Uppsala University
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Systems and Control
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Final exam: System identification

Date: March 18, 2005

Responsible examiner: Torsten Söderström

Preliminary grades: 3 = 23–32p, 4 = 33–42p, 5 = 43–50p.

Instructions

The solutions to the problems can be given in Swedish or in English.

Problem 6 is an alternative to the homework assignment. (In case you choose to hand in a solution to Problem 6, you will be accounted for the best performance of the homework assignments and Problem 6.)

Solve each problem on a separate page.

Write your name on every page.

Provide motivations for your solutions. Vague or lacking motivations may lead to a reduced number of points.

Aiding material: Textbooks in system identification, automatic control, statistics, signal processing, mathematical handbooks, handwritten lecture notes, collection of formulas (formelsamlingar), calculators. Note that the following are **not allowed**: Solutions manual, copies of OH transparencies, old exams.

Good luck!

Problem 1

Consider a system given by

$$y(t) = b_1 u(t-1) + b_2 u(t-2) + e(t)$$

where the measurement noise $e(t)$ is white with zero mean and variance σ^2. The parameters b_1 and b_2 are estimated using the least squares method.

(a) Express the asymptotic variances of the estimates $\hat{b}_1$ and $\hat{b}_2$ as functions of the input covariance function $r_u(\tau) = Eu(t+\tau)u(t)$. **3 points**

(b) Assume that the variance of the input must be bounded

$$Eu^2(t) \leq 1$$

Determine the covariance function of the input so that the parameter estimates $\hat{b}_1$ and $\hat{b}_2$ have as low variance as possible. **3 points**

Problem 2

Consider the model

$$y(t) = b_1 u(t-1) + b_2 u(t-2)$$

or written as a linear regression

$$\begin{aligned} y(t) &= \varphi^T(t)\theta, \quad t = 1, \ldots, N \\ \varphi^T(t) &= \begin{pmatrix} u(t-1) & u(t-2) \end{pmatrix} \end{aligned}$$

(a) Assume $u(t)$ is a step

$$u(t) = \begin{cases} 0 & t < 0 \\ a & t \geq 0 \end{cases}$$

and that the data are

$$y(t) = b_1 u(t-1) + b_2 u(t-2) + e(t), \quad t = 1, \ldots, N$$

where $e(t)$ is white noise of zero mean and variance λ^2.

Determine $\text{cov}(\hat{\theta})$, $\text{var}(\hat{b}_1)$ and $\text{var}(\hat{b}_2)$ for finite values of N. **2 points**

(b) Introduce

$$\beta_1 = b_1 + b_2, \quad \beta_2 = b_1 - b_2$$

Determine (for finite N) $\text{var}(\hat{\beta}_1)$ and $\text{var}(\hat{\beta}_2)$. **1 point**

(c) Rewrite the linear regression as

$$y(t) = \psi^T(t)\beta, \qquad \beta = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}$$

with β as in part (b). What is $\psi(t)$? **2 points**

(d) Determine how the least squares estimate β depends on data. Also find $\text{cov}(\hat{\beta})$. **3 points**

(e) If $u(t)$ were white noise of variance σ^2, determine (for large values of N) $\text{cov}(\hat{\theta})$ and $\text{cov}(\hat{\beta})$. **2 points**

Problem 3

Consider identification of a system as a finite impulse response model

$$y(t) = b_1 u(t-1) + \ldots + b_n u(t-n) + \varepsilon(t)$$

Determine for what model order n the parameters $\{b_i\}_{i=1}^n$ can be uniquely determined for the following input signals. (Treat the asymptotic case with data available for $t = 1, \ldots, N$, where $N \to \infty$.)

