## Lecture 5

**Instrumental Variable Methods (IVM) (Ch. 8)**

Main Idea: Modify the LS method to be consistent also for correlated disturbances.



## Least Squares Method

Consider the ARX model,

$$A(q^{-1})y(t) = B(q^{-1})u(t) + \varepsilon(t)$$

or, equivalently,

$$y(t) = \boldsymbol{\varphi}^T(t)\boldsymbol{\theta} + \varepsilon(t)$$

where $\varepsilon(t)$ is the equation error $(y(t) - y_m(t))$, and

$$\boldsymbol{\varphi}(t) = [-y(t-1) \dots - y(t-n_a)\, u(t-1) \dots u(t-n_b)]^T$$
$$\boldsymbol{\theta} = [a_1 \dots a_{n_a}\, b_1 \dots b_{n_b}]^T$$



The least squares estimate

$$\hat{\boldsymbol{\theta}} = \left[\frac{1}{N}\sum_{t=1}^{N} \boldsymbol{\varphi}(t)\boldsymbol{\varphi}^T(t)\right]^{-1} \left[\frac{1}{N}\sum_{t=1}^{N} \boldsymbol{\varphi}(t)\mathbf{y}(t)\right]$$

has the estimation error (when $N \to \infty$)

$$\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0 = E\left[\boldsymbol{\varphi}(t)\boldsymbol{\varphi}^T(t)\right]^{-1} E\left[\boldsymbol{\varphi}(t)\varepsilon(t)\right]$$

Consequently, for $\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0 = 0$ to hold, we must have

$$E\left[\boldsymbol{\varphi}(t)\varepsilon(t)\right] = 0,$$

which is satisfied if, and essentially only if, $\varepsilon(t)$ is white noise. Hence, the least squares estimate is not consistent for correlated noise sources!



Cure:

- PEM (last lecture). Model the noise.
  - Applicable to general model structures.
  - Generally very good properties of the estimates.
  - Computationally quite demanding.
- Instrumental variable methods (IVM). Do not model the noise.
  - Retain the simple LS structure.
  - Simple and computationally efficient approach.
  - Consistent for correlated noise.
  - Less robust and statistically less effective than PEM.

## The IV method

Introduce a vector $\boldsymbol{z}(t) \in \mathbb{R}^{n_\theta}$ with entries uncorrelated with $\varepsilon(t)$. Then (for large values of $N$)

$$0 = \frac{1}{N}\sum_{t=1}^{N} \boldsymbol{z}(t)\varepsilon(t) = \frac{1}{N}\sum_{t=1}^{N} \boldsymbol{z}(t)\left[y(t) - \boldsymbol{\varphi}^T(t)\boldsymbol{\theta}\right]$$

which yields (if the inverse exists)

$$\hat{\boldsymbol{\theta}} = \left[\frac{1}{N}\sum_{t=1}^{N} \boldsymbol{z}(t)\boldsymbol{\varphi}^T(t)\right]^{-1} \left[\frac{1}{N}\sum_{t=1}^{N} \boldsymbol{z}(t)y(t)\right]$$

The elements of $\boldsymbol{z}(t)$ are usually called the **instruments**. Note that if $\boldsymbol{z}(t) = \boldsymbol{\varphi}(t)$, the IV estimate reduces to the LS estimate.



## Choice of Instruments

Obviously, the choice of instruments is very important. They have to be chosen

(i) such that $\boldsymbol{z}(t)$ is uncorrelated with $\varepsilon(t)$ ($E\boldsymbol{z}(t)\varepsilon(t) = 0$), and

(ii) such that the matrix

$$\frac{1}{N}\sum_{t=1}^{N} \boldsymbol{z}(t)\boldsymbol{\varphi}^T(t) \quad \rightarrow \quad E\boldsymbol{z}(t)\boldsymbol{\varphi}^T(t)$$

has full rank. In other words it is essential that $\boldsymbol{z}(t)$ and $\boldsymbol{\varphi}(t)$ are correlated.



