Automatic control II

Computer Exercise 4

LQ and Model Predictive Control (MPC) of a tank process

Preparation exercises: Preparation exercises 1 and 2.

Reading instructions: Chapter 5, 9 and 16 in Glad–Ljung

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1 Introduction

This laboratory work is partially based on Computer Exercise 3. Therefore it is advisable to have Computer Exercise 3 fresh in mind before starting this laboratory work. The goal of this laboratory work is to illustrate how physical modeling can be used to form a mathematical model of a real process and how this model can be used for controller design.

Two different control methods, LQ control and MPC, will be compared. Both methods use state-space models in the controller design.

1.1 Survey

In this laboratory work a tank process is modeled using physical modeling. The resulting non linear model is linearized around a stationary operating point and the model is used in the design of the controllers. The water level in the lower tank is controlled using these controllers.

The laboratory work consists of the following steps:

- Modeling using physical relations from fluid mechanics and experimental estimation of the parameters in the model. The model is described in state-space form.

- LQ control of the process. Knowing the model of the system and using the experience about LQ from Computer Exercise 3, an LQ controller is designed. The water level in the lower tank is then controlled.

- Model Predictive Control of the process. With the knowledge of the system in forms of model and physical limitations an MPC is designed. Comparisons are then made between the LQ controller and the MPC.

The tank system consists of two identical water tanks placed on top of each other. Each tank has a small hole in the bottom so that the water can flow from the upper tank to the lower and from the lower tank to a container below the system. The holes are approximately equally large in both tanks. An electrical pump moves the water from the container to the upper tank. Each tank is equipped with a sensor that delivers a voltage that is approximately proportional to the water level in the tank. The process is depicted schematically in Figure 1.
The input signal to the process is the voltage $u$ (volts) to the electrical pump. The flow of water through the pump is approximately proportional to the voltage $u$:

$$q_{\text{pump}} = K_p u \quad (1)$$

The output signals from the process are the voltages $y_1$ and $y_2$ (volts) from the water level sensors in the upper and lower tank respectively. Since the sensors are nearly linear the following relations between the output signals and the water levels $h_1$ and $h_2$ in the both tanks are obtained:

$$y_1 = K_1 h_1 \quad (2)$$

$$y_2 = K_2 h_2 \quad (3)$$

where $K_1$ and $K_2$ are the proportionality constants of the water level sensors.
2 Modeling of the process

2.1 Physical modeling

In this section a non linear model of the tank system is derived. The model is then linearized around a stationary operating point.

The tank process can be described by the two coupled differential equations

\[
A_1 \frac{dh_1}{dt} = q_{1\text{in}} - q_{1\text{out}} \tag{4}
\]

and

\[
A_2 \frac{dh_2}{dt} = q_{2\text{in}} - q_{2\text{out}} \tag{5}
\]

where \( A \) is the area of the cross sections of the tanks. The Equations (4) and (5) simply state that the net change of volume in a tank is equal to the difference between the volume entering the tank and the volume leaving it. The flow into the upper tank, \( q_{1\text{in}} \), is equal to the flow out of the electrical pump, \( q_{\text{pump}} = K_p u \), which gives

\[
q_{1\text{in}} = K_p u. \tag{6}
\]

Furthermore, the flow into the lower tank is equal to the flow out of the upper tank, so

\[
q_{2\text{in}} = q_{1\text{out}}. \tag{7}
\]

The flows out from the two tanks, \( q_{\text{out}} \) and \( q_{2\text{out}} \) are given by Torricelli’s principle:

\[
q_{\text{out}} = a_1 \sqrt{2gh_1} \tag{8}
\]

and

\[
q_{2\text{out}} = a_2 \sqrt{2gh_2}. \tag{9}
\]

where \( a \) is the area of the holes. By putting all this together the following non linear model for the system is obtained:

\[
\frac{A_1}{a_1} \frac{dh_1}{dt} = -\sqrt{2gh_1} + \frac{K_p u}{a_1} \tag{10}
\]

\[
\frac{A_2}{a_2} \frac{dh_2}{dt} = \sqrt{2gh_1} - \sqrt{2gh_2} \tag{11}
\]

