LQG for discrete-time systems
Starting point and control strategy

- “Standard” state space representation:

\[
\begin{align*}
q x &= F x + G u + N v_1, \\
\dot{z} &= M x, \\
y &= H x + v_2,
\end{align*}
\]

\[
\eta = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}, \quad \Phi_{\eta}(\omega) = \begin{bmatrix} R_1 & R_{12} \\ R_{12}^T & R_2 \end{bmatrix}.
\]

- Minimize the criterion

\[
V = \| z \|^2_{Q_1} + \| u \|^2_{Q_2} = E \left[ z^T Q_1 z + u^T Q_2 u \right].
\]

- The weighting matrices \( Q_1 = Q_1^T \geq 0 \) and \( Q_2 = Q_2^T > 0 \) are design parameters.

- Control law:

\[
u(k) = -L \hat{x}(k) + \tilde{r}(k), \quad \dot{q} \hat{x} = F \hat{x} + G u + K(y - H \hat{x}).\]
LQG: The optimal controller

**Theorem 9.4**

- The optimal control law is $u(t) = -L\hat{x}(k|k - 1)$,
- $\hat{x}(k|k - 1)$ is obtained from the corresponding *Kalman filter*.
- The optimal state feedback gain is
  \[ L = (G^T S G + Q_2)^{-1} G^T S F. \]
- The matrix $S = S^T \geq 0$ is the solution to the discrete-time *Riccati equation* (DARE)
  \[ S = F^T S F + M^T Q_1 M - F^T S G (G^T S G + Q_2)^{-1} G^T S F. \]
- **N.B.** There are two *different* DAREs involved, the one above and the one for the Kalman filter!
- **Corollary 9.2:** For the case when $R_{12} = 0$ the control law $u(k) = -L\hat{x}(k|k)$ is the optimal controller with a direct/feed through term (so that $u(k)$ depends on $y(k)$ also).
Example: Sampled LQ control of a DC motor
The effect of $Q_1$ and $Q_2$

- The DC-motor $Y(s) = \frac{1}{s(s+1)} U(s)$,
- sampled with $h = 0.1 \Rightarrow y(k) = \frac{0.4837 \cdot 10^{-2} (q+0.9672)}{(q-1)(q-0.9048)} u(k)$.
- $u(k) = -Lx(k) + mr(k)$, $Q_1 = 1 = \text{constant}$ and $Q_2$ varied.

- Simulations: Step responses ($r = \text{unit step}$) for the closed loop systems. The output $y$ to the left, the input $u$ to the right.
Cross term in the criterion

- Sometimes there is a cross term in the criterion:

\[ V = E \left[ z^T Q_1 z + 2 x^T Q_{12} u + u^T Q_2 u \right] \]

- Can be treated explicitly.

- **Alternative:** Bring it back to the standard case by a preliminary state feedback,

\[ \tilde{u} = u + Q_2^{-1} Q_{12}^T x. \]

- This leads to the modified LQ-problem with

\[ \tilde{M} = I, \quad \tilde{Q}_1 = M^T Q_1 M - Q_{12} Q_2^{-1} Q_{12}^T, \quad \tilde{Q}_2 = Q_2. \]

- The optimal control law is then \( u = -Lx = -\tilde{L}x - Q_2^{-1} Q_{12}^T x \), i.e. \( L = \tilde{L} + Q_2^{-1} Q_{12}^T \).
Continuous-time LQG by sampling controller

Sampling the criterion function

- Continuous-time system \((A, B, M, C)\).
- Sampling controller with sampling period \(h\).
- Minimize the continuous-time criterion

\[
V_c = E \left[ z^T(t)Q_1 z(t) + u^T(t)Q_2 u(t) \right].
\]

- This is equivalent to minimizing the discrete-time criterion

\[
V_d = E \left[ x^T(k)Q_1 x(k) + 2x^T(k)Q_{12} u(k) + u^T(k)Q_2 u(k) \right],
\]

with

- \(\bar{Q}_1 = \int_0^h \Psi^T Q_1 \Psi dt\),
- \(\bar{Q}_{12} = \int_0^h \Gamma^T Q_1 \Psi dt\),
- \(\bar{Q}_2 = \int_0^h \Gamma^T Q_1 \Gamma dt + hQ_2\),

where \(\Psi(t) = Me^{At}\) and \(\Gamma(t) = \int_0^t \Psi(s)Bds\).
**Example: The effect of the sampling period**

**Intersample behaviour**

- LQ control of the DC-motor.
- Two different sampling periods: $h = 1$ (left) and $h = 0.1$ (right).

![Graphs showing output for discrete-time and continuous-time systems with and without sampling controllers.](image)

- Identical weighting matrices, $Q_1 = 1$ and $Q_2 = 0.01$, were used for:
  - continuous-time,
  - discrete-time and
  - sampled continuous-time versions of the criterion.
LQG: Comments

- LQG optimal for given $Q_1$ and $Q_2$, and under idealized conditions (exact model, $R_1$ and $R_2$ known etc)
- $Q_1$ and $Q_2$ are design parameters — use the optimization as a design tool, adjust $Q_1$ and $Q_2$ to get desirable properties for the closed loop system
- $Q_1$ penalizes $z$ and $Q_2$ penalizes $u$, and only their relative sizes matter
- $Q_1$ and $Q_2$ usually diagonal matrices: if $z_i = [z]_i$ is too big, increase the corresponding diagonal element $q^1_i = [Q_1]_{ii}$ etc
- Too see any change of significance, $Q_1/Q_2$ should be increased or decreased at least one order of magnitude (say 10 times)
LQG: Comments, cont’d

- LQG gives an explicit solution for the optimization problem, and in the form of LTI filters $F_y(q)$ and $F_r(q)$ ($F_y(s)$ and $F_r(s)$ in cont. time), which are easily implemented in a computer.
- The design is an iterative process: check e.g. the sensitivity functions $S$ and $T$, and simulate the closed loop system.
- LQG requires a linear model, so limitations (e.g. bounds on $u$) cannot be accounted for explicitly.
- Useful Matlab functions:
  - `lqr, lqry, dlqr`: compute $L$
  - `lqg, lqgreg`: return the controller as LTI object
  - `lqrd`: computes $L$ for sampled cont.-time criterion