HOMEWORK ASSIGNMENTS
AUTOMATIC CONTROL II

HOMEWORK ASSIGNMENT I

Deadline: Tuesday September 22, 17.00

Problem 1 The continuous-time system

\[ Y(s) = \frac{1}{s+1} U(s) \quad \Leftrightarrow \quad \begin{cases} \dot{x}(t) = -x(t) + u(t), \\ y(t) = x(t), \end{cases} \]

is to be controlled by a PI-controller. For a continuous-time PI-controller,

\[ F_{PI}^c(s) = C_p + \frac{C_i}{s}, \]

the corresponding closed loop system will be stable for all \( C_p, C_i > 0 \) (you may verify this). However, the controller must be implemented in a computer, and hence the PI-controller must be converted to a discrete-time controller. One commonly used implementation for a discrete-time PI-controller is

\[
\begin{align*}
    e(kh) &= r(kh) - y(kh), \\
    a(kh) &= a(kh-h) + e(kh), \\
    u(kh) &= K_p e(kh) + K_i a(hk)
\end{align*}
\]

\[
\Leftrightarrow \begin{cases} U(z) = F_{PI}^d(z) E(z), \\ F_{PI}^d(z) = K_p + K_i h \frac{z}{z-1}, \end{cases}
\]

(1)

where \( h \) is the sampling period, and \( a(kh) \) is the accumulated control error (corresponds to the integral of \( e \)).

There are several approaches to discretizing a continuous-time controller. One way is to use some discrete-time approximation of the time derivative. Commonly used approximations are:

- Forward difference: \( \frac{d}{dt} y(kh) \approx \frac{y(kh+h)-y(kh)}{h}, \) set \( s = \frac{z-1}{h} \)

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- Tustin’s approximation:
  \( \frac{d}{dt} y(kh) \approx d(kh) \) where \( d(kh+h)+d(kh) = \frac{y(kh+h)-y(kh)}{h}, \) set \( s = \frac{2}{h} \frac{z-1}{z+1} \)
For example, the forward difference approximation of the continuous-time PI controller is obtained by setting

\[ F^d_{PI}(z) = F^c_{PI}(s)|_{s = \frac{z-1}{h}} = C_p + C_i \frac{h}{z-1} = \frac{C_pz + hC_i - C_p}{z-1}. \]

Comparing this with (1), which can be rewritten as

\[ F^d_{PI}(z) = K_p + K_ih \frac{z}{z-1} = \frac{(K_p + K_ih)z - K_p}{z-1}, \]

and equating the powers of \( z \) in the numerator, one gets the equation system

\[
\begin{align*}
K_p + K_ih &= C_p, \\
-K_p &= hC_i - C_p,
\end{align*}
\]

\[ \leftrightarrow \left\{ \begin{array}{l}
K_p = C_p - C_ih, \\
K_i = C_i.
\end{array} \right. \tag{2} \]

Equation (2) tells how the discrete-time PI parameters \( K_p \) and \( K_i \) should be chosen to approximate \( F^c_{PI}(s) \) by use of the forward difference approximation.

(a) Assume that the PI-controller is tuned in continuous time, so that some feasible \( C_p \) and \( C_i \) are found. In the same way as above, convert \( F^c_{PI}(s) \) to \( F^d_{PI}(z) \) using the backward difference and Tustin’s approximations: Express \( K_p \) and \( K_i \) in \( F^d_{PI}(z) \) in terms of \( C_p, C_i \) and \( h \) for both cases. (1 pt)

Next the stability of the closed loop system will be examined, and for simplicity this analysis will be performed in discrete time.

(b) Determine the sampled version of the system (using zero order hold), and show that the corresponding transfer function can be written as

\[ G^d(z) = \frac{1 - \alpha}{z - \alpha}. \]

Determine \( \alpha \) in terms of the sampling period \( h \). (1 pt)

(c) Based on a continuous-time design, and one of the time derivative approximations, the discrete-time PI parameters are set to \( K_p = 1 \) and \( K_i = 2 \). Analyse the stability of the closed loop system, using \( G^d(z) \) and \( F^d_{PI}(z) \). For which sampling periods \( h > 0 \) is the closed loop system stable? Hint: Both roots of \( z^2 + az + b = 0 \) lies within the unit circle if and only if \(|a| - 1 < b < 1 \) holds. (You may verify this.) (2 pts)

(d) Assume that the PI-controller is implemented in Matlab. Fill in the missing lines of code in the following Matlab code snippet, so that the controller works properly:

```matlab
t = GetTime; % Initiate controller time
while true % Control loop begins
    t = t + h; % Increment controller time to next sample instant
    WaitUntil(t); % Wait for next sample instant
    r = ReadAD(0); % Read present setpoint value on A/D-converter, channel 0
    y = ReadAD(1); % Read present output value on A/D-converter, channel 1
    % Compute the control input u //FILL IN HERE!!//
    WriteDA(0,u); % Write control input u on D/A-converter, channel 0
end % Control loop ends here
```

2
Problem 2 An inverted pendulum has the (linearized) continuous-time model

\[
\begin{align*}
\dot{x}(t) &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t), \\
y(t) &= \begin{bmatrix} 0 & 1 \end{bmatrix} x(t),
\end{align*}
\]

where \( u \) is the applied torque about the hinged (sv. “ledade”) lower end of the pendulum, and \( y \) is the angular deviation from the vertical line. An initial, continuous-time control design yields the control law

\[
u(t) = -\begin{bmatrix} 2 & 3 \end{bmatrix} x(t) + 2r(t).
\]

(a) Determine the transfer function for the closed loop system, \( Y(s) = G_c(s)R(s) \), when the continuous-time control law (3) is used.

