Intro. Computer Control Systems: F8
Properties of state-space descriptions and feedback

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1) The state-space description of a system is
   a unique
   b unique
   c stable

2) The eigenvalues of the system matrix $A$
   reveals something
   a poles
   b zeros
   c the closed-loop system

3) Solution to $\dot{x} = Ax + Bu$
   with initial condition $x_0$
   is obtained using
   a a linear system of equations
   b the matrix exponential
   c the Nyquist contour
1) The state-space description of a system is
   a  not unique ↑
   b  unique ↑
   c  stable ↓
F7: Quiz!

1) The state-space description of a system is
   a) not unique ↑
   b) unique ↑
   c) stable ↓

2) The eigenvalues of the system matrix $A$ reveals something about
   a) poles ↑
   b) zeros ↑
   c) the closed-loop system ↓
1) The state-space description of a system is
   a not unique ↑
   b unique ↑
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2) The eigenvalues of the system matrix $A$ reveals something about
   a poles ↑
   b zeros ↑
   c the closed-loop system ↓

3) Solution to $\dot{x} = Ax + Bu$ with initial condition $x_0$ is obtained using
   a a linear system of equations ↑
   b the matrix exponential ↑
   c the Nyquist contour ↓
Nonlinear systems and states

Most systems are nonlinear!

Nonlinear differential equations:

\[ \dot{x} = f(x, u) \]
\[ y = h(x, u) \]

Linearize around operating point \( x_0, u_0 \). Typically use a stationary point: \( \dot{x} = f(x_0, u_0) = 0 \)
Nonlinear systems and states

Nonlinear differential equations:

\[
\begin{align*}
\dot{x} &= f(x, u) \\
y &= h(x, u)
\end{align*}
\]

Taylor series expansion around stationary point \( x_0, u_0 \) with \( y_0 = h(x_0, u_0) \) results in linear deviation model:

\[
\begin{align*}
\dot{\Delta x} &= A\Delta x + B\Delta u \\
\Delta y &= C\Delta x + D\Delta u
\end{align*}
\]

- Linear state-space description of the deviations around the operating point of system.
- Matrices \( A, B, C \) and \( D \) given by derivatives of \( f(x, u) \) and \( h(x, u) \) with respect to \( x \) and \( u \). See ch. 8.4 G&L.
State-feedback control

State space description of linear time-invariant system

\[
\begin{align*}
\dot{x} & = Ax + Bu \\
y & = Cx \\
\Rightarrow & \quad Y(s) = G(s)U(s)
\end{align*}
\]
State-feedback control

State space description of linear time-invariant system

\[
\dot{x} = Ax + Bu \\
y = Cx \\
\Rightarrow \quad G(s) = C(sI - A)^{-1}B
\]
State-feedback control

Idea: Feedback control using states

\[ u = -Lx + \ell_0 r, \]

where \( L \) and \( \ell_0 \) are design parameters.
State-feedback control

Closed-loop system from $r$ to $y$ comes:

\[
\begin{align*}
\dot{x} &= Ax + B (-Lx + l_0r) = (A - BL)x + Bl_0r \\
y &= Cx
\end{align*}
\]

Is it possible to

- control the system to all states $x^*$ in $\mathbb{R}^n$?
- design the closed-loop system’s poles?
- (estimate the state $x(t)$?)
Controllability

A sought state $x^*$ is **controllable** if some input $u(t)$ can move the system from $x(0) = 0$ to $x(T) = x^*$.
For $x_0 = 0$, we can compute the state at $t = T$

$$x(T) = e^{At}x_0 + \int_{0}^{T} e^{A\tau} Bu(T - \tau) d\tau$$
Controllability

Med \( x_0 = 0 \) är tillståndet vid \( t = T \)

\[
x(T) = \int_0^T e^{A\tau} Bu(T - \tau) d\tau
\]

\[
= \text{[via Cayley-Hamiltons theorem]}
\]

\[
= B\gamma_0 + AB\gamma_1 + \cdots + A^{n-1}B\gamma_{n-1}
\]

Therefore:

\( x(T) \) is a linear combination of \( B, AB, \ldots, A^{n-1}B \).

