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Covariance formulas

1. Assume that $e(t)$ is white noise of zero mean and unit variance. Assume further that $|a| < 1,\ |d| < 1$. Then

$$\begin{aligned} & E\left[\frac{1+cq^{-1}}{1+aq^{-1}}e(t)\right]\left[\frac{b_0+b_1q^{-1}}{1+dq^{-1}}e(t)\right] \\ = \quad & \frac{b_0+cb_1-ab_1-b_0cd}{1-ad} \end{aligned} \tag{0.1}$$

2. Assume that $e(t)$ is white noise of zero mean and unit variance. Assume further that the polynomial $A(z) = z^2 + a_1 z + a_2$ has all zeros strictly inside the unit circle. Then

$$E\left[\frac{b_0+b_1q^{-1}+b_2q^{-2}}{1+a_1q^{-1}+a_2q^{-2}}e(t)\right]^2 = \frac{(b_0^2+b_1^2+b_2^2)Q_0+2(b_0b_1+b_1b_2)Q_1+2b_0b_2Q_2}{D} \tag{0.2}$$

where

$$\begin{aligned} Q_0 &= 1+a_2 \\ Q_1 &= -a_1 \\ Q_2 &= a_1^2 - a_2(1+a_2) \\ D &= (1-a_2)[(1+a_2)^2 - a_1^2] \end{aligned}$$

When the polynomial $A(z)$ is factorized, an alternative form of D is obtained:

$$\begin{aligned} A(z) &= z^2 + a_1 z + a_2 \equiv (z-p_1)(z-p_2) \\ D &= (1-p_1^2)(1-p_2^2)(1-p_1p_2) \end{aligned}$$

Example 1

Compute the covariance function of the ARMA(1,1) process

$$y(t) + ay(t-1) = e(t) + ce(t-1), \qquad Ee(t) = 0, \quad Ee^2(t) = 1$$

Solution. Apply (0.1) with $b_0 = 1,\ b_1 = c,\ d = a$. This gives

$$r_y(0) = Ey^2(t) = \frac{1 + c(c-a) - ca}{1-a^2} = \frac{1 + c^2 - 2ac}{1-a^2}$$

Alternatively, apply (0.2) with $b_0 = 1,\ b_1 = c,\ b_2 = 0,\ a_1 = a,\ a_2 = 0$. This gives

$$Q_0 = 1,\ Q_1 = -a,\ Q_2 = a^2,\ D = 1 - a^2$$

and

$$r_y(0) = \frac{(1+c^2) \times 1 + 2c \times (-a) + 0 \times a^2}{1-a^2} = \frac{1 + c^2 - 2ac}{1-a^2}$$

To find the remaining covariance elements, set

$$v(t) \quad = \quad \frac{1}{1 + aq^{-1}} e(t)$$

and note that

$$y(t) = v(t) + cv(t-1)$$

Hence

$$\begin{aligned} r_y(0) &= (1+c^2) r_v(0) + 2c r_v(1) \\ r_y(1) &= (1+c^2) r_v(1) + c r_v(0) + c r_v(2) \\ r_y(k) &= (1+c^2) r_v(k) + c r_v(k-1) + c r_v(k+1), \qquad k \geq 1 \end{aligned}$$

Either (0.1) or (0.2) gives easily

$$r_v(0) = Ev^2(t) = \frac{1}{1-a^2}$$

Apply (0.1) with $c = 0,\ b_0 = 0,\ b_1 = 1,\ d = a$ to get

$$r_v(1) = \frac{-a}{1-a^2}$$

Alternatively, apply (0.2) with $a_1 = a,\ a_2 = 0$ and note that $r_v(1)$ is the coefficient for $(2b_0b_1 + 2b_1b_2)$. It still gives

$$r_v(1) = \frac{-a}{1-a^2}$$

Similarly, $r_v(2)$ is the coefficient for $2b_0b_2$. Hence,

$$r_v(2) = \frac{a_1^2}{1-a^2} = \frac{a^2}{1-a^2}$$

Using the above results gives

$$\begin{aligned} r_y(0) &= (1+c^2)\frac{1}{1-a^2} + 2c\frac{-a}{1-a^2} = \frac{1+c^2-2ac}{1-a^2} \\ r_y(1) &= (1+c^2)\frac{-a}{1-a^2} + c\frac{1}{1-a^2} + c\frac{a^2}{1-a^2} \\ &= \frac{-a-ac^2+c+a^2c}{1-a^2} = \frac{(c-a)(1-ac)}{1-a^2} \end{aligned}$$

To treat the general case, consider $k \geq 1$. Then

$$\begin{aligned} r_v(k) &= Ev(t+k)v(t) \\ &= E[-av(t+k-1)+e(t+k)]v(t) \\ &= -ar_v(k-1) \\ &= \cdots = (-a)^k r_v(0) = \frac{(-a)^k}{1-a^2} \end{aligned}$$

Hence for $k \geq 1$

$$\begin{aligned} r_y(k) &= (1+c^2)\frac{(-a)^k}{1-a^2} + c\frac{(-a)^{k-1}}{1-a^2} + c\frac{(-a)^{k+1}}{1-a^2} \\ &= \frac{(-a)^{k-1}}{1-a^2}\left[(1+c^2)(-a)+c+a^2c\right] \\ &= \frac{(-a)^{k-1}(c-a)(1-ac)}{1-a^2} \end{aligned}$$

