State-space descriptions of systems

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1) The bandwidth $\omega_B$ of the closed-loop system $G_c(s)$ affects its 
  a quickness $\uparrow$ 
  b damping $\uparrow$ 
  c stability $\downarrow$

2) Higher static gain $G_0(0)$ improves $G_c(s)$: 
  a quickness $\uparrow$ 
  b stability $\uparrow$ 
  c accuracy $\downarrow$

3) A non-minimum phase system is 
  a always unstable $\uparrow$ 
  b may have a zero in the right half-plane $\uparrow$ 
  c easy to control $\downarrow$ 

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   a. quickness ↑
   b. damping ↑
   c. stability ↓
F6: Quiz!

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   a always unstable ↑
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Linear time-invariant systems

Different mathematical forms of same model:

- ODE:
  \[
  \frac{d^n}{dt^n} y + \cdots + a_{n-1} \frac{d}{dt} y + a_n y = b_0 \frac{d^m}{dt^m} u + \cdots + b_{m-1} \frac{d}{dt} u + b_m u
  \]
Linear time-invariant systems

\[ y(t) = G(u(t)) \]

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- **Transfer function:**
  \[
  Y(s) = G(s)U(s) \quad \text{ignoring initial conditions}
  \]
Linear time-invariant systems

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  \[
  Y(s) = G(s) U(s) \quad \text{ignoring initial conditions}
  \]

- **State-space description:**
  \[
  \dot{x}(t) = Ax(t) + Bu(t) \\
  y(t) = Cx(t) + Du(t)
  \]
System states

**System states at time** $t$ = summary of its history to predict effect of input $u(t^*)$ where $t^* > t$. 
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- Transfer function yields

$$y(t) = \mathcal{L}^{-1}\{G(s)U(s)\} = \int_{\tau=0}^{t} g(\tau) u(t - \tau) \, d\tau$$

Note: matrix multiplication and eigenvalues necessary!
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  \[
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- Alternative description, state-space form
  
  \[
  y(t) = \sum_{i=1}^{n} c_i x_i(t) + Du(t) = Cx(t) + Du(t)
  \]  
  
  Note: matrix multiplication and eigenvalues necessary!
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- Alternative description, state-space form

$$y(t) = \sum_{i=1}^{n} c_i x_i(t) + Du(t) = Cx(t) + Du(t)$$

where

$$C = \begin{bmatrix} c_1 & \cdots & c_n \end{bmatrix} \quad \text{and} \quad x(t) \triangleq \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}$$
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- Alternative description, state-space form

$$y(t) = \sum_{i=1}^{n} c_i x_i(t) + D u(t) = C x(t) + D u(t)$$

where

$$C = \begin{bmatrix} c_1 & \cdots & c_n \end{bmatrix} \quad \text{and} \quad x(t) \triangleq \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}$$

Note: matrix multiplication and eigenvalues necessary!
Build intuition from simple systems

Example: State vector in space $\mathbb{R}^2$

Figur: Force $u(t)$ and position $y(t)$.

Standard form of linear ODE model:

$$\frac{d^2}{dt^2} y + \left( \frac{K}{m} \right) y = \left( \frac{1}{m} \right) u$$

[Board: derive state-space description]
Build intuition from simple systems
Example: State vector in space $\mathbb{R}^2$

Input $u(t)$ impulse

Output $y(t) = Cx(t)$
Build intuition from simple systems
Example: State vector in space $\mathbb{R}^2$

$x(t)$ at $t = 0^+$
Build intuition from simple systems

Example: State vector in space \( \mathbb{R}^2 \)

\[ x(t) \text{ at } t = 20 \]
Build intuition from simple systems
Example: State vector in space $\mathbb{R}^2$

$x(t)$ at $t = 100$
Build intuition from simple systems

Example: State vector in space $\mathbb{R}^2$

Input $u(t)$ sine

Output $y(t) = Cx(t)$
Build intuition from simple systems

Example: State vector in space $\mathbb{R}^2$

$x(t)$ at $t = 200$
Linear time-invariant system models

Relations between mathematical descriptions

Descriptions with different strengths

\[ \frac{d^n}{dt^n} y + \cdots = \frac{d^m}{dt^m} u + \cdots \]

\[ Y(s) = G(s)U(s) \]

\[ \dot{x} = Ax + Bu \]
\[ y = Cx + Du \]
Linear time-invariant system models

Relations between mathematical descriptions

Translate from one description to the next

\[
\frac{d^n}{dt^n} y + \cdots \quad \quad Y(s) = G(s)U(s)
\]

\[
= \frac{d^m}{dt^m} u + \cdots
\]

\[
\dot{x} = Ax + Bu \\
y = Cx + Du
\]
Relations between descriptions
State-space form $\rightarrow$ transfer function

\[
\begin{align*}
\dot{x} &= Ax + Bu \\
y &= Cx + Du
\end{align*}
\]

\[
Y(s) = G(s)U(s)
\]

[Board: Laplace transform and solve $G(s)$]
Relations between descriptions

State-space form $\rightarrow$ transfer function

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

$$Y(s) = G(s)U(s)$$

Transfer function obtained by

$$G(s) = C \begin{pmatrix} (sI - A)^{-1} B \end{pmatrix} + D = \frac{b(s)}{a(s)}$$

Important property:

- System matrix $A$:s eigenvalues $\{\lambda_i\}$ given by solution to
  $$\det(\lambda I - A) = \lambda^n + a_1 \lambda^{n-1} + \cdots + a_n = 0$$
- $a(s) = \det(sI - A)$ is a polynomial of order $n$
Relations between descriptions

State-space form $\rightarrow$ transfer function

\[
G(s): \text{s poles } p_i \subseteq A: \text{s eigenvalues } \lambda_j
\]
Choice of state variables and system matrices **not** unique!