\( A \) state \( x^* \) is controllable if it can be expressed as such a linear combination, i.e., if \( x^* \) is in the column space of

\[
S \triangleq [B \ AB \ \cdots \ A^{n-1}B]
\]
Controllability

Controllable system

All states $x^*$ are controllable $\iff$ $S$'s columns are linearly independent

Note: $\text{rank}(S) = n$ or $\det(S) \neq 0$
Assume $u(t) \equiv 0$. A state $x^* \neq 0$ is **unobservable** if the output $y(t) \equiv 0$ when system starts at $x(0) = x^*$.
Observability

When \( u(t) \equiv 0 \) we obtain

\[
y(t) = Cx(t)
= Ce^{At}x^* + 0
\]

When \( y(t) \equiv 0 \) we do not observe any changes in the output:

\[
\frac{d^k}{dt^k} y(t) \bigg|_{t=0} = CA^k x^* = 0.
\]

That is,

\[
Cx^* = 0, \quad CAx^* = 0, \quad \ldots, \quad CA^{n-1}x^* = 0
\]
Observability

When \( u(t) \equiv 0 \) and \( y(t) \equiv 0 \) we observe no changes:

\[
Cx^* = 0, \quad CAx^* = 0, \quad \ldots, \quad CA^{n-1}x^* = 0
\]

or

\[
Ox^* = 0
\]

where

\[
O \triangleq \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}
\]

Therefore:

- A state \( x^* \neq 0 \) is **unobservable** if it belongs to the **null space** of \( O \).
Observability

\[ y(t) = Cx(t) \]

**Figur:** Example null space of \( \mathcal{O} \) and unobservable state \( x^* \).

**Observable system**

All states \( x^* \) are observable \( \Leftrightarrow \mathcal{O} \)‘s columns are linearly independent

**Note:** \( \text{rank}(\mathcal{O}) = n \) or \( \text{det}(\mathcal{O}) \neq 0 \)
Build intuition from simple systems

Exemple: controllable system

System on controllable canonical form $\iff$ controllable

\[
\dot{x}(t) = \begin{bmatrix} -2 & -1 \\ 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t)
\]

\[
y(t) = \begin{bmatrix} 1 & 1 \end{bmatrix} x(t)
\]

Transfer function:

\[
G(s) = C(sI - A)^{-1} B = \frac{s + 1}{s^2 + 2s + 1} = \frac{s + 1}{(s + 1)^2} = \frac{1}{s + 1}
\]

[Board: investigate observability using $O$]
Build intuition from simple systems
Exemple: controllable system

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[Board: investigate observability using $O$]

\[
O = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \implies \det O = 0 \iff \text{unobservable}
\]
Build intuition from simple systems
Example: observable system

System on observable canonical form ⇔ observable

\[
\begin{align*}
\dot{x}(t) &= \begin{bmatrix} -2 & 1 \\ -1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t) \\
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Transfer function:

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[Board: investigate controllability using \( S \)]
Build intuition from simple systems

Example: observable system

System on observable canonical form $\Leftrightarrow$ observable

$$\dot{x}(t) = \begin{bmatrix} -2 & 1 \\ -1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t)$$

Transfer function:

$$G(s) = C(sI - A)^{-1}B = \frac{s + 1}{s^2 + 2s + 1} = \frac{s + 1}{(s + 1)^2} = \frac{1}{s + 1}$$

[Board: investigate controllability using $S$]

$$S = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \Rightarrow \det S = 0 \Leftrightarrow \text{non-controllable}$$
Build intuition from simple systems

**Exemple: controllable and observable system**

Systems in previous examples have the same transfer function

\[ G(s) = \frac{1}{s + 1}. \]

Can also be written in state-space form

\[
\begin{align*}
\dot{x}(t) &= -x(t) + u(t), \\
y(t) &= x(t).
\end{align*}
\]

where \( x(t) \) is a scalar.

[Board: investigate \( S \) and \( O \)]
Build intuition from simple systems

Example: controllable and observable system

Systems in previous examples have the same transfer function

\[ G(s) = \frac{1}{s + 1}. \]

Can also be written in state-space form

\[ \dot{x}(t) = -x(t) + u(t), \]
\[ y(t) = x(t). \]

where \( x(t) \) is a scalar.

[Board: investigate \( S \) and \( O \)]

\[ S = 1 \quad \Rightarrow \quad \det S = 1 \]
\[ O = 1 \quad \Rightarrow \quad \det O = 1 \quad \Leftrightarrow \quad \text{controllable and observable} \quad (1) \]

Note: we eliminated “invisible states”
Minimal realization

System with transfer function $G(s)$ and state-space form

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

Definition 8.2 G&L

State-space form of $G(s)$ is a **minimal realization** if vector $x$ has the smallest possible dimension.
Minimal realization

System with transfer function $G(s)$ and state-space form

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

Definition 8.2 G&L

State-space form of $G(s)$ is a minimal realization if vector $x$ has the smallest possible dimension.

