# System identification
# Homework assignment 1

In this homework assignment you will investigate some properties of linear regression applied to polynomial trends, by using Matlab. Both constructed and real data will be used.

1. Consider the polynomial model

$$y(t) = a_0 + a_1 t + \ldots + a_n t^n$$

which is quite common for simple modelling of trends. The model can easily be written as a linear regression by choosing

$$\varphi(t) = (1 \;\; t \ldots t^n)^T \qquad \theta = (a_0 \ldots a_n)^T$$

giving

$$y(t) = \varphi^T(t)\theta$$

Consider the regression model for $t = 1, \ldots, N$, giving the overdetermined system of equations $Y = \Phi\theta$. The least squares solution can be computed numerically in different ways.

(a) One possibility is to compute the solution by firt forming the normal equations $\Phi^T\Phi = \Phi^T Y$ and then solving this set of equations. The Matlab code would have the structure

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(a'*a)\(a'*b)
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(b) Another possibility is to compute the solution using a singular value decomposition or an QR factorization of the matrix $\Phi$. Such an approach is obtained with the Matlab code

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a\b
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The task is to compare the numerical properties of these two approaches. Set $Y = \Phi\theta_0$ where $\theta_0 = (1 \ldots 1)^T$. Try some different values of $N$ (say in the interval [20,50]) and $n$ (say in the interval [4,10]). Compute $Y$ and the solutions, here denoted $\theta_1$ and $\theta_2$, respectively. The numerical precision of the approaches can be evaluated using the norms

$$||\theta_0 - \theta_1||, \qquad ||\theta_0 - \theta_2||, \qquad ||\theta_1 - \theta_2||$$

Try also to find an *explanation* of your findings.

2. Polynomial models should be used with considerable care, especially outside the region of the measured data. This is illustrated in the following.

In the table below the number of inhabitants of Uppsala is given for a couple of years.

| Year | Population |
|---|---|
| 1920 | 69 547 |
| 1930 | 73 537 |
| 1940 | 78 804 |
| 1950 | 93 497 |
| 1960 | 104 569 |
| 1970 | 129 672 |
| 1980 | 146 192 |

Fit some polynomial trends to this set of data. As there are 7 measured values, the polynomial can have at most degree 6. Use the polynomial to interpolate and extrapolate between the measured data points. In particular, use the polynomial to predict (from the data and the model) the population of Uppsala in 1990. Preferably present your results also in graphical form. What conclusions can be drawn on the choice of model order (degree of the polynomial)?

# System identification
# Homework assignment 2

This homework assignment is intended to be solved by means of pen and pencil. It might as a complement be illustrated by means of numerical experimentation.

Consider a linear regession model

$$\mathcal{M}_1 : \qquad y(t) = a + bt + \varepsilon_1(t)$$

Assume that the aim is to estimate the parameter $b$. One alternative is of course to use linear regression and estimate both $a$ and $b$.

Another alternative is to work with *differenced* data. For this purpose introduce $z(t) = y(t) - y(t-1)$. Then $\mathcal{M}_1$ gives the new model structure

$$\mathcal{M}_2 : \qquad z(t) = b + \varepsilon_2(t)$$

where $\varepsilon_2(t) = \varepsilon_1(t) - \varepsilon_1(t-1)$. Linear regression can be applied to $\mathcal{M}_2$ for estimating $b$ only, treating $\varepsilon_2(t)$ as the equation error.

Derive and compare the variances of the estimate of $b$ in the two cases. Assume that data are collected at times $t = 1, \ldots, N$, and that they obey

$$\mathcal{S} : \qquad y(t) = a_0 + b_0 t + e(t)$$

where $e(t)$ is white noise of zero mean and unit variance.

# System identification
# Homework assignment 3

This homework assignment is intended to illustrate how the experimental condition influences the accuracy or quality of the estimated model.

Consider the system

$$\mathcal{S}: \quad y(t) + a_0 y(t-1) = b_0 u(t-1) + e(t)$$

where $e(t)$ is white noise of zero mean and variance $\lambda^2$. The system is to be identified in the model structure

$$\mathcal{M}: \quad y(t) + a y(t-1) = b u(t-1) + \varepsilon(t)$$

using the least squares method. Two different input signals, both of the same magnitude should be used, namely

$\mathcal{X}_1$: $u(t)$ zero mean white noise of variance $\sigma^2$
$\mathcal{X}_2$: $u(t)$ a step function of size $\sigma$

Simulate the system using the two above experimental conditions, and identify it. Call the obtained models $\mathcal{M}_1$ and $\mathcal{M}_2$, respectively. Illustrate the found results in the following way.

