Covariance function: \( r_w(\tau) \triangleq EW(t + \tau)w^T(t) \Rightarrow r_w(\tau) = r_w^T(-\tau) \)

Spectral density:

Discrete time: \( \Phi_w(\omega) \triangleq \sum_{\tau=-\infty}^{\infty} r_w(\tau)e^{-i\omega\tau} \) (for \( -\pi \leq \omega < \pi \))

\[ r_w(\tau) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \Phi_w(\omega)e^{i\omega\tau} d\omega \quad \text{and} \quad r_w(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \Phi_w(\omega)d\omega \]

Continuous time: \( \Phi_w(\omega) \triangleq \int_{-\infty}^{\infty} r_w(\tau)e^{-i\omega\tau} d\tau \) (for \( \omega \in \mathbb{R} \))

\[ r_w(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi_w(\omega)e^{i\omega\tau} d\omega \quad \text{and} \quad r_w(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi_w(\omega)d\omega \]

In both cases \( \Phi_w(\omega) = \Phi_w^*(\omega) \geq 0 \).

\[ \Rightarrow \text{for scalar } w: \Phi_w(\omega) = \Phi_w(-\omega) \geq 0 \text{ for all } \omega.\]
Repetition: Linear filtering and spectral factorization

**White noise:** \( w(t) \) white noise \( \Leftrightarrow \Phi_w(\omega) = R_w = \text{constant} \)

Discrete time: \( r_w(\tau) = 0 \) for all \( \tau \neq 0 \)

**Linear filtering:**

Discrete time: \( y(k) = G(q)u(k) \Rightarrow \Phi_y(\omega) = G(e^{i\omega})\Phi_u(\omega)G^*(e^{i\omega}) \)

Continuous time: \( y(t) = G(p)u(t) \Rightarrow \Phi_y(\omega) = G(i\omega)\Phi_u(\omega)G^*(i\omega) \)

\( \Rightarrow \Phi_y(\omega) = |G|^2\Phi_u(\omega) \) for the scalar case

**Spectral factorization:** If \( 0 \leq \Phi_w(\omega) < \infty \) is rational in

- (discrete time:) \( \cos \omega \) there exists a rational \( G(z) \)
- (continuous time:) \( \omega^2 \) there exists a rational \( G(s) \)

which is stable and minimum phase, such that \( \Phi_w(\omega) = |G|^2 \) (with \( z = e^{i\omega}/s = i\omega \) respectively).
State space models

**Discrete time:** Let \( v(k) \) be white noise with \( Ev(k) = 0 \) and \( r_v(0) = R_v \). Let \( x(k+1) = Fx(k) + Gv(k) \), with \( F \) stable. Then \( x(k) \) is a stationary stochastic process, with

\[
rx(\tau) = Ex(k+\tau)x^T(k) = F^\tau \Pi_x, \quad \text{where} \quad \Pi_x = F\Pi_x F^T + GR_v G^T.
\]

\( \Rightarrow rx(0) = \Pi_x. \) The eq. is called the discrete-time Lyapunov equation.

**Continuous time:** Let \( v(t) \) be white noise with \( Ev(t) = 0 \) and intensity \( R_v \). Let \( \dot{x}(t) = Ax(t) + Bv(t) \), with \( A \) stable. Then \( x(t) \) is a stationary stochastic process, with

\[
Ex(t)x^T(t) = \Pi_x, \quad \text{where} \quad A\Pi_x + \Pi_x A^T + BR_v B^T = 0.
\]

The equation is called the continuous-time Lyapunov equation.
Cross-covariance and cross-spectrum

Consider two stationary stochastic processes, \(x(t)\) and \(y(t)\), with \(Ex(t) = Ey(t) = 0\).

**Cross-covariance:** \(r_{xy}(\tau) \triangleq Ex(t + \tau)y^T(t)\)

**Cross-spectrum:**

- **Discrete time:** \(\Phi_{xy}(\omega) \triangleq \sum_{\tau=-\infty}^{\infty} r_{xy}(\tau)e^{-i\omega\tau}\)

- **Continuous time:** \(\Phi_{xy}(\omega) \triangleq \int_{-\infty}^{\infty} r_{xy}(\tau)e^{-i\omega\tau} d\tau\)

\(x(t)\) and \(y(t)\) independent \(\Rightarrow r_{xy}(\tau) \equiv 0\) and \(\Phi_{xy}(\omega) \equiv 0\).

Discrete time: \(y(k) = G(q)u(k) \Rightarrow \Phi_{yu}(\omega) = G(e^{i\omega})\Phi_u(\omega)\)

Continuous time: \(y(t) = G(p)u(t) \Rightarrow \Phi_{yu}(\omega) = G(i\omega)\Phi_u(\omega)\)
Let $v$ and $\epsilon$ be white noise of zero mean and $\Phi_v(\omega) = R_v$, $\Phi_\epsilon(\omega) = R_\epsilon$ and $\Phi_{v\epsilon}(\omega) = 0$.

$$v \rightarrow G_w \rightarrow w \underset{\Sigma}{\rightarrow} y \Rightarrow \epsilon \rightarrow G_\epsilon \rightarrow y$$

Then $\Phi_y(\omega) = |G_w|^2 R_v + R_\epsilon$ is a rational function.

By spectral factorization we get $\Phi_y(\omega) = |G_\epsilon|^2$ for some stable, minimum phase, rational $G_\epsilon \Rightarrow$ we can compute $\epsilon = G_\epsilon^{-1} y$ (since $G_\epsilon^{-1}$ is stable).
Full model: Incorporating the noise in the system model

$u$ - input, $z$ - controlled/performance variable, $y$ - measured output,

$w$ - system/process noise, $n$ - measurement noise,

$v_1$ - white process noise, $v_2$ - white measurement noise.