Your task now is to re-design for a sampling controller so that the closed loop system gets an equivalent performance as \( G_c(s) \) in (a). The system is sampled with the sampling period \( h = 0.1 \) seconds.

(b) Determine the sampled discrete-time model of the system (using zero order hold). That is, determine \( F, G \) and \( H \) in the state space model

\[
\begin{align*}
x(kh + h) &= Fx(kh) + Gu(kh), \\
y(kh) &= Hx(kh),
\end{align*}
\]

(c) Determine the vector \( L \) and the scalar \( m \) in the discrete-time state feedback

\[
u(kh) = -Lx(kh) + mr(kh),
\]

so that the closed loop system is equivalent to \( G_c(s) \) in (a). A thorough motivation is required.

Hint: Use pole placement: Determine the continuous-time poles, and then “translate” these to corresponding discrete-time poles.

(d) Simulate the step responses of both the continuous-time closed loop system, \( G_c(s) \), and the discrete-time closed system with the controller in (c). Plot both step responses in the same plot and verify that they are equivalent.
HOMEWORK ASSIGNMENT II

Deadline: Friday October 9, 17.00

Problem 1
(a) Consider the continuous-time system
\[ \dot{x}(t) = \begin{bmatrix} -2\zeta & -1 \\ 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} v(t), \quad 0 < \zeta < 1 \]
\[ y(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} x(t), \]
where \( v(t) \) is white noise with intensity \( \Phi_v(\omega) = R_v = 1 \). Determine the covariance matrix for the state vector, \( \Pi_x = \mathbb{E} x(t) x(t)^T \). Next, determine the spectrum \( \Phi_y(\omega) \) for the output \( y \). Also, plot the spectrum for some \( 0 < \zeta < 1 \).

(b) A discrete-time, stationary stochastic process \( w(k) \) has the spectral density
\[ \Phi_w(\omega) = \frac{10}{2.6 - \cos \omega}. \]
The stochastic process can be modeled as \( w(k) = G_w(q)v(k) \), where \( v(k) \) is zero mean white noise with unit variance, \( \mathbb{E} v(k)^2 = 1 \), and where \( G_w(q) \) is a stable, minimum phase transfer operator. Find \( G_w(q) \).

Problem 2 The block diagram below represents a certain industrial process.

The control objective is to keep the output \( y \) as close as possible to the constant setpoint \( r \). The output must not exceed a critical level for more than 1 % of the time on average — this is a strict requirement. On the other hand, the lower the output is kept, the more expensive is the operation of the process. Thus, the setpoint is chosen as close to the critical level as possible, without violating the 1 % of the time requirement.

The system is modeled as
\[ \dot{x}(t) = \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} (u(t) + d(t)), \]
\[ y(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} x(t), \]
where \( d \) is a resonant random disturbance. It is found that the disturbance can be modeled as
\[ \dot{z}(t) = \begin{bmatrix} -0.4 & -4.04 \\ 1 & 0 \end{bmatrix} z(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} v(t), \]
\[ d(t) = \begin{bmatrix} 1 & 0.2 \end{bmatrix} z(t), \]
where $v$ is white noise with the intensity $\Phi_v(\omega) = R_v = 1$.

An initial design utilizes proportional control in order to stabilize the system (the control law is $u = r - y$, chosen to give reasonable stability margins). It is then found that the setpoint must be 0.07 units below the critical level — see the figure below, showing a simulation of this case.

Your task is to design a controller that performs better than the proportional controller above, and thereby save money for your company (every 0.01 unit the setpoint gets closer to the critical level reduces the cost by SEK 100 000 monthly). Use the LQG design technique:

- Minimize the output $y$ (assuming $r = 0$)
- The constraint $|u(t)| \leq 0.30 \forall t$ must not be violated (due to physical limitations)

Your choices of the design parameters ($Q_1$, $Q_2$ etc) should be well motivated.

(a) Combine the models of the system and the disturbance into an augmented state space model, with $u$ and $v$ as inputs and $y$ as output. (1 pt)

(b) Design the controller, that is find a state feedback gain $L$ and a Kalman filter gain $K$ such that the requirements are fulfilled. (4 pts)

(c) Simulate the obtained closed loop system and plot the results to verify that the requirements are met. Use the pre-defined realizations $d_1$, $d_2$ and $d_3$ of the disturbance $d$ in the simulations — these are available in a .mat-file at Studentportalen. (2 pts)

How much closer to the critical level can the setpoint be with your controller? The distance should be reduced with at least 50% compared to the proportional control above.