Appendix

1 Propositional Logic - Axioms and Inference Rules

Axioms

Axiom 1.1 **[Commutativity]**

$$\begin{array}{rcl} (p \land q) & = & (q \land p) \\ (p \lor q) & = & (q \lor p) \\ (p = q) & = & (q = p) \end{array}$$

Axiom 1.2 **[Associativity]**

$$\begin{array}{rcl} p \land (q \land r) & = & (p \land q) \land r \\ p \lor (q \lor r) & = & (p \lor q) \lor r \end{array}$$

Axiom 1.3 **[Distributivity]**

$$\begin{array}{rcl} p \lor (q \land r) & = & (p \lor q) \land (p \lor r) \\ p \land (q \lor r) & = & (p \land q) \lor (p \land r) \end{array}$$

Axiom 1.4 **[De Morgan]**

$$\begin{array}{rcl} \neg(p \land q) & = & \neg p \lor \neg q \\ \neg(p \lor q) & = & \neg p \land \neg q \end{array}$$

Axiom 1.5 **[Negation]**

$$\neg\neg p \quad = \quad p$$

Axiom 1.6 **[Excluded Middle]**

$$p \lor \neg p \quad = \quad T$$

Axiom 1.7 **[Contradiction]**

$$p \land \neg p \quad = \quad F$$

Axiom 1.8 **[Implication]**

$$p \Rightarrow q \quad = \quad \neg p \vee q$$

Axiom 1.9 **[Equality]**

$$(p = q) \quad = \quad (p \Rightarrow q) \wedge (q \Rightarrow p)$$

Axiom 1.10 **[or-simplification]**

$$\begin{aligned} p \vee p &= p \\ p \vee T &= T \\ p \vee F &= p \\ p \vee (p \wedge q) &= p \end{aligned}$$

Axiom 1.11 **[and-simplification]**

$$\begin{aligned} p \wedge p &= p \\ p \wedge T &= p \\ p \wedge F &= F \\ p \wedge (p \vee q) &= p \end{aligned}$$

Axiom 1.12 **[Identity]**

$$p \quad = \quad p$$

Inference Rules

$$\frac{p_1 = p_2 \quad , \quad p_2 = p_3}{p_1 = p_3} \quad \text{Transitivity}$$

$$\frac{p_1 = p_2}{E(p_1) = E(p_2) \quad , \quad E(p_2) = E(p_1)} \quad \text{Substitution}$$

$$\frac{q_1 \;,\; q_2 \;,\; \ldots \;,\; q_n \;,\; q_1 \wedge q_2 \wedge \ldots \wedge q_n \Rightarrow (p_1 = p_2)}{E(p_1) = E(p_2) \quad , \quad E(p_2) = E(p_1)} \quad \text{Conditional Substitution}$$

2 Propositional Logic - Derived Theorems

Equivalence and Truth

Theorem 2.1 **[Associativity of $=$]**

$$((p = q) = r) \quad = \quad (p = (q = r))$$

Theorem 2.2 **[Identity of $=$]**

$$(T = p) \quad = \quad p$$

Theorem 2.3 **[Truth]**

$$T$$

Negation, Inequivalence, and False

Theorem 2.4 **[Definition of F]**

$$F \quad = \quad \neg T$$

Theorem 2.5 **[Distributivity of $\neg$ over $=$]**

$$\begin{aligned} \neg(p = q) \quad &= \quad (\neg p = q) \\ (\neg p = q) \quad &= \quad (p = \neg q) \end{aligned}$$

Theorem 2.6 **[Negation of F]**

$$\neg F \quad = \quad T$$

Theorem 2.7 **[Definition of $\neg$]**

$$\begin{aligned} (\neg p = p) \quad &= \quad F \\ \neg p \quad &= \quad (p = F) \end{aligned}$$

Disjunction

Theorem 2.8 **[Distributivity of $\vee$ over $=$]**

$$\begin{aligned} (p \vee (q = r)) \quad &= \quad ((p \vee q) = (p \vee r)) \\ ((p \vee (q = r)) = (p \vee q)) \quad &= \quad (p \vee r) \end{aligned}$$

