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System identification

Computer exercise 1

Stationary stochastic processes

Preparation exercises:
1. Preparation exercise 1-3.

Name		Assistant's comments
Program	Year of reg.	
Date		
Passed prep. ex.	Sign	
Passed comp. ex.	Sign	

1 Introduction

System identification deals with the problem of building mathematical models of dynamic systems based on observed data collected from the system. This is a basic scientific methodology and since dynamic models of systems are used in almost all disciplines, system identification has a very broad application area.

The basic set-up in system identification is that given measurement of the input and the output of the system finding an appropriate model of the system. A key problem is that in almost all cases, the output is not only affected by the input (via the system) but also by unmeasurable disturbances, 'noise', see Figure 1. The disturbance v can for example be sensor noise and/or a unmeasured load disturbances.

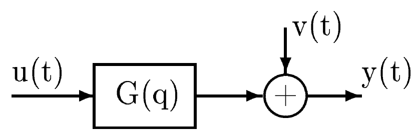


Figure 1: A system $G(q)$ with input u and output y, where the output y is affected by an unmeasurable disturbance v.

In order to analyze and understand system identification it is a prerequisite to understand some basic properties of disturbances and in particular stochastic disturbances. In this computer laboratory, properties of discrete-time stationary stochastic processes will therefore be studied/"refreshed", in order to give a feasible background for understanding system identification.

2 Basic concepts and preparations

2.1 Deterministic vs stochastic signals

The most characteristic feature of a disturbance is that its value is not known beforehand. Hence a deterministic model like, for example, $w(t) = \sin(t)$ is seldom a good way of describing a disturbance. Instead, it is more natural to use stochastic (or random) concepts to describe disturbances.

As a simple example of a stochastic process consider the output signal generated by tossing a coin 5000 times (+1 for heads, -1 for tails). We will obtain a different output sequence every time we do the experiment. The output from each experiment is called a *realization* of the stochastic process.

As a stochastic process represents a whole family of signal realizations, the deterministic signal descriptions can not be directly applied on a stochastic process. The search for good ways of characterizing a stochastic process has been a long and arduous one, but the mathematicians and statisticians of this century have been able to develop a whole framework for stochastic processes that allows us to describe them in a way that resembles that of deterministic signals in many ways. The whole framework is built around the so called *covariance function.*

For a stochastic signal[1] $w(t)$ the function

$$m_w = E\, w(t) \tag{1}$$

is called the *mean value function.* The (auto) covariance function is defined as

$$r_w(\tau) \;= \operatorname{cov}\left(w(t+\tau), w(t)\right) = \; E\,\left(\left(w(t+\tau) - m_w\right)\left(w(t) - m_w\right)\right) \tag{2}$$

Note that $r_w(0)$ is the variance of the process and tells how large the fluctuations are (the standard deviation is the square root of $r_w(0)$). It follows from Schwartz's inequality that $|r_w(\tau)| \leq r_w(0)$. The value of $r_w(\tau)$ gives the correlation between values of the process with a time spacing of τ. Values close to $r_w(0)$ mean a strong correlation, zero values indicate no correlation and negative values indicate negative correlation.

2.2 Characterizing the coin tossing stochastic process

The coin tossing process is a typical stochastic processes, so let us take a closer look at it. We know that we get different realizations every time we run the process, but what is it that characterizes all the realizations? The key characteristic is that each toss is independent from the others, i.e. there is no correlation between one toss and another. In mathematical terms we can describe this process as a sequence of independent identically distributed (iid) random variables, with mean value $m = 0$, and variance $\sigma^2 = 1$. The covariance function hence becomes

$$r_w(\tau) \;=\; E\,\left(\left(y(t+\tau) - m\right)\left(y(t) - m\right)\right) \;=\; \begin{cases} 0 & \text{for } \tau \neq 0 \\ \sigma^2 & \text{for } \tau = 0 \end{cases} \tag{3}$$

The correlation function for the coin tossing process simply says that the coin tossing is uncorrelated (i.e. the chance of getting a head or tail does not depend on what you got in the previous tossing) and has a variance of $\sigma^2 = 1$.

2.3 Spectral density of stochastic process

In the time domain a stochastic process $w(t)$ is normally characterized by its mean m and covariance function $r_w(\tau)$. If we take the Fourier transform of the covariance function we obtain the spectra of the process which describes the frequency content of the stochastic process in a similar way that the Fourier transform does for deterministic signals. The spectral density of a stochastic process with covariance function $r_w(\tau)$ is defined as

$$\phi_w(\omega) = \frac{1}{2\pi} \sum_{\tau=-\infty}^{\infty} r_w(\tau) e^{-i\tau\omega} \tag{4}$$

[1] We will only consider stochastic processes that is *wide sense stationarity*, then both the mean and covariance functions are independent of time.