$\mathcal{X}_1$	$u_1(t) = 1$	
$\mathcal{X}_2$	$u_2(t) = (-1)^t$	
$\mathcal{X}_3$	$u_3(t) = 1 + (-1)^t$	
$\mathcal{X}_4$	$u_4(t) = \sin \omega_o t$	$0 < \omega_o < \pi$
$\mathcal{X}_5$	$u_5(t) = \sin \omega_1 t + 2 \sin \omega_2 t$	$0 < \omega_1 < \omega_2 < \pi$
$\mathcal{X}_6$	$u_6(t)$ white noise	

6 points

Problem 4

Consider prediction error identification of an ARMA(1,1) process

$$y(t) + ay(t-1) = e(t) + ce(t-1)$$

(a) What is the asymptotic variances of the estimates $\hat{a}$ and $\hat{c}$? **4 points**

(b) What is the asymptotic variance of the difference $\hat{a} - \hat{c}$? **4 points**

Problem 5

Consider a first order system

$$y(t) + ay(t-1) = bu(t-1) + e(t)$$

where the input $u(t)$ and the disturbance $e(t)$ are mutually independent white noise sequences of zero mean, and variances σ^2 and λ^2, respectively.
The parameter vector

$$\theta = (\ a\ b\ )^T$$

is to be estimated, and its asymptotic covariance matrix to be determined.

(a) Assume that the least squares method is applied. Determine the asymptotic covariance matrix. **2 points**

(b) Assume that an instrumental variable method is applied with instruments

$$z(t) = (\ u(t-1)\ u(t-2)\ )^T$$

Determine the asymptotic covariance matrix. **3 points**

(c) Assume that an instrumental variable method is applied with instruments

$$z(t) = (\ -x(t-1)\ u(t-1)\ )^T$$

where $x(t) = \frac{bq^{-1}}{1+aq^{-1}} u(t)$. (We assume that a and b are known when $x(t)$ is generated). Determine the asymptotic covariance matrix. **3 points**

(d) Compare the covariance matrices determined in parts (a) - (c). Can they be sorted in some increasing order? **2 points**

Problem 6

Consider a pure sinewave signal

$$y(t) = A \sin(\omega_o t), \qquad 0 < \omega_o < \pi, \qquad t = 1, 2, \ldots$$

(a) Determine the covariance function

$$r_y(\tau) = \lim_{N \to \infty} \frac{1}{N} \sum_{t=1}^{N} y(t+\tau) y(t)$$

of the signal. **2 points**

(b) Of what order is the signal $y(t)$ persistently exciting? **2 points**

(c) Assume that a second order AR model

$$y(t) + a_1 y(t-1) + a_2 y(t-2) = e(t)$$

is fitted to measurements of the signal. What are the asymptotic values ($N \to \infty$) of the parameter values? **4 points**

(d) Where are the zeros of the polynomial $A(z)$ determined in part (c) located. Please interprete! **2 points**

System identification, March 18, 2005 — Answers and brief solutions

Problem 1

(a)

$$\mathrm{cov}\begin{pmatrix}\hat b_1\\ \hat b_2\end{pmatrix}=\frac{\sigma^2}{N}\begin{pmatrix}r_u(0) & r_u(1)\\ r_u(1) & r_u(0)\end{pmatrix}^{-1}=\frac{\sigma^2}{N}\frac{1}{r_u^2(0)-r_u^2(1)}\begin{pmatrix}r_u(0) & -r_u(1)\\ -r_u(1) & r_u(0)\end{pmatrix}$$

Hence

$$\mathrm{var}(\hat b_1)=\mathrm{var}(\hat b_2)=\frac{\sigma^2}{N}\frac{r_u(0)}{r_u^2(0)-r_u^2(1)}$$

(b) Under the constraints

$$|r_u(1)|\le r_u(0)\le 1$$

the variances are minimized if

$$r_u(0)=1,\ r_u(1)=0$$

This is achieved, for example, if the input signal is white noise.