Now introduce the stationary working point

\[
u = u_0
\]

\[
h_1 = h_{0_1}
\]

\[
h_2 = h_{0_2}
\]
where the last two sets of equalities should ideally be approximately the same following the fact that the areas of the holes in the two tanks are equal. Note that a stationary working point is defined by the water levels being constant so that \( \frac{dh_1}{dt} = \frac{dh_2}{dt} = \frac{dh_0}{dt} = 0 \), and from Equation (10) we obtain

\[
0 = \frac{dh_0}{dt} = -\sqrt{2gh_0} + \frac{K_p}{a_1}u_0. \tag{12}
\]

The non-linear model (10)–(11) is now linearized. This is done by a Taylor series expansion around the working point. From (10)

\[
\frac{A_1}{a_1} \frac{dh_1}{dt} = -\sqrt{2gh_0} - \sqrt{\frac{g}{2h_0_1}}(h_1 - h_{01}) + \frac{K_p}{a_1}u_0 + \frac{K_p}{a_1}(u - u_0) + \text{Higher order terms} \tag{13}
\]

\[
\frac{A_2}{a_2} \frac{dh_2}{dt} = \sqrt{2gh_0} + \sqrt{\frac{g}{2h_0_1}}(h_1 - h_{01}) - \sqrt{2gh_0_2} - \sqrt{\frac{g}{2h_0_2}}(h_2 - h_{02}) + \text{Higher order terms} \tag{15}
\]

where, in the last equality, Equation (12) has been used. From (11)

\[
\frac{A_2}{a_2} \frac{dh_2}{dt} = \sqrt{\frac{g}{2h_0_2}}(h_1 - h_{01}) - \sqrt{\frac{g}{2h_0_2}}(h_2 - h_{02}) + \text{Higher order terms} \tag{16}
\]

Introducing the notations for the deviations from the operating point

\[
\Delta h_1 = h_1 - h_{01} \\
\Delta h_2 = h_2 - h_{02} \\
\Delta u = u - u_0
\]

the linearized model (neglecting terms of second order or higher)

\[
\frac{d\Delta h_1}{dt} = -\frac{1}{T_1} \Delta h_1 + \frac{K_p}{A_1} \Delta u \tag{17}
\]

\[
\frac{d\Delta h_2}{dt} = \frac{1}{T_1} \Delta h_1 - \frac{1}{T_2} \Delta h_2 \tag{18}
\]

is obtained, where

\[
T_i = \frac{A_i}{a_i} \sqrt{\frac{2h_0}{g}} \tag{19}
\]

Using Equation (10) with \( u \) equal to zero, or Equation (11) with \( h_1 \) equal to zero, gives

\[
\frac{dh_i}{\sqrt{h_i}} = -\frac{a_i}{A_i} \sqrt{2gd} \tag{20}
\]
Integrating the left hand side from the height \( h_{1i} \) to the height \( h_{2i} \) and the right hand side from time \( t_1 \) to \( t_2 \) gives an expression for \( \frac{a_i}{A_i} \) as

\[
a_i = \frac{2(\sqrt{h_{1i}} - \sqrt{h_{2i}})}{\sqrt{2g} \Delta t_i}
\]

Hence

\[
T_i = \frac{\Delta t_i \sqrt{h_{0i}}}{\sqrt{h_{1i} - h_{2i}}}
\]  \quad (22)

The Equations (2) and (3) now give the final model:

\[
\frac{d\Delta y_1}{dt} = -\frac{1}{T_1} \Delta y_1 + \frac{K_p K_1}{A_1} \Delta u
\]  \quad (23)

\[
\frac{d\Delta y_2}{dt} = \frac{1}{T_1 K_1} \Delta y_1 - \frac{1}{T_2} \Delta y_2
\]  \quad (24)

Note that the model is linearized around a certain working point \((h_0, u_0)\). The linearized system will thus not behave exactly as the original system, but the approximation is reasonable for small deviations from the working point.

**Preparation exercise 1:**

Give a model of the system in state-space form. Let the states be the (voltage) deviation of the water levels from the working point and let the output be the level (voltage) deviation in the lower tank.

**Answer:**

Execute the script `tankParameters.m` to obtain the parameters for the model.

**Exercise:**

Give the state space model for the system with the numerical values for the parameters.

**Answer:**
Exercise:
Are there any physical limitations on the system?
If yes, what are they?

Answer:

We now have the continuous-time model of the process. In Matlab linear systems can be represented as LTI objects (LTI is an abbreviation for Linear Time Invariant). Thus, a linear system may be treated as a single object rather than being described by the system matrices. In order for your model to be used for control design, it is necessary to convert the model into this form. The controllers will then convert the model into discrete time using the sampling time chosen by you.