The covariance function can be found from the spectral density by the inverse relation

$$r_w(\tau) = \int_{-\pi}^{\pi} \phi_w(\omega) e^{i\omega\tau} d\omega \tag{5}$$

The area under the spectral density curve represents the mean signal power in a certain frequency band. In particular we have that the variance of the signal is given by

$$r_w(0) = \int_{-\pi}^{\pi} \phi_w(\omega) d\omega \tag{6}$$

The coin tossing process has a covariance function that is a unit impulse sequence. The Fourier transform of that is a constant, i.e., it has equal amounts of all frequency components.

The coin tossing process is an example of a white noise process, i.e., a sequence of independent random variables with a certain probability distribution. However, as a disturbance model, the coin tossing is not very realistic (a random sequence with -1 and 1). A more suitable pattern is obtained if the disturbance is modeled as a white Gaussian process with zero mean and variance σ^2. Such a process is often denoted

$$w(t) \sim N(0, \sigma^2) \tag{7}$$

2.4 Filtering of stochastic processes

In system identification it is of utmost importance to know and understand how the characteristics of a stationary stochastic process change as the process is filtered by a linear system or filter

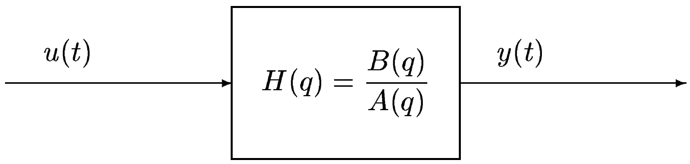
$$H(q) = \frac{B(q)}{A(q)} = \sum_{n=0}^{\infty} h(n) q^{-n} \tag{8}$$

In particular, we need to know how the mean, covariance function and spectrum change. The main results are summarized in Figure 2. Notice, though, that in practice it is seldom a good way to compute auto/cross covariance functions using the formulas in Figure 2.

A large class of disturbances can be described by filtering a white noise process. That is obtained by letting $u(t)$ in Figure 1 be white noise. We then have an ARMA process.

Preparation exercise 1. Assume that white noise with zero mean and unit variance is filtered. Determine a first order filter

$$H(z) = \frac{b}{z + a} \tag{9}$$



$$r_u(\tau) = \operatorname{cov}\left(u(t+\tau), u(t)\right)$$

$$r_{yu}(\tau) = \operatorname{cov}\left(y(t+\tau), u(t)\right) = \sum_{n=0}^{\infty} h(n) r_u(\tau - n)$$

$$r_y(\tau) = \operatorname{cov}\left(y(t+\tau), y(t)\right) = \sum_{n=0}^{\infty}\sum_{l=0}^{\infty} h(n)h(l) r_u(\tau + l - n)$$

$$\phi_u(\omega) = \frac{1}{2\pi} \sum_{\tau=-\infty}^{\infty} r_u(\tau) e^{-i\tau\omega}$$

$$\phi_{yu}(\omega) = \frac{1}{2\pi} \sum_{\tau=-\infty}^{\infty} r_{yu}(\tau) e^{-i\tau\omega} = H(e^{i\omega})\phi_u(\omega)$$

$$\phi_y(\omega) = \frac{1}{2\pi} \sum_{\tau=-\infty}^{\infty} r_y(\tau) e^{-i\tau\omega} = H(e^{i\omega})H(e^{-i\omega})\phi_u(\omega)$$

Figure 2: Summary of filtering results for stochastic processes.

that generates a signal with the spectral density

$$\phi(\omega) = \frac{1}{2\pi} \cdot \frac{0.75}{1.25 - \cos\omega} \tag{10}$$

What is the variance of the signal? In Exercise 1, you will need the results from here.

Answer:

3 Filtering a white stochastic process

In this section we will analyze the covariance function, the spectrum and different realizations of a stochastic process that is obtained by filtering a white stochastic process with the first and second order systems below.

$$H_1(z) = \frac{b}{z+a} \qquad H_2(z) = \frac{b_0 z + b_1}{z^2 + a_1 z + a_2} \tag{11}$$

In particular, we will look at how system parameters such as poles and zeros influence the results.