Problem 2

(a)

$$\Phi=\begin{pmatrix}u(0) & u(-1)\\ u(1) & u(0)\\ \vdots & \vdots\\ u(N-1) & u(N-2)\end{pmatrix}=\begin{pmatrix}1 & 0\\ 1 & 1\\ \vdots & \vdots\\ 1 & 1\end{pmatrix}$$

$$\mathrm{cov}(\hat\theta)=\lambda^2\left(\Phi^T\Phi\right)^{-1}=\lambda^2\begin{pmatrix}N & N-1\\ N-1 & N-1\end{pmatrix}^{-1}=\lambda^2\begin{pmatrix}1 & -1\\ -1 & \frac{N}{N-1}\end{pmatrix}$$

$$\mathrm{var}(\hat b_1)=\lambda^2\qquad \mathrm{var}(\hat b_2)=\lambda^2\frac{N}{N-1}$$

(b)

$$\begin{aligned}\mathrm{var}(\hat\beta_1)&=\begin{pmatrix}1 & 1\end{pmatrix}\mathrm{cov}(\hat\theta)\begin{pmatrix}1\\ 1\end{pmatrix}=\lambda^2\frac{1}{N-1}\\ \mathrm{var}(\hat\beta_2)&=\begin{pmatrix}1 & -1\end{pmatrix}\mathrm{cov}(\hat\theta)\begin{pmatrix}1\\ -1\end{pmatrix}=\lambda^2\frac{4N-3}{N-1}\end{aligned}$$

(c)

$$b_1=\frac{\beta_1+\beta_2}{2}\qquad b_2=\frac{\beta_1-\beta_2}{2}$$

$$\begin{aligned}y(t)&=\frac{\beta_1+\beta_2}{2}u(t-1)+\frac{\beta_1-\beta_2}{2}u(t-2)\\ \psi^T(t)&=\frac{1}{2}\begin{pmatrix}u(t-1)+u(t-2) & u(t-1)-u(t-2)\end{pmatrix}\\ \Psi&=\begin{pmatrix}\psi^T(1)\\ \psi^T(2)\\ \vdots\\ \psi^T(N)\end{pmatrix}=\begin{pmatrix}0.5 & 0.5\\ 1 & 0\\ \vdots & \vdots\\ 1 & 0\end{pmatrix}\end{aligned}$$

(d)

$$\begin{aligned}\hat\beta&=(\Psi^T\Psi)^{-1}\Psi^TY\\ &=\begin{pmatrix}0.25+N-1 & 0.25\\ 0.25 & 0.25\end{pmatrix}^{-1}\begin{pmatrix}0.5 & 1 & \dots & 1\\ 0.5 & 0 & \dots & 0\end{pmatrix}Y\\ &=\frac{1}{0.25(N-1)}\begin{pmatrix}0.25 & -0.25\\ -0.25 & 0.25+N-1\end{pmatrix}\begin{pmatrix}\sum_1^N y(t)-0.5y(1)\\ 0.5y(1)\end{pmatrix}\\ &=\begin{pmatrix}\frac{1}{N-1}\sum_2^N y(s)\\ -\frac{1}{N-1}\sum_2^N y(s)+2y(1)\end{pmatrix}\end{aligned}$$

$$\mathrm{cov}(\hat\beta)=\lambda^2\left(\Psi^T\Psi\right)^{-1}=\lambda^2\begin{pmatrix}0.25+N-1 & 0.25\\ 0.25 & 0.25\end{pmatrix}^{-1}=\frac{\lambda^2}{N-1}\begin{pmatrix}1 & -1\\ -1 & 4N-3\end{pmatrix}$$

$$\mathrm{var}(\hat\beta_1)=\frac{\lambda^2}{N-1}\qquad \mathrm{var}(\hat\beta_2)=\lambda^2\frac{4N-3}{N-1}$$

as in part (b)!

(e)

$$\begin{aligned}\mathrm{cov}(\hat\theta)&=\frac{\lambda^2}{N}\begin{pmatrix}r_u(0) & r_u(1)\\ r_u(1) & r_u(0)\end{pmatrix}^{-1}=\frac{\lambda^2}{N\sigma^2}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\\ \mathrm{cov}(\hat\beta)&=\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\mathrm{cov}(\hat\theta)\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}=\frac{\lambda^2}{N\sigma^2}\begin{pmatrix}2 & 0\\ 0 & 2\end{pmatrix}\end{aligned}$$