Theorem 2.9 **[Distributivity of $\vee$ over $\vee$]**

$$p \vee (q \vee r) = (p \vee q) \vee (p \vee r)$$

Conjunction

Theorem 2.10 **[Mutual definition of $\wedge$ and $\vee$]**

$$\begin{aligned}
(p \wedge q) &= (p = (q = (p \vee q))) \\
(p \wedge q) &= ((p = q) = (p \vee q)) \\
((p \wedge q) = p) &= (q = (p \vee q)) \\
((p \wedge q) = (p = q)) &= (p \vee q) \\
(((p \wedge q) = p) = q) &= (p \vee q)
\end{aligned}$$

Theorem 2.11 **[Distributivity of $\wedge$ over $\wedge$]**

$$p \wedge (q \wedge r) = (p \wedge q) \wedge (p \wedge r)$$

Theorem 2.12 **[Absorption]**

$$\begin{aligned}
&p \wedge (\neg p \vee q) = p \wedge q \\
&p \vee (\neg p \wedge q) = p \vee q
\end{aligned}$$

Theorem 2.13 **[Distributivity of $\wedge$ over $=$]**

$$\begin{aligned}
(p \wedge q) &= ((p \wedge \neg q) = \neg p) \\
((p \wedge q) = (p \wedge \neg q)) &= \neg p \\
p \wedge (q = p) &= (p \wedge q)
\end{aligned}$$

Theorem 2.14 **[Replacement]**

$$(p = q) \wedge (r = p) \quad = \quad (p = q) \wedge (r = q)$$

Theorem 2.15 **[Definition of $=$]**

$$(p = q) \quad = \quad (p \wedge q) \vee (\neg p \wedge \neg q)$$

Theorem 2.16 **[Exclusive or]**

$$\neg(p = q) \quad = \quad (\neg p \wedge q) \vee (p \wedge \neg q)$$

Implication

Theorem 2.17 [Definition of Implication]

$$
\begin{array}{rcl}
(p \Rightarrow q) & = & ((p \vee q) = q) \\
((p \Rightarrow q) = (p \vee q)) & = & q \\
(p \Rightarrow q) & = & ((p \wedge q) = p) \\
((p \Rightarrow q) = (p \wedge q)) & = & p
\end{array}
$$

Theorem 2.18 [Contrapositive]

$$(p \Rightarrow q) \quad = \quad (\neg q \Rightarrow \neg p)$$

Theorem 2.19 [Distributivity of $\Rightarrow$ over $=$]

$$p \Rightarrow (q = r) \quad = \quad ((p \Rightarrow q) = (p \Rightarrow r))$$

Theorem 2.20 [Shunting]

$$p \wedge q \Rightarrow r \quad = \quad p \Rightarrow (q \Rightarrow r)$$

Theorem 2.21 [Elimination/Introduction of $\Rightarrow$]

$$
\begin{array}{rcl}
p \wedge (p \Rightarrow q) & = & p \wedge q \\
p \wedge (q \Rightarrow p) & = & p \\
p \vee (p \Rightarrow q) & = & T \\
p \vee (q \Rightarrow p) & = & \neg q \vee p \\
(p \vee q) \Rightarrow (p \wedge q) & = & (p = q) \\
p \Rightarrow F & = & \neg p \\
F \Rightarrow p & = & T
\end{array}
$$

Theorem 2.22 [Right Zero of $\Rightarrow$]

$$(p \Rightarrow T) \quad = \quad T$$

Theorem 2.23 [Left Identity of $\Rightarrow$]

$$(T \Rightarrow p) \quad = p$$

Theorem 2.24 [Weakening/Strengthening]

$$\begin{array}{rcl} p & \Rightarrow & p \vee q \\ p \wedge q & \Rightarrow & p \\ p \wedge q & \Rightarrow & p \vee q \\ p \vee (q \wedge r) & \Rightarrow & p \vee q \\ p \wedge q & \Rightarrow & p \wedge (q \vee r) \end{array}$$

Theorem 2.25 [Modus Ponens]

$$p \wedge (p \Rightarrow q) \quad \Rightarrow \quad q$$

Theorem 2.26 [Proof by Cases]

$$\begin{array}{rcl} (p \Rightarrow r) \wedge (q \Rightarrow r) & = & (p \vee q \Rightarrow r) \\ (p \Rightarrow r) \wedge (\neg p \Rightarrow r) & = & r \end{array}$$

Theorem 2.27 [Mutual Implication]

$$(p \Rightarrow q) \wedge (q \Rightarrow p) \quad = \quad (p = q)$$

Theorem 2.28 [Antisymmetry]

$$(p \Rightarrow q) \wedge (q \Rightarrow p) \quad \Rightarrow \quad (p = q)$$

Theorem 2.29 [Transitivity]

$$\begin{array}{rcl} (p \Rightarrow q) \wedge (q \Rightarrow r) & \Rightarrow & (p \Rightarrow r) \\ (p = q) \wedge (q \Rightarrow r) & \Rightarrow & (p \Rightarrow r) \\ (p \Rightarrow q) \wedge (q = r) & \Rightarrow & (p \Rightarrow r) \end{array}$$

Theorem 2.30 [Monotonicity of $\vee$]

$$(p \Rightarrow q) \quad \Rightarrow \quad (p \vee r \Rightarrow q \vee r)$$

Theorem 2.31 [Monotonicity of $\wedge$]

$$(p \Rightarrow q) \quad \Rightarrow \quad (p \wedge r \Rightarrow q \wedge r)$$

Substitution

Theorem 2.32 [Leibniz]

$$(e = f) \quad \Rightarrow \quad (E(e) = E(f))$$

Theorem 2.33 [Substitution]

$$\begin{array}{rcl} (e = f) \wedge E(e) & = & (e = f) \wedge E(f) \\ (e = f) \Rightarrow E(e) & = & (e = f) \Rightarrow E(f) \\ q \wedge (e = f) \Rightarrow E(e) & = & q \wedge (e = f) \Rightarrow E(f) \end{array}$$

Theorem 2.34 [Replace by T]

$$\begin{array}{rcl} p \wedge E(p) & = & p \wedge E(T) \\ p \Rightarrow E(p) & = & p \Rightarrow E(T) \\ q \wedge p \Rightarrow E(p) & = & q \wedge p \Rightarrow E(T) \end{array}$$

Theorem 2.35 [Replace by F]

$$\begin{array}{rcl} p \vee E(p) & = & p \vee E(F) \\ E(p) \Rightarrow p & = & E(F) \Rightarrow p \\ E(p) \Rightarrow p \vee q & = & E(F) \Rightarrow p \vee q \end{array}$$

Theorem 2.36 [Shannon]

$$E(p) \quad = \quad (p \wedge E(T)) \vee (\neg p \wedge E(F))$$

3 Predicate Logic - Axioms

Axiom 3.1 [Definition of $\exists$]

$$(m \geq n) \quad \Rightarrow \quad \left(\begin{array}{c} \exists i \, : \, m \leq i < n \, : \, p_i \\ = \\ F \end{array} \right)$$

$$(m < n) \quad \Rightarrow \quad \left(\begin{array}{c} \exists i \, : \, m \leq i < n \, : \, p_i \\ = \\ \left(\begin{array}{c} \exists i \, : \, m \leq i < n-1 \, : \, p_i \\ \vee \\ p_{n-1} \end{array} \right) \end{array} \right)$$

Axiom 3.2 [Definition of ∀]

$$(m \geq n) \quad \Rightarrow \quad \left(\begin{array}{c} \forall i \ : \ m \leq i < n \ : \ p_i \\ = \\ T \end{array} \right)$$

$$(m < n) \quad \Rightarrow \quad \left(\begin{array}{c} \forall i \ : \ m \leq i < n \ : \ p_i \\ = \\ \left(\begin{array}{c} \forall i \ : \ m \leq i < n-1 \ : \ p_i \\ \wedge \\ p_{n-1} \end{array} \right) \end{array} \right)$$