In practice these demands are fulfilled by choosing the instruments to consist of delayed and/or filtered inputs. The instruments are commonly chosen such that

$$\boldsymbol{z}(t) = \begin{bmatrix} -\eta(t-1) & \ldots & -\eta(t-n_a) & u(t-1) & \ldots & u(t-n_b) \end{bmatrix}^T$$

where the signal $\eta(t)$ is obtained by filtering the input as

$$C(q^{-1})\eta(t) = D(q^{-1})u(t).$$

In the special case when $C(q^{-1}) = 1$ and $D(q^{-1}) = -q^{-n_b}$,

$$\boldsymbol{z}(t) = \begin{bmatrix} u(t-1) & \ldots & u(t-n_a-n_b) \end{bmatrix}^T$$

**Rem:** Notice that $u(t)$ and the noise $\varepsilon(t)$ are assumed to be independent.



## Extended IV methods

Recall that the basic IV estimate is derived from

$$\min_{\boldsymbol{\theta}} \left\| \sum_{t=1}^{N} \boldsymbol{z}(t)\varepsilon(t) \right\|^2$$

More flexibility is obtained if the instrument vector $\boldsymbol{z}(t)$ is augmented to dimension $n_z$ ($n_z \geq n_\theta$), and if we allow for a weighting and a prefiltering of the residuals by some stable filter $F(q^{-1})$, i.e.,

$$\min_{\boldsymbol{\theta}} \left\| \sum_{t=1}^{N} \boldsymbol{z}(t)F(q^{-1})\varepsilon(t) \right\|_{\mathbf{Q}}^2$$

where $||\boldsymbol{x}||_{\boldsymbol{Q}}^2 = \boldsymbol{x}^T\boldsymbol{Q}\boldsymbol{x}$ and $\boldsymbol{Q}$ is a positive definite weighting matrix.

Inserting

$$\varepsilon(t) = y(t) - \boldsymbol{\varphi}^T(t)\boldsymbol{\theta}$$

yields the so-called extended IV method

$$\hat{\boldsymbol{\theta}} = \arg\min_{\boldsymbol{\theta}} \left\| \left[\sum_{t=1}^{N} \boldsymbol{z}(t)F(q^{-1})\boldsymbol{\varphi}^T(t)\right]\boldsymbol{\theta} - \left[\sum_{t=1}^{N} \boldsymbol{z}(t)F(q^{-1})y(t)\right] \right\|_{\boldsymbol{Q}}^2$$

When $F(q^{-1}) \equiv 1$ and $\boldsymbol{Q} = \mathbf{I}$, the basic IV method is obtained.
Introduce

$$\begin{aligned} \boldsymbol{R}_N &= \frac{1}{N}\sum_{t=1}^{N} \boldsymbol{z}(t)F(q^{-1})\boldsymbol{\varphi}^T(t) \\ \boldsymbol{r}_N &= \frac{1}{N}\sum_{t=1}^{N} \boldsymbol{z}(t)F(q^{-1})y(t) \end{aligned}$$



Then

$$\begin{aligned} \hat{\boldsymbol{\theta}} &= \arg\min_{\boldsymbol{\theta}} \|\mathbf{R}_N\boldsymbol{\theta} - \mathbf{r}_N\|_{\mathbf{Q}}^2 \\ &= \arg\min_{\boldsymbol{\theta}} (\mathbf{R}_N\boldsymbol{\theta} - \mathbf{r}_N)^T \mathbf{Q} (\mathbf{R}_N\boldsymbol{\theta} - \mathbf{r}_N) \\ &= \left[\mathbf{R}_N^T\mathbf{Q}\mathbf{R}_N\right]^{-1} \mathbf{R}_N^T\mathbf{Q}\mathbf{r}_N \end{aligned}$$

Note that due to numerical instability the algorithm should **not** be implemented in this manner.