Problem 3

If $u(t)$ is p.e. of order n but not of order $n+1$, then FIR models of order n (but not of order $n+1$) can be identified. Hence,

$$u_1\Rightarrow n\le 1,\qquad u_2\Rightarrow n\le 1,\qquad u_3\Rightarrow n\le 2,$$

$$u_4\Rightarrow n\le 2,\qquad u_5\Rightarrow n\le 4,\qquad u_6\Rightarrow n\ \text{arbitrary}$$

Problem 4

(a)

$$\mathrm{cov}\begin{pmatrix}\hat a\\ \hat c\end{pmatrix} = \frac{1}{N}\begin{pmatrix}\frac{1}{1-a^2} & -\frac{1}{1-ac}\\ -\frac{1}{1-ac} & \frac{1}{1-c^2}\end{pmatrix}^{-1}$$
$$= \frac{1}{N}\frac{1}{(c-a)^2}\begin{pmatrix}(1-a^2)(1-ac)^2 & (1-a^2)(1-c^2)(1-ac)\\ (1-a^2)(1-c^2)(1-ac) & (1-c^2)(1-ac)^2\end{pmatrix}$$

(b)

$$\mathrm{var}(\hat a-\hat c) = \begin{pmatrix}1 & -1\end{pmatrix}\mathrm{cov}\begin{pmatrix}\hat a\\ \hat c\end{pmatrix}\begin{pmatrix}1\\ -1\end{pmatrix}$$
$$= \frac{1}{N}\frac{1}{(c-a)^2}\Big[(1-a^2)(1-ac)^2+(1-c^2)(1-ac)^2$$
$$-2(1-ac)(1-a^2)(1-c^2)\Big]$$
$$= \frac{1}{N}(1-a^2c^2)$$

Problem 5

(a)

$$\mathrm{cov}(\hat\theta) = \frac{\lambda^2}{N}\begin{pmatrix}\frac{b^2\sigma^2+\lambda^2}{1-a^2} & 0\\ 0 & \sigma^2\end{pmatrix}^{-1} = \frac{1}{N}\begin{pmatrix}\frac{\lambda^2(1-a^2)}{b^2\sigma^2+\lambda^2} & 0\\ 0 & \frac{\lambda^2}{\sigma^2}\end{pmatrix}$$

(b) Use the results on page 268 in the textbook. The matrix R is as follows

$$R = Ez(t)\varphi^T(t) = E\begin{pmatrix}u(t-1)\\ u(t-2)\end{pmatrix}\begin{pmatrix}-y(t-1) & u(t-1)\end{pmatrix} = \begin{pmatrix}0 & \sigma^2\\ -b\sigma^2 & 0\end{pmatrix}$$

The covariance matrix of the estimates becomes

$$P_{\mathrm{IV}} = \frac{1}{N}\lambda^2R^{-1}\mathrm{cov}(z(t))R^{-T}$$
$$= \frac{1}{N}\frac{\lambda^2}{\sigma^2}\begin{pmatrix}0 & 1\\ -b & 0\end{pmatrix}^{-1} I \begin{pmatrix}0 & -b\\ 1 & 0\end{pmatrix}^{-1}$$
$$= \frac{1}{N}\frac{\lambda^2}{b^2\sigma^2}\begin{pmatrix}1 & 0\\ 0 & b^2\end{pmatrix}$$

(c) The matrix R is as follows

$$R = Ez(t)\varphi^T(t) = E\begin{pmatrix}-x(t-1)\\ u(t-1)\end{pmatrix}\begin{pmatrix}-y(t-1) & u(t-1)\end{pmatrix} = \begin{pmatrix}\frac{b^2\sigma^2}{1-a^2} & 0\\ 0 & \sigma^2\end{pmatrix}$$