Axiom 3.3 [Range Split]

$$(m_1 \le m_2 \le m_3) \quad \Rightarrow \quad \left(\begin{array}{c} \left(\begin{array}{c} \forall i \ : \ m_1 \le i < m_2 \ : \ p_i \\ \wedge \\ \forall i \ : \ m_2 \le i < m_3 \ : \ p_i \end{array} \right) \\ = \\ \forall i \ : \ m_1 \le i < m_3 \ : \ p_i \end{array} \right)$$

$$(m_1 \le m_2) \ \wedge \ (n_1 \ge n_2) \quad \Rightarrow \quad \left(\begin{array}{c} \left(\begin{array}{c} \forall i \ : \ m_1 \le i < n_1 \ : \ p_i \\ \wedge \\ \forall i \ : \ m_2 \le i < n_2 \ : \ p_i \end{array} \right) \\ = \\ \forall i \ : \ m_1 \le i < n_1 \ : \ p_i \end{array} \right)$$

$$(m_1 \le m_2 \le m_3) \quad \Rightarrow \quad \left(\begin{array}{c} \left(\begin{array}{c} \exists i \ : \ m_1 \le i < m_2 \ : \ p_i \\ \vee \\ \exists i \ : \ m_2 \le i < m_3 \ : \ p_i \end{array} \right) \\ = \\ \exists i \ : \ m_1 \le i < m_3 \ : \ p_i \end{array} \right)$$

$$(m_1 \le m_2) \ \wedge \ (n_1 \ge n_2) \quad \Rightarrow \quad \left(\begin{array}{c} \left(\begin{array}{c} \exists i \ : \ m_1 \le i < n_1 \ : \ p_i \\ \vee \\ \exists i \ : \ m_2 \le i < n_2 \ : \ p_i \end{array} \right) \\ = \\ \exists i \ : \ m_1 \le i < n_1 \ : \ p_i \end{array} \right)$$

Axiom 3.4 [Interchange of Dummies]

$$\begin{array}{c} \forall i \ : \ m_1 \le i < n_1 \ : \ (\forall j \ : \ m_2 \le j < n_2 \ : \ p_{i,j}) \\ = \\ \forall j \ : \ m_2 \le j < n_2 \ : \ (\forall i \ : \ m_1 \le i < n_1 \ : \ p_{i,j}) \end{array}$$

$$\begin{array}{c} \exists i \ : \ m_1 \le i < n_1 \ : \ (\exists j \ : \ m_2 \le j < n_2 \ : \ p_{i,j}) \\ = \\ \exists j \ : \ m_2 \le j < n_2 \ : \ (\exists i \ : \ m_1 \le i < n_1 \ : \ p_{i,j}) \end{array}$$

Axiom 3.5 [Dummy Renaming]

$$\forall i \; : \; m \leq i < n \; : \; p_i \quad = \quad \forall j \; : \; m \leq j < n \; : \; p_j$$

Axiom 3.6 [Distributivity of $\vee$ over $\forall$]

$$(p \; \vee \; (\forall i \; : \; m \leq i < n \; : \; q_i)) \quad = \quad \forall i \; : \; m \leq i < n \; : \; (p \; \vee \; q_i)$$

Axiom 3.7 [Distributivity of $\wedge$ over $\forall$]

$$(m < n) \quad \Rightarrow \quad \left(\begin{array}{c} p \; \wedge \; (\forall i \; : \; m \leq i < n \; : \; q_i) \\ = \\ \forall i \; : \; m \leq i < n \; : \; p \; \wedge \; q_i \end{array} \right)$$

$$\left(\begin{array}{c} \forall i \; : \; m \leq i < n \; : \; p_i \\ \wedge \\ \forall i \; : \; m \leq i < n \; : \; q_i \end{array} \right) \quad = \quad \forall i \; : \; m \leq i < n \; : \; (p_i \; \wedge \; q_i)$$