1. Plot (in the same diagram) the true step response, and the step responses of $\mathcal{M}_1$ and $\mathcal{M}_2$,

2. Plot (in the same diagram) the true frequency response (discrete time Bode plot) and the frequency responses of $\mathcal{M}_1$ and $\mathcal{M}_2$,

3. Compare the true static gain with the static gains of $\mathcal{M}_1$ and $\mathcal{M}_2$,

You may try the numerical values $a = -0.8$, $b = 1.0$, $\sigma^2 = 1$, $\lambda^2 = 0.3$, $N = 100$. Interprete the results in terms of model quality of $\mathcal{M}_1$ and $\mathcal{M}_2$. It may be useful to test several realizations.

# System identification
# Homework assignment 4

This homework assignment is intended to illustrate some different ways to cope with nonzero means in the data. (See also Problem 12.4 in *System Identification*).

Generate data for the system

$$\begin{aligned} A(q^{-1})z(t) &= B(q^{-1})u(t) + e(t) \\ y(t) &= z(t) + m \end{aligned}$$

and regard $u(t)$ and $y(t)$ as the recorded input and output measurements, respectively.

You may try the numerical values $A(q^{-1}) = 1 - 0.8q^{-1}$, $B(q^{-1}) = 1.0q^{-1}$, $u(t)$ and $e(t)$ being mutually independent white noise sequences of zero mean and variances $\sigma^2 = 1.0$ and $\lambda^2 = 0.1$, respectively. Set $m = 5$, $N = 100$. Identify the system in the following ways. Compare the results and try to give explanations. In all cases polynomials like $A(q^{-1})$ etc is assumed to have an appropriate structure.

1. Use the least squares method and the model structure

$$\mathcal{M} : A(q^{-1})y(t) = B(q^{-1})u(t) + \varepsilon(t)$$

2. Use the least squares method and the model structure

$$\mathcal{M} : A(q^{-1})y(t) = B(q^{-1})u(t) + \mu + \varepsilon(t)$$

   i e append an additional parameter (called $\mu$ above) in the parameter vector.

3. First compute the output average

$$\overline{y} = \sum_{t=1}^{N} y(t)/N$$

   Then apply the least squares method with the model structure

$$\mathcal{M} : A(q^{-1})[y(t) - \overline{y}] = B(q^{-1})u(t) + \varepsilon(t)$$

4. First differentiate the data in the sense

$$\Delta y(t) = y(t) - y(t-1)$$

   Then apply the least squares method with the model structure

$$\mathcal{M} : A(q^{-1})\Delta y(t) = B(q^{-1})\Delta u(t) + \varepsilon(t)$$

5. Use some instrumental variable method with the model structure

$$\mathcal{M} : A(q^{-1})\Delta y(t) = B(q^{-1})\Delta u(t) + \varepsilon(t)$$

6. Use some the prediction error method with the model structure

$$\mathcal{M} : A(q^{-1})\Delta y(t) = B(q^{-1})\Delta u(t) + C(q^{-1})\varepsilon(t)$$

# System identification
# Homework assignment 5

This homework assignment is intended to illustrate some different approaches to tracking parameter variations in a dynamic system.

Use Matlab to simulate the time–varying system

$$y(t) + ay(t-1) = b(t-1)u(t-1) + e(t)$$

You may try the following numerical values: $a = -0.8$, $u(t)$ and $e(t)$ mutually independent white Gaussian noise sequences, of zero mean and variances $Eu^2(t) = 1$, $Ee^2(t) = 0.5$, $N = 300$. Let the time–varying parameter $b(t)$ shift between the two values 2 and 4 every 60:th sample.

Simulate data and apply some variants of the RLS algorithm to estimate the system parameters. Illustrate your results by plotting (for each variant of RLS) in the same diagram the true and the estimated parameters. It is important to use the same data series for all the identification algorithms. Try the following variants of RLS:

1. A standard RLS with no forgetting ($\lambda = 1$).
2. A standard RLS with forgetting facotr $\lambda < 1$.
3. A Kalman filter variant of real–time RLS with

$$R_1 = \begin{pmatrix} 0 & 0 \\ 0 & r \end{pmatrix}$$

4. The above Kalman filter variant but with the algorithm parameter $r$ possibly time–varying. You can assume that it is known *when* the parameter $b$ is changing.

Try to find appropriate values of the algorithm parameters $\lambda$ and $r$.