
Sensitivity and robustness

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F9: Quiz!

1) When a system is observable
   a the states can be estimated arbitrarily well
   b the states can be controlled arbitrarily well
   c the system is also stable

2) State estimation using an observer
   a does not handle initial errors of the state
   b can be described as a differential equation
   c is an unstable process

3) The transfer function for a control system with estimated states
   a is different from that of control system with known states
   b is the same as that of control system with known states
   c is real-valued
F9: Quiz!

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   b the states can be controlled arbitrarily well ↑
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Control system with disturbances and noise
Using general linear feedback

How will the control system cope with unknown disturbances and noise?
How will the control system cope with *unknown* disturbances and noise?

[Board: the closed-loop system with $V(s)$ and $N(s)$]
Sensitivity functions

- Open-loop system: $G_o(s) \triangleq F_y(s)G(s)$
- **Sensitivity function:**

$$S(s) \triangleq \frac{1}{1 + G_o(s)}$$
Sensitivity functions

- Open-loop system: \( G_o(s) \triangleq F_y(s)G(s) \)
- **Sensitivity function:**
  \[
  S(s) \triangleq \frac{1}{1 + G_o(s)}
  \]
- **Complementary sensitivity function:**
  \[
  T(s) \triangleq 1 - S(s) = \frac{G_o(s)}{1 + G_o(s)}
  \]
Sensitivity functions

- **Open-loop system:** 
  \[ G_o(s) \triangleq F_y(s)G(s) \]

- **Sensitivity function:** 
  \[ S(s) \triangleq \frac{1}{1 + G_o(s)} \]

- **Complementary sensitivity function:** 
  \[ T(s) \triangleq 1 - S(s) = \frac{G_o(s)}{1 + G_o(s)} \]

- **Consequence:** 
  \[ S(s) + T(s) \equiv 1, \quad \forall s \]

- **S(s) and T(s) affected by controller \( F_y(s) \).**
Closed-loop system and the sensitivity functions

\[ Y(s) = G_c(s)R(s) + S(s)V(s) - T(s)N(s) \]
Closed-loop system and the sensitivity functions

- **Closed-loop system:**
  
  \[ Y(s) = G_c(s)R(s) + S(s)V(s) - T(s)N(s) \]

- **Want both** \(|S(i\omega)|\) and \(|T(i\omega)| \ll 1\) *simultaneously*...
Closed-loop system and the sensitivity functions

Closed-loop system:

\[ Y(s) = G_c(s)R(s) + S(s)V(s) - T(s)N(s) \]

Want both \(|S(i\omega)|\) and \(|T(i\omega)| \ll 1\) simultaneously...

...but impossible since

\[ |S(i\omega)| + |T(i\omega)| \geq |S(i\omega) + T(i\omega)| \equiv 1 \]
Closed-loop system and sensitivity functions
Design trade-off

Example:
- **Disturbance** \( v(t) \) with energy at low frequencies
- **Noise** \( n(t) \) with energy at high frequencies

\[
\begin{align*}
|V(i\omega)| & \quad |N(i\omega)| \\
\end{align*}
\]
Closed-loop system and sensitivity functions

Design trade-off

Example:

- **Disturbance** $v(t)$ with energy at low frequencies
- **Noise** $n(t)$ with energy at high frequencies

Typical design trade-off is then:

- **Low** $\omega$: $|S(i\omega)| \ll 1$ to suppress $V(i\omega)$.
- **High** $\omega$: $|T(i\omega)| \ll 1$ to suppress $N(i\omega)$.
Simultaneously we want Nyquist contour

\[ G_o(i\omega) = F_y(i\omega)G(i\omega) = \frac{T(i\omega)}{S(i\omega)}, \quad -\infty \leq \omega \leq \infty \]

far from $-1$. (Cf. F6 and F7.)
Control systems with model errors

All models are approximate

\[ y = ur - G0e + F + y0 \]

Assume that the real system can be written as:

\[ G(s) = G_0(s)(1 + \Delta G(s)) \]

The relative model error for \( G(s) \):

\[ \Delta G(s) = G_0(s) - G(s) \]

How is stability affected by unknown error \( \Delta G(s) \)?
Control systems with model errors

All models are approximate

\[ y = y^0 \]

\[ u = -G^0 e + F r \]

\[ r \rightarrow e \rightarrow F \rightarrow u \rightarrow G^0 \rightarrow y^0 \]

- Assume that the real system can be written as

\[ G^0(s) = G(s)(1 + \Delta G(s)) \]

- The relative model error for \( G(s) \):

\[ \Delta G(s) = \frac{G^0(s) - G(s)}{G(s)} \]
Control systems with model errors

All models are approximate

Assume that the real system can be written as

\[ G^0(s) = G(s)(1 + \Delta_G(s)) \]

The relative model error for \( G(s) \):

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How is stability affected by unknown error \( \Delta_G(s) \)?
Model errors and stability

Assume:

1. Controller $F(s)$ stabilizes the assumed system $G(s)$
2. $G(s)$ and $G^0(s)$ have same number of poles in right half-plane.
3. Open-loop: $F(s)G(s) \to 0$ and $F(s)G^0(s) \to 0$ where $|s| \to \infty$
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(Result 6.2) Robustness criterium

If the assumptions are valid and complementary sensitivity function fulfills

$$|T(i\omega)| < \frac{1}{|\Delta_G(i\omega)|}, \quad -\infty \leq \omega \leq \infty$$

$\Rightarrow$ the real closed-loop system is also stable!
Model errors and stability

\( \Delta_G(i\omega) \) is unknown but is we know limitation \( |\Delta_G(i\omega)| < g(\omega) \)
Model errors and stability

$\Delta_G(i\omega)$ is unknown but is we know limitation $|\Delta_G(i\omega)| < g(\omega)$

$|T(i\omega)| < \frac{1}{g(\omega)} < \frac{1}{|\Delta_G(i\omega)|}$

$\Rightarrow$ real closed-loop system is also stable
Summary and recap

- Sensitivity with respect to disturbances and noise
- Sensitivity functions and their impact on control
- Robustness with respect to model errors
- Robustness criterion in the frequency domain