Axiom 3.8 [Distributivity of $\wedge$ over $\exists$]

$$(p \; \wedge \; (\exists i \; : \; m \leq i < n \; : \; q_i)) \quad = \quad \exists i \; : \; m \leq i < n \; : \; (p \; \wedge \; q_i)$$

Axiom 3.9 [Distributivity of $\vee$ over $\exists$]

$$(m < n) \quad \Rightarrow \quad \left(\begin{array}{c} p \; \vee \; (\exists i \; : \; m \leq i < n \; : \; q_i) \\ = \\ \exists i \; : \; m \leq i < n \; : \; (p \; \vee \; q_i) \end{array} \right)$$

$$\left(\begin{array}{c} \exists i \; : \; m \leq i < n \; : \; p_i \\ \vee \\ \exists i \; : \; m \leq i < n \; : \; q_i \end{array} \right) \quad = \quad \exists i \; : \; m \leq i < n \; : \; (p_i \; \vee \; q_i)$$

Axiom 3.10 [Universality of T]

$$\forall i \; : \; m \leq i < n \; : \; T \quad = \quad T$$

Axiom 3.11 [Existence of F]

$$\exists i \; : \; m \leq i < n \; : \; F \quad = \quad F$$

Axiom 3.12 [Generalized De Morgan]

$$\begin{array}{rcl} \neg\,(\exists i\ :\ m \leq i < n\ :\ p_i) & = & \forall i\ :\ m \leq i < n\ :\ \neg p_i \\ \neg\,(\forall i\ :\ m \leq i < n\ :\ p_i) & = & \exists i\ :\ m \leq i < n\ :\ \neg p_i \end{array}$$

Axiom 3.13 [Trading]

$$\begin{array}{rcl} (m \leq i < n)\ \Rightarrow\ p_i & = & \forall i\ :\ m \leq i < n\ :\ p_i \\ (m \leq i < n)\ \wedge\ p_i & \Rightarrow & \exists i\ :\ m \leq i < n\ :\ p_i \end{array}$$

Axiom 3.14 [Definition of Numerical Quantification]

$$(m \geq n) \quad \Rightarrow \quad \left(\begin{array}{c} \mathcal{N} i\ :\ m \leq i < n\ :\ p_i \\ = \\ 0 \end{array} \right)$$

$$(m < n)\ \wedge\ \neg p_{n-1} \quad \Rightarrow \quad \left(\begin{array}{c} \mathcal{N} i\ :\ m \leq i < n\ :\ p_i \\ = \\ \mathcal{N} i\ :\ m \leq i < n-1\ :\ p_i \end{array} \right)$$

$$(m < n)\ \wedge\ p_{n-1} \quad \Rightarrow \quad \left(\begin{array}{c} \mathcal{N} i\ :\ m \leq i < n\ :\ p_i \\ = \\ \left(\begin{array}{c} \mathcal{N} i\ :\ m \leq i < n-1\ :\ p_i \\ + \\ 1 \end{array} \right) \end{array} \right)$$

Axiom 3.15 [Definition of Σ]

$$(m \geq n) \quad \Rightarrow \quad \left(\begin{array}{c} \Sigma i\ :\ m \leq i < n\ :\ e_i \\ = \\ 0 \end{array} \right)$$

$$(m < n) \quad \Rightarrow \quad \left(\begin{array}{c} \Sigma i\ :\ m \leq i < n\ :\ e_i \\ = \\ \left(\begin{array}{c} \Sigma i\ :\ m \leq i < n-1\ :\ e_i \\ + \\ e_{n-1} \end{array} \right) \end{array} \right)$$

Axiom 3.16 [Definition of Π]

$$(m \geq n) \quad \Rightarrow \quad \left(\begin{array}{c} \Pi i \ : \ m \leq i < n \ : \ e_i \\ = \\ 1 \end{array} \right)$$

$$(m < n) \quad \Rightarrow \quad \left(\begin{array}{c} \Pi i \ : \ m \leq i < n \ : \ e_i \\ = \\ \left(\begin{array}{c} \Pi i \ : \ m \leq i < n-1 \ : \ e_i \\ * \\ e_{n-1} \end{array} \right) \end{array} \right)$$