Form the LTI object contsys using the ss command. Note that it is possible to get help in Matlab for the different functions by typing help fname.

3 LQ control of the tank process

3.1 Theory

An LQ controller as described in Computer Exercise 3 and in Glad-Ljung Chapter 9 will be used. We will use the LQ controller to find a suitable desired closed-loop system. The system is described in the form of a block diagram.

\[ \sum \begin{bmatrix} m \\ -L \end{bmatrix} + u(k) = \begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} x(k) \\ u(k) \end{bmatrix} + \begin{bmatrix} C \end{bmatrix} y(k) \]
in Figure 2. The main idea for the LQ controller is to minimize a quadratic criterion.

Consider a system in state space form

\[ \begin{align*}
    x(k+1) &= Ax(k) + Bu(k) \\
    y(k) &= Cx(k)
\end{align*} \tag{25} \]

where all the states are measured. The system can be controlled with a state-feedback:

\[ u(k) = -Lx(k) + mr(k) \tag{26} \]

where \( m \) is chosen so that the closed loop system has unit static gain. In the LQ design, the gain matrix, \( L \), is determined so that the following cost function is minimized:

\[ V = \sum_{k=1}^{\infty} [r(k) - y(k)]^2 Q_1 + [u(k)]^2 Q_2 \tag{27} \]

where \( r \) and \( u \) are scalars. Eq (27) can be compared with eq (9.35) in Glad-Ljung.

The closed loop system in state space form is then

\[ \begin{align*}
    x(k+1) &= (A - BL)x(k) + Bmr(k) \\
    y(k) &= Cx(k)
\end{align*} \tag{28} \]

The gain matrix, \( L \), hence depends on the weights \( Q_1 \) and \( Q_2 \) in eq. (27).

**Preparation exercise 2:**
Consider the criterion in eq. (27).

Explain how \( Q_1 \) and \( Q_2 \) affect the closed loop system. How can \( Q_1 \) and \( Q_2 \) affect the fastness of the closed loop system and why?

**ANSWER:**

3.2 Controller design

The controller is designed in the same way as the LQ controller in Computer Exercise 3. The Matlab function **lq_sim**

- load your continuous-time state space model of the system, \texttt{contsys} and the parameters \( K_1, K_2, u_0, h_0 \) in \texttt{tankParameters.m}.
- convert the model into discrete time using a sampling interval \texttt{tsamp}.
• calculates $L$ from the LQ criteria.
• calculates $m$ so that the static gain ($r \to y = x_2$) equals 1.
• control the tank system given an external reference signal, $r(t)$.

The LQ control of the tank process is activated with the function:

$$\text{lq\_sim( } Q_1, Q_2, r, \text{Tscale, tsamp) }$$

where

$Q_1$ is the weight on the output signal and must be given. Note that only the ratio $Q_1/Q_2$ determines the characteristics of the controller.

$Q_2$ is the weight on the manipulated signal and must be given.

$r$ is the reference signal to be specified to the system. Keep in mind that it is given as a deviation with respect to the working point.

$\text{Tscale}$ is the time scale for the plot. Default value is [360] seconds.

$\text{tsamp}$ is the sampling time. Default is [1] second.

Alternatively, one can simulate how the system responds when the input has a maximum and/or minimum value, which is quite common in real systems. In this case, the pump can take values between 0 and 9V. The LQ control with bounded inputs is activated with the function:

$$\text{lq\_sim\_cap( } Q_1, Q_2, r, \text{Tscale, tsamp) }$$

whose arguments work in the same way as in the unbounded version.

Exercise
How does bounding the input affect the response of the system?

ANSWER:
Exercise
How does the ratio $Q_1/Q_2$ (for example 1, 100 and the result you had in Computer Exercise 3) affect the response of the system? Are there any benefits or drawbacks with big values of $Q_1/Q_2$?

**ANSWER:**

Exercise
How can overflow be prevented?

**ANSWER:**

4 Model Predictive Control of the tank process

As seen in the LQ control of the process the tanks can easily overflow which in a real situation might be very expensive and/or dangerous and not at all optimal. The most desirable feature in MPC is the way it allows and implements constraints into the control design. Here you will try some different MPC design strategies.