Exercise 1. Answer the following questions, with the help of the MATLAB macro `noise`.

a) Test if the filter you calculated in Preparation exercise 1 is correct. Note that $b \equiv \sqrt{3}/2$ is a fix value.

b) Vary the poles of $H_2(z)$ on a radius out from the origin. What happens?

c) Where should you place the poles of $H_2(z)$ to get a low-pass filter?

d) Where should you place the poles of $H_2(z)$ to get a resonance top at $\omega = 1$? What can you say about the frequency content of the signal by just looking at the realization?

e) How does a resonant system manifest its properties in the different diagrams?

f) What happens when $H_2(z)$ has a zero close to the unit circle?

Answer:

4 Calculating and estimating covariance functions

Preparation exercise 2. Determine the covariance function for an AR(1) process:

$$y(t) + ay(t-1) = e(t) \tag{12}$$

where $e(t)$ is white noise with zero mean and unit variance.

Answer:

Preparation exercise 3. Determine the covariance function for an MA(1) process:

$$y(t) = e(t) + ce(t-1) \tag{13}$$

where $e(t)$ is white noise with zero mean and unit variance. Moreover, let us consider a general MA(n) process; for what values of τ is it generally true that $r(\tau) = 0$?

Answer:

For a stationary stochastic process[2] with mean m and covariance function $r(\tau)$, the covariance function can be estimated from data by

$$\hat{r}(\tau) \triangleq \frac{1}{N-\tau} \sum_{t=1}^{N-\tau} [y(t+\tau) - \hat{m}][y(t) - \hat{m}] \tag{14}$$

[2]Rather an ergodic process. However, for the processes studied in this course stationarity will be sufficient for expressions as (14) and (15) to converge to their expected values.

where N is the number of data and

$$\hat{m} \triangleq \frac{1}{N} \sum_{t=1}^{N} y(t) \tag{15}$$

is the estimated mean.

Exercise 2. With the macro `lab1b` you can simulate ARMA(1) processes (or, especially, AR(1) and MA(1) processes)

$$y(t) + ay(t-1) = e(t) + ce(t-1) \tag{16}$$

and estimate its covariance functions. The macro has the following syntax: `lab1b([1 c],[1 a],N,tau,nr);`
Type `help lab1b` for details. Default values are `N=100, tau=50` and `nr=1`. For example, to generate an AR(1) process with a pole in 0.9 use the syntax `lab1b([1 ],[1 -0.9]);`.

Inspect how the quality of the estimated covariance function vary with the number of data N and the time shift τ for different pole locations. Especially, verify that the estimated covariance functions tends to the true ones, as the number of data tends to infinity, for some AR(1) and MA(1) processes. Also check if the results you obtained in prep. exc. 2-3 are correct. Repeat the estimation procedure to study the effect of the individual realizations.

Answer:

5 (Spectral factorization)

Remark: This exercise is optional!

Spectral factorization can be viewed as a way to aggregate different noise sources. Assume here that an ARMA process

$$A(q)x(t) = C(q)v(t) \tag{17}$$

is observed in white measurement noise

$$y(t) = x(t) + e(t) \tag{18}$$

and that $v(t)$ and $e(t)$ are uncorrelated white noise sequences of zero mean and variances λ_v^2 and λ_e^2, respectively. As far as the second order properties (the spectrum and the covariance function) are concerned, $y(t)$ can be viewed as generated from one single noise source as

$$A(q)y(t) = D(q)\varepsilon(t) \tag{19}$$

The polynomial D and the noise variance λ_ε^2 are derived by equating the expressions for the spectra of the output signals in (18) and (19). The two representations (17) and (18), and (19) of the process $y(t)$, are shown schematically in Figure 3.

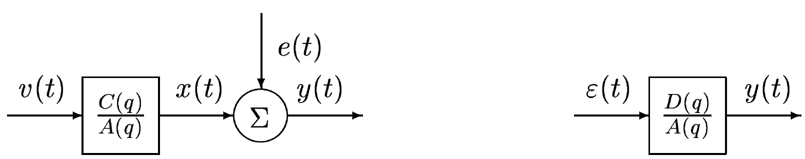


Figure 3: Two representations of an ARMA process with noisy observations.

Exercise 3. Let (17) be an AR(2) process. Then (19) will be an ARMA(2,2) process. Investigate how the poles and zeros are varying with the signal-to-noise ratio (SNR) defined as

$$\text{SNR} = \frac{\text{E}x^2(t)}{\text{E}e^2(t)} \tag{20}$$

by using the file `lab1c`. In particular, what happens when

a) SNR $\to 0$

b) SNR $\to \infty$

Also comment on the shape of the spectrum.

Answer: