Uppsala University
Dept. of Systems and Control

Final Exam in System Identification for F4

Date and time: Friday, March 14, 2008, at 14–19
Aiding material: The course book *System Identification* by Söderström and Stoica, the mathematical handbook *BETA*, the material "Covariance formulas", and a calculator. *Note that the solution manual to the course book is not allowed and that no copies of any material are allowed whatsoever.*
Examiner: Magnus Mossberg

Read the following information carefully.

- Give clear and structured solutions to the problems in Swedish or in English. Do not hand in anything else but one solution to each problem. Do not solve more than one problem on each sheet of paper and do not write on the back of the sheets. Write your name on every sheet of paper. Hand in the solutions in numerical order. In your solutions, you can refer to results and equations in the book by giving page- or equation numbers.
- If you find anything in a problem formulation unclear, see to that the assumptions you make when solving the problem are clearly stated in your solution.
- The problems are not necessarily given in an increased order of difficulty and subproblems are not necessarily dependent in the sense that their solutions build upon each other.
- The last problem is an alternative to the homework assignments. In case you hand in a solution to the last problem, you will be accounted for the best performance of the homework assignments and the last problem.

Good luck!

1. The model structure

$$i(t) + ai(t-1) = bu(t-1)$$

is considered for describing the relation between the input voltage $u(t)$ and the resulting current $i(t)$ in an electric circuit. An experiment is performed which results in the measurements $\{u(t)\}_{t=1}^{N}$ and $\{i(t)\}_{t=1}^{N}$ (assume that $u(0) = 0$ and $i(0) = 0$). Give the system of equations that defines the least squares estimate of a and b. (6p)

2. Consider the model

$$y(t) = \sum_{k=0}^{\infty} g(k)u(t-k) + v(t).$$

The covariance between $u(t)$ and $y(t)$ is given by $r_{yu}(\tau) = \mathrm{E}\{y(t+\tau)u(t)\}$. Assume that $u(t)$ and $v(t)$ are independent and that $u(t)$ is zero mean white noise with variance λ^2.

   a) Show that $r_{yu}(\tau) = \lambda^2 g(\tau)$.
   b) Propose an estimator of $g(\tau)$ based on the relation in a).

(7p)

3. A system is given by

$$y(t) + a_0 y(t-1) = e(t),$$

where $e(t)$ is zero mean white noise with variance λ^2. It holds that

$$r_y(\tau) = \mathrm{E}\{y(t)y(t-\tau)\} = (-a_0)^{\tau} \frac{\lambda^2}{1-a_0^2}, \quad \tau \geqslant 0.$$

Assume that $\hat{a}$ is the estimate of a that the prediction error method gives for the model

$$y(t) + ay(t-1) = e(t)$$

from the available data $\{y(t)\}_{t=1}^{N}$ (assume that $y(0) = 0$).

   a) Compute $\hat{a}$ in the limiting case when $N \to \infty$.
   b) Determine the (approximative) variance of $\hat{a}$ for a large but finite N.

(7p)

4. The constant c is to be estimated recursively from the measurements

$$y(t) = c + e(t),$$

where $e(t)$ is white noise of zero mean and variance σ^2. Therefore, the criterion function

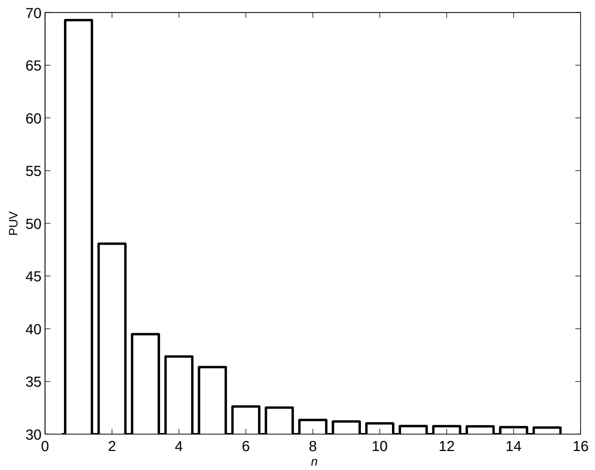
$$J_t(c) = \sum_{k=1}^{t} 0.9^{t-k}(y(k) - c)^2$$