[Board: alternative states $z = T x$]
Relations between descriptions

Transfer function $\rightarrow$ State-space form

Given transfer function

$$G(s) = \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_n}{s^n + a_1 s^{n-1} + \cdots + a_n}$$

one can choose e.g. controllable canonical form

$$
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_3 \\
\vdots \\
\dot{x}_n
\end{bmatrix} =
\begin{bmatrix}
-a_1 & -a_2 & -a_3 & \cdots & -a_n \\
1 & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 0
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
\vdots \\
x_n
\end{bmatrix} +
\begin{bmatrix}
1 \\
0 \\
0 \\
\vdots \\
0
\end{bmatrix} u
$$

$$y = \begin{bmatrix}
b_1 - a_1 b_0 & b_2 - a_2 b_0 & \cdots & b_n - a_n b_0
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
\vdots \\
x_n
\end{bmatrix} + b_0 u$$
Relations between descriptions

Transfer function $\rightarrow$ State-space form

Given transfer function

$$G(s) = \frac{b_0s^n + b_1s^{n-1} + \cdots + b_n}{s^n + a_1s^{n-1} + \cdots + a_n}$$

one can choose e.g. observable canonical form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} -a_1 & 1 & 0 & \cdots & 0 \\ -a_2 & 0 & 1 & \cdots & 0 \\ -a_3 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_n & 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} b_1 - a_1b_0 \\ b_2 - a_2b_0 \\ b_3 - a_3b_0 \\ \vdots \\ b_n - a_nb_0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + b_0u$$
Relations between descriptions

Transfer function $\rightarrow$ State-space form

\[ Y(s) = G(s)U(s) \]
\[ \dot{x} = Ax + Bu \]
\[ y = Cx + Du \]

Given transfer function

\[ G(s) = \frac{b_0s^n + b_1s^{n-1} + \cdots + b_n}{s^n + a_1s^{n-1} + \cdots + a_n} \]

we have general state-space form

\[ \dot{x} = Ax + Bu \]
\[ y = Cx + Du \]

Note: when order of numerator $< n$ we obtain $D = 0$
Solution to state-space equation
First-order system

Output $y(t) = cx(t) + du(t)$ where $x(t)$ given by solution to

$$\dot{x}(t) = ax(t) + bu(t), \quad x(0) = x_0.$$
Solution to state-space equation
First-order system

Output $y(t) = cx(t) + du(t)$ where $x(t)$ given by solution to

$$\dot{x}(t) = ax(t) + bu(t), \quad x(0) = x_0.$$ 

Via Laplace transform of both sides one obtains

$$X(s) = \frac{1}{s - a} x_0 + \frac{b}{s - a} U(s)$$
Solution to state-space equation

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Via Laplace transform of both sides one obtains

$$X(s) = \frac{1}{s - a}x_0 + \frac{b}{s - a}U(s)$$

and inverse transform yields solution

$$x(t) = e^{at}x_0 + \int_0^t e^{a(\tau)}b \underbrace{u(t - \tau)h(\tau)}_{h(t)}d\tau$$
Solution to state-space equation

Matrix exponential

- Exponential $e^{at}$ is a function which fulfills

$$\dot{f}(t) = af(t), \quad f(0) = 1$$
Solution to state-space equation

Matrix exponential

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  $$\dot{f}(t) = af(t), \quad f(0) = 1$$

- Via Laplace transform

  $$F(s) = \frac{1}{s - a} \quad \xleftarrow{\mathcal{L}} \quad e^{at} = \mathcal{L}^{-1} \left\{ \frac{1}{s - a} \right\}$$
Solution to state-space equation

Matrix exponential

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- Matrix exponential $e^{At}$ is a matrix function which fulfills
  \[ \dot{F}(t) = AF(t), \quad F(0) = I \]
Solution to state-space equation

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- Via Laplace transform
  \[ F(s) = (sI - A)^{-1} \quad \mathcal{L} \quad e^{At} = \mathcal{L}^{-1}\left\{ (sI - A)^{-1} \right\} \]
Solution to state-space equation

General

Output $y(t) = Cx(t) + Du(t)$ where $x(t)$ given by solution to

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0.$$ 

[Board: Laplace + inverse transform]
Solution to state-space equation

General

Output \( y(t) = Cx(t) + Du(t) \) where \( x(t) \) given by solution to

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\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0.
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[Board: Laplace + inverse transform]

Via Laplace transform of both sides

\[
X(s) = (sI - A)^{-1}x_0 + (sI - A)^{-1}BU(s)
\]
Solution to state-space equation

**General**

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**[Board: Laplace + inverse transform]**

Via Laplace transform of both sides

$$X(s) = (sI - A)^{-1}x_0 + (sI - A)^{-1}BU(s)$$

and inverse transform gives solution

$$x(t) = e^{At}x_0 + \int_{0}^{t} e^{A\tau}Bu(t-\tau) d\tau$$

Note: matrix exponential $e^{At}$
Stability

The state evolve according to:

\[ x(t) = e^{At} x_0 + \int_0^t e^{A\tau} B u(t - \tau) d\tau \]

Asymptotically stable if

\[ \lim_{t \to \infty} x(t) = 0 \quad \text{when } u(t) \equiv 0 \]
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Asymptotic stability of states

\(A\):s eigenvalues are strictly in left half-plane \(\Leftrightarrow\) system is asymptotically stable
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Asymptotic stability of states

\( A \):s eigenvalues are strictly in left half-plane ⇔ system is asymptotically stable

Input-output stability of system

\( A \):s eigenvalues are strictly in left half-plane ⇒ system is input-output stable
Summary and recap

- State-space description using vectors and matrices
- System matrices and transfer functions
- Solution to state-space equation
- Stability concepts