4.1 Theory

Theory was presented during the lectures and will therefore only be mentioned briefly.

The tank system is given in state space form:

$$x(k+1) = Ax(k) + Bu(k)$$

$$y(k) = Cx(k)$$

(29)
where both the states $x_1$ and $x_2$ are assumed measurable. The basic criterion to be minimized, with respect to the input signal, is given by:

$$V(k) = \sum_{j=1}^{P} Q_1(r(k + j) - \hat{y}(k + j))^2 + \sum_{j=1}^{M} Q_2(\Delta \hat{u}(k + j))^2$$

where
- $Q_1$ is the weight on the output error.
- $Q_2$ is the weight on the increment of the input signal, $\Delta u$.
- $r$ is the reference signal (target).
- $\hat{y}$ is the predicted output, see example 16.1 in Glad-Ljung.
- $\Delta \hat{u}$ is the increment of the input ($u(k) = u(k - 1) + \Delta \hat{u}(k)$, and $\hat{u}(k + j) = \hat{u}(k + j - 1) + \Delta \hat{u}(k + j)$ for $j = 1, \ldots, M$).

For a schematic view of the prediction horizon (P) and the control horizon (M), see Figure 3. As in the LQ controller the ratio of the weights $Q_1/Q_2$ is of importance for the minimization.

When the constraints are added to the controller the minimization problem becomes nonlinear and numerical methods are needed to solve the minimization problem.

**Figure 3**: A schematic view of the horizons of Model Predictive Control.
4.2 Controller design

The design will use the Matlab function \texttt{mpc\_sim}, which:

- loads your continuous-time state space model of the system, \texttt{contsys in tankParameters.m}
- converts the model into discrete time using a sampling interval \texttt{tsamp}.
- designs the controller according to the parameters you choose.
- controls the tank system given an external reference signal, \( r(t) \).

The Model Predictive Controller of the tank process is activated with the Matlab function:

\[ \texttt{mpc\_sim(Q_1, Q_2, r, M, P, ylim, ulim, Tscale, tsamp)} \]

where

- \( Q_1 \) is the weight on the output signal, the water level in the lower tank. It must be given.
- \( Q_2 \) is the weight on the manipulated signal. It must be given.
- \( \textbf{ylim} \) are the constraints on the two output signals, the tank levels: \([C_{y1\text{\_min}} \ C_{y1\text{\_max}} \ C_{y2\text{\_min}} \ C_{y2\text{\_max}}]\). The default values are \([-\infty \ -\infty \ \infty \ \infty]\), i.e. no constraints.
- \( \textbf{ulim} \) are the constraints on the manipulated signal, the pump voltage: \([C_{\text{umin}} \ C_{\text{umax}}]\). The default values are \([-\infty \ \infty]\), i.e. no constraints.
- \( M \) is the control horizon (called \( N \) in Glad-Ljung). It is usually much smaller than \( P \). Can be from 1 up to \( P \). The size of the optimization problem is determined by the size of \( M \). Default value is \([3]\).
- \( P \) is the prediction horizon (called \( M \) in Glad-Ljung) and should cover the rise time of the system. Default value is \([30]\).
- \( \texttt{Tscale} \) is the duration of the simulation. Default value is \([360]\) seconds.
- \( \texttt{tsamp} \) is the sampling time. Default is \([1]\) second.
**Exercise**
How do the constraints affect the control of the system? Are there any major differences between this control and the LQ control (consider approximate rise times, sensitivity for measurement noise and risk of overflow)?

**ANSWER:**

Try different horizons and sampling time. For example try some of the following combinations:

- \( M = 3 \quad P = 30 \quad tsamp = 2 \) (default)
- \( M = 3 \quad P = 10 \quad tsamp = 2 \)
- \( M = 3 \quad P = 10 \quad tsamp = 0.5 \)
- \( M = 3 \quad P = 15 \quad tsamp = 5 \)

Of course you can try other values instead/as well. Try different values for all the parameters. Make some step changes and evaluate the results!

**Exercise**
How do the different tuning parameters affect the controller and thereby the closed loop system?
- **a**, The effect of different control horizons, \( M \)?
- **b**, The effect of different prediction horizons, \( P \)?
- **a**, The effect of different sampling times, \( tsamp \)?

**ANSWER:**