Example 2. Assume

$$y(t) = \frac{bq^{-1}}{1+aq^{-1}}u(t)$$

and that $u(t)$ is white noise of zero mean and variance σ^2. Determine the covariance elements

$$r_y(0),\ r_y(1),\ r_{yu}(0),\ r_{yu}(1)$$

Solution. All the covariance elements will contain σ^2 as a scaling factor. We get directly from Example 1

$$r_y(0) = \frac{b^2\sigma^2}{1-a^2}, \qquad r_y(1) = \frac{-ab^2\sigma^2}{1-a^2}$$

To get $r_{yu}(0)$, we apply (0.1) with $e(t) = u(t),\ c = a,\ b_0 = 0,\ b_1 = b,\ d = a$. Then

$$r_{yu}(0) = \frac{ab-ab}{1-a^2}\sigma^2 = 0$$

This result can also be realized from the definition of $y(t)$ as

$$y(t) = bu(t-1) - abu(t-2) + a^2bu(t-3) + \ldots$$

which is a sum of terms that all are uncorrelated with $u(t)$. Hence $r_{yu}(0) = 0$. The formula can also be used to get

$$r_{yu}(1) = Ey(t)u(t-1) = b\sigma^2$$

Alternatively, use (0.1) with $e(t) = u(t),\ c = a,\ b_0 = b,\ b_1 = 0,\ d = a$. Then we have

$$Eu(t)y(t+1) = r_{yu}(1) = \frac{b - ba^2}{1 - a^2}\sigma^2 = b\sigma^2$$

Derivation of (0.1).

Alternative 1: Using a state space formulation

First introduce the two signals

$$\begin{aligned} y(t) &= \frac{1 + cq^{-1}}{1 + aq^{-1}} e(t) = \left[1 + \frac{(c-a)q^{-1}}{1 + aq^{-1}}\right] e(t) \\ z(t) &= \frac{b_0 + b_1 q^{-1}}{1 + dq^{-1}} e(t) = \left[b_0 + \frac{(b_1 - b_0 d)q^{-1}}{1 + dq^{-1}}\right] e(t) \end{aligned}$$

We want to compute $Ey(t)z(t)$. Next, introduce the state variables

$$\begin{aligned} x_1(t) &= \frac{q^{-1}}{1 + aq^{-1}} e(t) \\ x_2(t) &= \frac{q^{-1}}{1 + dq^{-1}} e(t) \end{aligned}$$

We then get the following state space model

$$\begin{aligned} x(t+1) &= \begin{pmatrix} -a & 0 \\ 0 & -d \end{pmatrix} x(t) + \begin{pmatrix} 1 \\ 1 \end{pmatrix} e(t) \\ \begin{pmatrix} y(t) \\ z(t) \end{pmatrix} &= \begin{pmatrix} c - a & 0 \\ 0 & b_1 - b_0 d \end{pmatrix} x(t) + \begin{pmatrix} 1 \\ b_0 \end{pmatrix} e(t) \end{aligned}$$

The Lyapunov equation for determining $P = Ex(t)x^T(t)$ becomes

$$\begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix} = \begin{pmatrix} -a & 0 \\ 0 & -d \end{pmatrix} \begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix} \begin{pmatrix} -a & 0 \\ 0 & -d \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \end{pmatrix}$$

which has the solution

$$P = \begin{pmatrix} \frac{1}{1-a^2} & \frac{1}{1-ad} \\ \frac{1}{1-ad} & \frac{1}{1-d^2} \end{pmatrix}$$

We then get

$$\begin{aligned} r_{yz}(0) &= Ey(t)z(t) \\ &= b_0 + (c-a)(b_1 - b_0 d)p_{12} \\ &= b_0 + \frac{(c-a)(b_1 - b_0 d)}{1 - ad} \\ &= \frac{1}{1 - ad}\left[b_0 - b_0 ad + b_1(c-a) - b_0(c-a)d\right] \\ &= \frac{1}{1 - ad}\left[b_0(1 - cd) + b_1(c-a)\right] \end{aligned}$$

Alternative 2: Using residue calculus

Straightforward calculations lead to

$$\begin{aligned}
Ey(t)z(t) &= \frac{1}{2\pi \mathrm{i}} \oint \frac{1+cz^{-1}}{1+az^{-1}} \frac{b_0+b_1 z}{1+dz} \frac{\mathrm{d}z}{z} \\
&= \frac{1}{2\pi \mathrm{i}} \oint \frac{z+c}{z+a} \frac{b_0+b_1 z}{1+dz} \frac{\mathrm{d}z}{z} \\
&= \left[\frac{cb_0}{a} + \frac{(c-a)(b_0-b_1 a)}{-a(1-ad)} \right] \\
&= \frac{1}{a(1-ad)} \left[cb_0(1-ad) - (c-a)(b_0-b_1 a) \right] \\
&= \frac{1}{a(1-ad)} \left[b_0(c-acd-c+a) + b_1 a(c-a) \right] \\
&= \frac{1}{1-ad} \left[b_0(1-cd) + b_1(c-a) \right]
\end{aligned}$$