Numerical studies of singularities in the vortex patch problem

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Background

In mathematical fluid dynamics, one of the most important open problems is whether the solutions to Euler and Navier-Stokes equations develop singularities in finite time. Studies [1, 3, 4] have shown indicatives of the existence of such for a family of related contour dynamics equations. One of these are the vortex patch problem that will be studied in this project.

A vortex patch, or a weak solution to the 2D Euler equation, consists of a simply connected and bounded region $D(t)$ of constant vorticity. A vortex patch evolves in time by

$$u(\vec{x}(\gamma,t),t) = \frac{\omega_0}{2\pi}\int_0^{2\pi} \log|\vec{x}(\gamma,t)-\vec{x}(\gamma',t)|\frac{\partial \vec{x}}{\partial \gamma}(\gamma',t)d\gamma', \tag{1}$$

where γ parametrizatrizes the boundary. The stream function is ω_0 inside $D(t)$, 0 outside. The relation between the stream function ψ and the vorticity ω is $\omega = -\Delta\psi$. In more general terms, the linear operator $L(\psi) = -\Delta\psi$ can be substituted with any other linear operator. For example, in the quasi-geostrophic equation the potential temperature θ, corresponding to the vorticity in the 2D Euler equation, is linked with the stream function by the linear operator $\theta = (-\Delta)^{\frac{1}{2}}\psi$.

Using an interpolation between the two, namely by letting the linear operator be $\theta = (-\Delta)^{1-\alpha/2}\psi$, $\alpha \in (0,1)$ Equation (1) for two patches can be transformed to

$$u(\vec{x}(\gamma,t),t) = \sum_{k=1}^{2}\frac{\theta_k}{2\pi}\int_{C_k(t)} \frac{\frac{\partial \vec{x}_k}{\partial \gamma}(\gamma',t)}{|\vec{x}(\gamma,t)-\vec{x}_k(\gamma',t)|^{\alpha}}d\gamma', \tag{2}$$

which will be the equation to study.

Problem Description

The vortex patch problem is whether or not singularities develop in finite time for the patches described above (it is known that there is no finite time blow up of solutions for the 2D Euler equations, but unknown for the generalized quasi-geostrophic equation). Numerically, this will be studied by evolving two patches in time to see if singularities seem to develop. This leads to evaluating singular integrals in each time step, taking the dynamics into account.

One way of evaluating the integral in Equation (2) is by rewriting it in to a coupled system of (functional) diffential equations and using Fast Fourier Transform (FFT). However, the non-adaptive grid along a curve $C(t)$ deteriorates and reduces the accuracy of the scheme. Thus, it would be of interest to study the performance of a FFT on an adaptive grid. This needs to be done at each time step. Finally, the time integration is done using Runge-Kutta 4-5.

Tasks

1. Formulate the problem in mathematical terms.
2. Numerically evolve two patches using FFT in C/C++.
3. Improve the FFT by performing an adaptive grid.
4. Implement the numerical method described in [2] and compare it with the proposed one.
5. Present convergence, stability and accuracy tests of all the implemented methods.

References

[1] Córdoba D, Fontelos MA, Mancho AM, Rodrigo JL. 2005. *Evidence of singularities for a family of contour dynamics equations.* Proc Natl Acad Sci 102(17)5949-52.

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[4] Majda A, Bertozzi A. 2002. *Vorticity and incompressible flow.* Cambridge Texts Appl Math 27.