Hidden structures in the eigenvectors of certain matrix sequences

The study of the spectral properties of Toeplitz(-like) matrix sequences is important for the understanding of, for example, discretizations of partial differential equations. Some useful references for this project, regarding the *eigenvalues* of these matrices, are [1, 2]. Your task in this project will be to study the *eigenvectors* of some particular Toeplitz matrix sequences.

An example of a Toeplitz matrix that you will study in this project is

$$T_n(f) = \begin{bmatrix} 6 & -4 & 2 & & & & \\ -4 & 6 & -4 & 2 & & & \\ 2 & -4 & 6 & -4 & 2 & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & 2 & -4 & 6 & -4 & 2 \\ & & & 2 & -4 & 6 & -4 \\ & & & & 2 & -4 & 6 \end{bmatrix} \in \mathbb{R}^{n \times n}, \tag{1}$$

and the associated matrix sequence (of matrices with increasing size $n \times n$) is denoted $\{T_n(f)\}_n$. D. Meadon, a former student of this course, has studied the eigenvectors of various Toeplitz(-like) matrix sequences [3], and you will explore and try to understand some properties discovered during the preparation of this thesis.

The matrix $T_n(f)$ in (1) is said to be the Toeplitz matrix, of order n, generated by the *symbol* f. The function f, which is 2π-periodic, is in this case

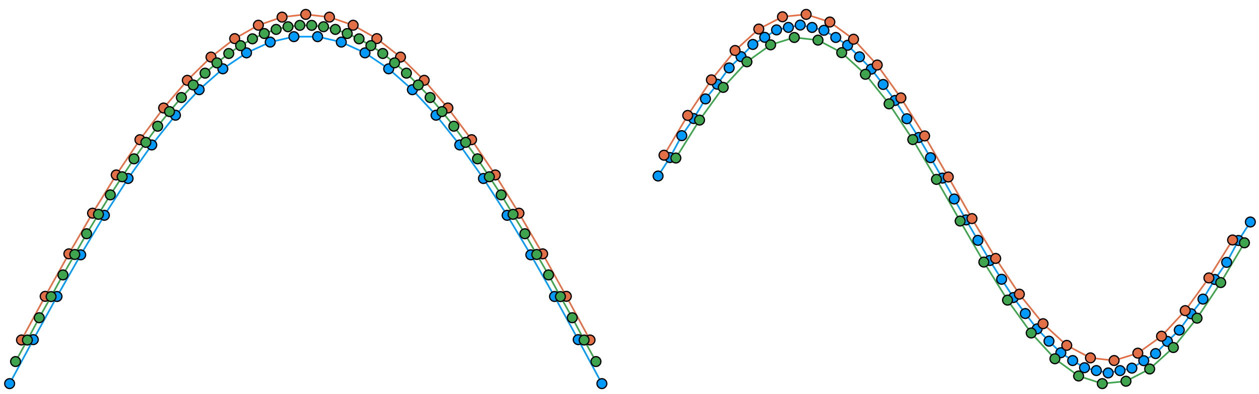
$$f(\theta) = 6 - 8\cos(\theta) + 4\cos(2\theta),$$

and the elements of (1) are given by the Fourier coefficients of f. Assume the more general symbol,

$$f(\theta) = 6 - 8\cos(\theta) + 2\gamma\cos(2\theta), \tag{2}$$

that is the 2:s in (1) are replaced by $\gamma \in \mathbb{R}$. Then, the symbol f is *monontone* for $\gamma \in [-1, 1]$. An example of this is $\gamma = 1$ which gives the bi-Laplace matrix from the second order finite difference discretization of the fourth derivative. However, for $\gamma = 2$ as in (1), then f is *non-monotone*. We will focus on this latter case, that is, $\gamma \notin [-1, 1]$.

By a special procedure and ordering scheme, which will be explained to and explored with the students of the project, the eigenvectors of the matrix $T_n(f)$, with $\gamma = 2$, have a special stucture. In the two panels below we show the first two eigenvectors $\mathbf{v}_j(T_n(f))$, $j = 1, 2$, for $n = 101$. The elements of the two eigenvectors seem to be three separate interlaced samplings of the sine function $\sin(j\theta)$ (plus a possible perturbation), whereas for monotone symbols they typically behave as single set of samplings; the eigenvectors of these matrices are closely related to the eigenvectors defined by the discrete sine transform, but we do not yet know when, how, or why the eigenvectors behave in this particular fashion. For further information, see [3].



You will study how these curves appear and if they can be constructed for different eigenvectors $\mathbf{v}_j$, different γ, and different sizes of matrices, n. Then, you will formulate hypotheses and conjectures from the numerical experiments that you have devised and executed.

This project relies heavily on experimental mathematics to gain insights on pure linear algebra topics. The motivation of this study is mainly to (a) understand the changing behavior of eigenvectors depending on j, γ in (2), and n and (b) as an extension use this information to construct new numerical methods in the future, both for computing eigenvalues and eigenvectors.

Planned Tasks (many extensions are possible)

1. Understand the basics of the theory of generalized locally Toeplitz (GLT) sequences [1, 2].
2. Study matrices generated by the symbol (2) and other non-monotone matrices (develop numerical tools and devise experiments changing j, γ, n, ...).
3. Describe different behaviors of the eigenvectors, formulate conjectures, and provide appropriate numerical evidence.

Practical details

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- Prerequisites: Basic understanding of linear algebra. Creativity, patience, and independence to propose, implement, and execute numerical experiments and present relevant data.
- Preferred programming language is JULIA [4], no experience required.
- Meetings and discussions to be carried out on platforms like IRC, Slack, Discord, or Zoom.

References

[1] S.-E. Ekström, *Matrix-Less Methods for Computing Eigenvalues of Large Structured Matrices*, Ph.D. Thesis, Uppsala University (2018) (`www.2pi.se/thesis.pdf`)
[2] C. Garoni and S. Serra-Capizzano, *Generalized Locally Toeplitz Sequences: Theory and Applications*, Springer, 2017 (`www.doi.org/10.1007/978-3-319-53679-8`)
[3] D. Meadon, *A Matrix-less Method for Approximating the Eigenvectors of Toeplitz-like Matrices*, M.Sc. Thesis, Uppsala University, 2021
[4] J. Bezanson, A. Edelman, S. Karpinski, and V. Shah, *Julia: A fresh approach to numerical computing*, SIAM review 59:1, pp. 65–98 (2017) (`www.julialang.org`)