The covariance matrix of the estimates becomes

$$P_{\mathrm{IV}} = \frac{1}{N}\lambda^2R^{-1}\mathrm{cov}(z(t))R^{-T}$$
$$= \frac{1}{N}\lambda^2\begin{pmatrix}\frac{b^2\sigma^2}{1-a^2} & 0\\ 0 & \sigma^2\end{pmatrix}^{-1}\begin{pmatrix}\frac{b^2\sigma^2}{1-a^2} & 0\\ 0 & \sigma^2\end{pmatrix}\begin{pmatrix}\frac{b^2\sigma^2}{1-a^2} & 0\\ 0 & \sigma^2\end{pmatrix}^{-1}$$
$$= \frac{1}{N}\frac{\lambda^2}{\sigma^2}\begin{pmatrix}\frac{1-a^2}{b^2} & 0\\ 0 & 1\end{pmatrix}$$

(d) The variance of the estimate $\hat b$ is the same for all three estimators. The variance of $\hat a$ differ though. The matrices can be ordered as follows.

$$P_{(b)} \geq P_{(c)} \geq P_{(a)}$$

Problem 6

(a)

$$r_y(\tau) = \lim_{N\to\infty}\frac{1}{N}\sum_{t=1}^{N}y(t+\tau)y(t)$$
$$= \lim_{N\to\infty}\frac{1}{N}\sum_{t=1}^{N}A^2\sin(\omega_o t+\omega_o\tau)\sin(\omega_o t)$$
$$= \frac{A^2}{2}\lim_{N\to\infty}\frac{1}{N}\sum_{t=1}^{N}[\cos(\omega_o\tau)-\cos(2\omega_o t+\omega_o\tau)]$$
$$= \frac{A^2}{2}\cos(\omega_o\tau)$$

(b) The signal $y(t)$ is persistently exciting of order 2. This can be seen as follows.

$$\det\begin{pmatrix}r_y(0) & r_y(1)\\ r_y(1) & r_y(0)\end{pmatrix} = r_y^2(0)-r_y^2(1)$$
$$= \frac{A^2}{2}(1-\cos^2(\omega_o)) = \frac{A^2}{2}\sin^2(\omega_o) > 0$$

$$\det\begin{pmatrix}r_y(0) & r_y(1) & r_y(2)\\ r_y(1) & r_y(0) & r_y(1)\\ r_y(2) & r_y(1) & r_y(0)\end{pmatrix} = r_y^3(0)+2r_y^2(1)r_y(2)-r_y(0)r_y^2(2)-2r_y^2(1)r_y(0)$$
$$= r_y(0)[r_y^2(0)-r_y^2(2)]-2r_y^2(1)[r_y(0)-r_y(2)]$$
$$= [r_y(0)-r_y(2)][r_y^2(0)+r_y(0)r_y(2)-2r_y^2(1)]$$
$$= \frac{A^6}{8}[1-\cos(2\omega_o)][1+\cos(2\omega_o)-2\cos^2(\omega_o)]$$
$$= 0$$

(c) The parameter estimates become

$$\begin{pmatrix}\hat a_1\\ \hat a_2\end{pmatrix} = \begin{pmatrix}r_y(0) & r_y(1)\\ r_y(1) & r_y(0)\end{pmatrix}^{-1}\begin{pmatrix}-r_y(1)\\ -r_y(2)\end{pmatrix}$$

$$
\begin{aligned}
&= \begin{pmatrix} 1 & \cos(\omega_o) \\ \cos(\omega_o) & 1 \end{pmatrix}^{-1} \begin{pmatrix} -\cos(\omega_o) \\ -\cos(2\omega_o) \end{pmatrix} \\
&= -\frac{1}{1-\cos^2(\omega_o)} \begin{pmatrix} 1 & -\cos(\omega_o) \\ -\cos(\omega_o) & 1 \end{pmatrix} \begin{pmatrix} \cos(\omega_o) \\ 2\cos^2(\omega_o) - 1 \end{pmatrix} \\
&= \begin{pmatrix} -2\cos(\omega_o) \\ 1 \end{pmatrix}
\end{aligned}
$$

d) The polynomial

$$A(z) = z^2 - 2\cos(\omega_o)z + 1$$

has zeros in $z = e^{\pm i\omega_o}$. These zeros lie on the unit circle. Their argument ω_o corresponds precisely to the angular frequency of the sine wave.