Result 8.11(+8.12) G&L

A state-space form is minimal realization $\Leftrightarrow$ controllable and observable $\Leftrightarrow A$:s eigenvalues $=$ $G(s)$:s poles
State-feedback control

State-space model with controller $u = -Lx + \ell_0r$ where

$$L = \begin{bmatrix} \ell_1 & \ell_2 & \cdots & \ell_n \end{bmatrix}$$

gives closed-loop system

$$\dot{x} = (A - BL)x + B\ell_0r$$
$$y = Cx$$
State-feedback control

State-space model with controller \( u = -Lx + \ell_0r \) where

\[
L = \begin{bmatrix} \ell_1 & \ell_2 & \cdots & \ell_n \end{bmatrix}
\]

gives closed-loop system

\[
\dot{x} = (A - BL)x + B\ell_0r
\]

\[
y =Cx
\]

That is, \( Y(s) = G_c(s)R(s) \) where

\[
G_c(s) = C(sI - A + BL)^{-1}B\ell_0
\]
State-feedback control

State-space model with controller \( u = -Lx + \ell_0r \) where

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y = Cx
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That is, \( Y(s) = G_c(s)R(s) \) where

\[
G_c(s) = C(sI - A + BL)^{-1}B\ell_0
\]

Eigenvalues/poles given by polynomial equation

\[
\det(sI - A + BL) = 0
\]

which we can design via \( L \)!
State-feedback control
Design of the gain $\ell_0$

$Y(s) = G_c(s)R(s)$ where

$$G_c(s) = C(sI - A + BL)^{-1}B\ell_0.$$ 

It is desirable to have at least $G_c(0) = 1$
State-feedback control
Design of the gain $\ell_0$

- $Y(s) = G_c(s)R(s)$ where
  
  $$G_c(s) = C(sI - A + BL)^{-1}B\ell_0.$$  

- It is desirable to have at least $G_c(0) = 1$

- $G_c(0) = C(-A + BL)^{-1}B\ell_0 = 1$ and so

  $$\ell_0 = \frac{1}{C(-A + BL)^{-1}B}$$
State-feedback control
Design of the gain $\ell_0$

- $Y(s) = G_c(s)R(s)$ where
  
  $$G_c(s) = C(sI - A + BL)^{-1}B\ell_0.$$ 

- It is desirable to have at least $G_c(0) = 1$

- $G_c(0) = C(-A + BL)^{-1}B\ell_0 = 1$ and so
  
  $$\ell_0 = \frac{1}{C(-A + BL)^{-1}B}$$ 

- More generally, replace $\ell_0 r$ with $F_r(s)R(s)$

- How to design $L$?
Build intuition from simple systems

Exemple: state-vector in $\mathbb{R}^2$

![Diagram of a simple system with a mass, spring, and force applied.]

Figur: Force $u(t)$ and position $y(t)$.

State-space form:

$$\begin{align*}
\dot{x} &= \begin{bmatrix} 0 & 1 \\ -k/m & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1/m \end{bmatrix} u \\
y &= \begin{bmatrix} 1 & 0 \end{bmatrix} x
\end{align*}$$

[Board: design $L$ so that closed-loop system has poles -2 and -3]
Result 9.1

State-space form is controllable $\iff L$ can be designed to yield arbitrarily placed poles (real and complex-conjugated) of the closed-loop system.
Result 9.1

State-space form is controllable $\Leftrightarrow L$ can be designed to yield arbitrarily placed poles (real and complex-conjugated) of the closed-loop system

- $L$ solved by $\det(sI - A + BL) = 0$ with desired roots
- $L$ very simple to solve for system on controllable canonical form
Pole placement

State-feedback control

\[ r\ell_0 + u \rightarrow (sI - A)^{-1}B \rightarrow x \rightarrow C \rightarrow y \]

Result 9.1

State-space form is controllable \( \Leftrightarrow L \) can be designed to yield arbitrarily placed poles (real and complex-conjugated) of the closed-loop system

- \( L \) solved by \( \det(sI - A + BL) = 0 \) with desired roots
- \( L \) very simple to solve for system on controllable canonical form
- What to do when we can’t measure \( x \) directly?
Summary and recap

- Linearization of nonlinear system models
- Properties:
  - Controllable
  - Observable
  - Minimal realization
- State-feedback control
- Pole placement for the closed-loop system