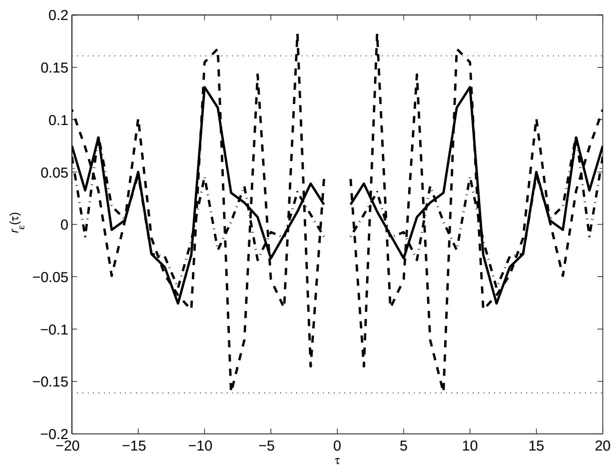
is considered. Give the estimator $\hat{c}(t)$ that minimizes $J_t(c)$, and find a recursive form

$$\hat{c}(t) = \hat{c}(t-1) + f\big(\hat{c}(t-1), y(t), t\big),$$

where $f\big(\hat{c}(t-1), y(t), t\big)$ is a function of $\hat{c}(t-1)$, $y(t)$, and t. (7p)

5. AR models of different orders are fitted to some data using the System Identification Toolbox in MATLAB, and the results are presented in the two figures below (the results are presented for the data used in the *estimation*).





The left figure presents the percentage of unexplained output variance (PUV) for AR(n) models, $n = 1, \ldots, 15$, whereas the right figure presents the autocorrelation function $r_\varepsilon(\tau)$ for the residuals $\varepsilon(t)$, with $r_\varepsilon(0) = 1$ omitted for clarity, for three different models.

a) Based on the information in the left figure, what model order would you choose? Motivate your choice.

b) Based on the information in the right figure, which model would you choose (that is, the model corresponding to the solid, dashed or dash-dotted curve)? Motivate your choice.

(6p)

6. Consider the observations

$$y(t) = c + e(t), \quad t = 1, \dots, N,$$

where $e(t)$ is white noise of zero mean and variance σ^2. Consider the estimator

$$\hat{c} = \frac{1}{2N} \sum_{t=1}^{N} y(t).$$

a) Is the estimator $\hat{c}$ unbiased? Motivate your answer.

b) Compute the variance of the estimator $\hat{c}$.

(7p)

7. *This problem is an alternative to the homework assignments.*

a) Consider the system

$$y(t) + a_0 y(t-1) = b_0 u(t-1) + e(t),$$

where $e(t)$ is white noise of zero mean and variance σ^2. The input signal $u(t)$ is given as the feedback

$$u(t) = r(t) - ky(t),$$

where $r(t)$ is white noise of zero mean and variance λ^2, independent of $e(t)$, and where k is a positive constant. Show that $\mathrm{E}\{z(t)\varphi^T(t)\}$ is nonsingular, where

$$\begin{aligned} z(t) &= \begin{bmatrix} -r(t-2) & r(t-1) \end{bmatrix}^T, \\ \varphi^T(t) &= \begin{bmatrix} -y(t-1) & u(t-1) \end{bmatrix}. \end{aligned}$$

Note that this is one of the conditions that must be satisfied when using the instrumental variable method.

b) Consider the system

$$y(t) = b_1 u(t-1) + b_2 u(t-2) + \ldots + b_n u(t-n) + e(t),$$

where $e(t)$ is white noise of zero mean and variance λ^2. What requirements has the input signal $u(t)$ to fulfill if the system parameters are to be estimated using the least squares method?

(10p)