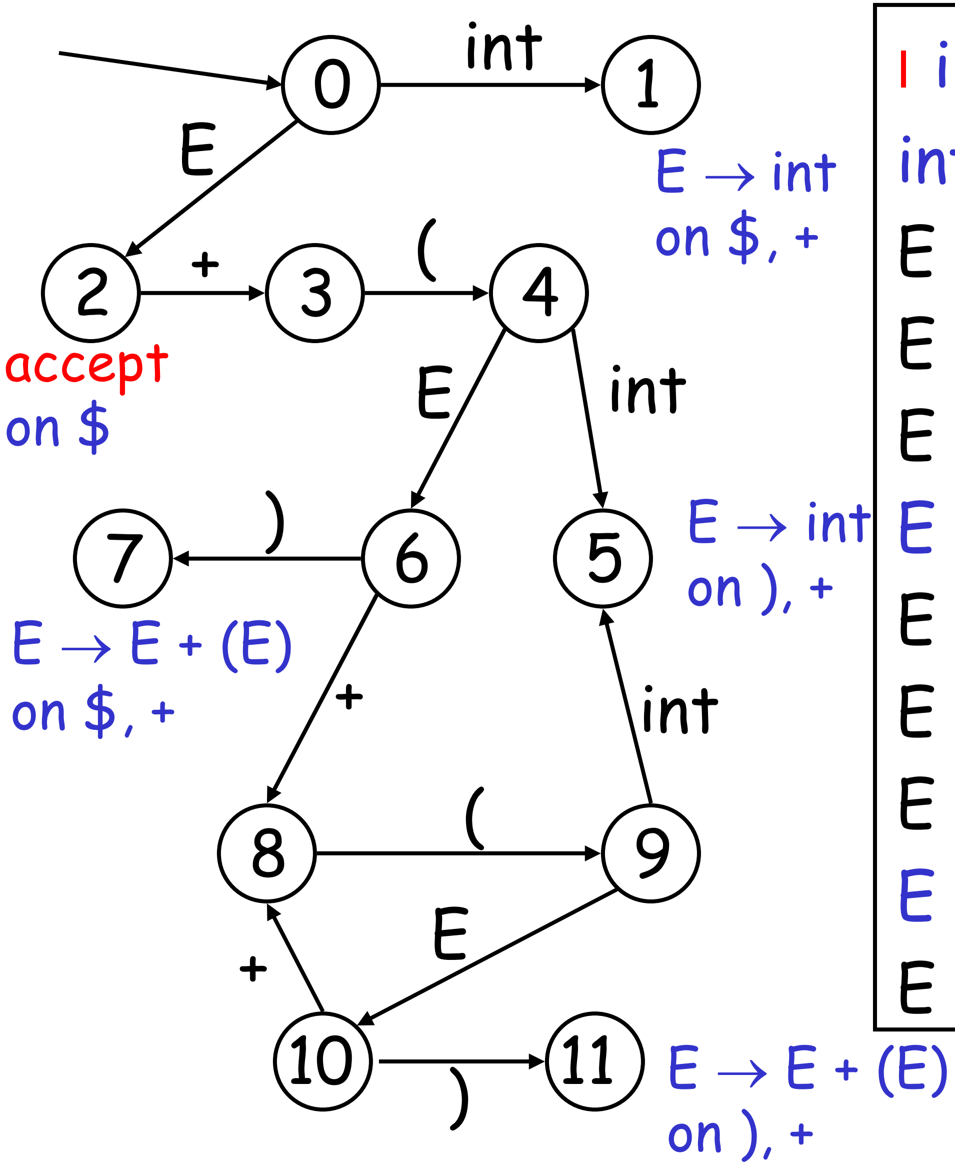
Key Question: To Shift or to Reduce?

Idea: use a finite automaton (DFA) to decide when to shift or reduce

- The input is the stack
- The language consists of terminals and non-terminals

- We run the DFA on the stack and examine the resulting state X and the token tok after |
  - If X has a transition labeled tok then shift
  - If X is labeled with "$A \rightarrow \beta$ on tok" then reduce

LR(1) Parsing: An Example



| int + (int) + (int)$	shift
int | + (int) + (int)$	E → int
E | + (int) + (int)$	shift(x3)
E + (int |) + (int)$	E → int
E + (E |) + (int)$	shift
E + (E) | + (int)$	E → E+(E)
E | + (int)$	shift (x3)
E + (int |)$	E → int
E + (E |)$	shift
E + (E) | $	E → E+(E)
E | $	accept

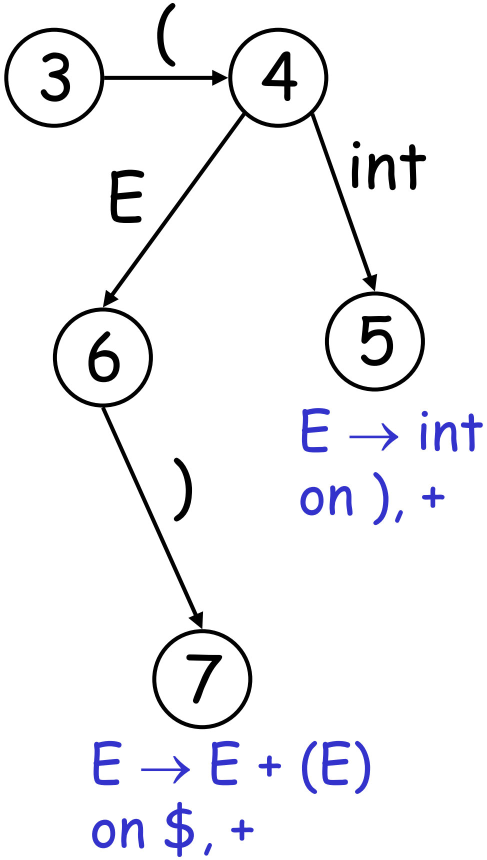
Representing the DFA

- Parsers represent the DFA as a 2D table

  (Recall table-driven lexical analysis)
- Lines correspond to DFA states
- Columns correspond to terminals and non-terminals
- Typically columns are split into:
  - Those for terminals: action table
    - action = shift or reduce
  - Those for non-terminals: goto table

Representing the DFA: Example

- The table for a fragment of our DFA:



	int	+	(	)	$	E
…						
3			s4			
4	s5					g6
5		$r_{E \to \text{int}}$		$r_{E \to \text{int}}$		
6		s8		s7		
7		$r_{E \to E+(E)}$			$r_{E \to E+(E)}$	
…						

The LR Parsing Algorithm

- After a shift or reduce action we rerun the DFA on the entire stack
  - This is wasteful, since most of the work is repeated

- Remember for each stack element on which state it brings the DFA

- LR parser maintains a stack

$$\langle \text{sym}_1, \text{state}_1 \rangle \ldots \langle \text{sym}_n, \text{state}_n \rangle$$

state_k is the final state of the DFA on $\text{sym}_1 \ldots \text{sym}_k$

The LR Parsing Algorithm

```
let I = w$ be initial input
let j = 0
let DFA state 0 be the start state
let stack = ⟨ dummy, 0 ⟩
  repeat
    case action[top_state(stack), I[j]] of
      shift k: push ⟨ I[j++], k ⟩
      reduce $X \to A$:
        pop $|A|$ pairs,
        push ⟨X, Goto[top_state(stack),X]⟩
      accept: halt normally
      error: halt and report error
```

LR Parsers

- Can be used to parse more grammars than LL
- Most programming languages grammars are LR
- LR Parsers can be described as a simple table
- There are tools for building the table
- How is the table constructed?