**Rem:** Notice that $\boldsymbol{R}_N$ is in general not a square matrix.



## Theoretical Analysis

Assumptions

(i) The system is strictly causal and asymptotically stable.

(ii) The input is persistently exciting of a sufficiently high order.

(iii) The disturbance is a stationary stochastic process with rational spectral density,

$$\varepsilon(t) = H(q^{-1})e(t), \qquad Ee^2(t) = \lambda^2$$

(iv) The input and the disturbance are independent.

(v) The model and the true system have the same transfer function if and only if $\boldsymbol{\theta} = \boldsymbol{\theta}_0$ (uniqueness).

(vi) The instruments and the disturbances are uncorrelated.



Consider the system

$$y(t) = \boldsymbol{\varphi}^T(t)\boldsymbol{\theta}_0 + \varepsilon(t)$$

Then

$$\begin{aligned} \mathbf{r}_N &= \frac{1}{N}\sum_{t=1}^{N} \boldsymbol{z}(t)F(q^{-1})y(t) \\ &= \underbrace{\frac{1}{N}\sum_{t=1}^{N} \boldsymbol{z}(t)F(q^{-1})\boldsymbol{\varphi}^T(t)}_{\boldsymbol{R}_N}\boldsymbol{\theta}_0 + \underbrace{\frac{1}{N}\sum_{t=1}^{N} \boldsymbol{z}(t)F(q^{-1})\varepsilon(t)}_{\mathbf{q}_N} \\ &= \mathbf{R}_N\boldsymbol{\theta}_0 + \mathbf{q}_N \end{aligned}$$

Thus

$$\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0 = \left[\mathbf{R}_N^T \mathbf{Q} \mathbf{R}_N\right]^{-1} \mathbf{R}_N^T \mathbf{Q} \mathbf{q}_N \to \left[\mathbf{R}^T \mathbf{Q} \mathbf{R}\right]^{-1} \mathbf{R}^T \mathbf{Q} \mathbf{q}$$

where

$$\begin{aligned}\mathbf{R} &\triangleq \lim_{N\to\infty} \mathbf{R}_N = E\left[\boldsymbol{z}(t) F(q^{-1}) \boldsymbol{\varphi}^T(t)\right] \\ \mathbf{q} &\triangleq \lim_{N\to\infty} \mathbf{q}_N = E\left[\boldsymbol{z}(t) F(q^{-1}) \varepsilon(t)\right]\end{aligned}$$

Therefore, the IV estimate will be consistent ($\lim_{N\to\infty} \hat{\boldsymbol{\theta}} = \boldsymbol{\theta}_0$) if

(i) $\mathbf{R}$ has full rank (inaccurate estimates will be obtained if $\mathbf{R}$ is nearly rank deficient).

(ii) $E\left[\boldsymbol{z}(t) F(q^{-1}) \varepsilon(t)\right] = 0$.



Furthermore, the parameter estimation errors are asymptotically Gaussian distributed with zero mean and variance $\mathbf{P}_{IV}$

$$\sqrt{N}(\hat{\boldsymbol{\theta}}_N - \boldsymbol{\theta}_0) \to N(0, \mathbf{P}_{IV})$$

where

$$\mathbf{P}_{IV} = \lambda^2 \left(\mathbf{R}^T \mathbf{Q} \mathbf{R}\right)^{-1} \mathbf{R}^T \mathbf{Q} \boldsymbol{S} \mathbf{Q} \mathbf{R} \left(\mathbf{R}^T \mathbf{Q} \mathbf{R}\right)^{-1}$$

where

$$\boldsymbol{S} = E\left[F(q^{-1}) H(q^{-1}) \boldsymbol{z}(t)\right] \left[F(q^{-1}) H(q^{-1}) \boldsymbol{z}(t)\right]^T$$

**Rem:** For multivariable systems $\boldsymbol{S}$ must be modified.



## Optimal IVM

The main usefulness in being able to express $\mathbf{P}_{IV}$ lies in the comparison to $\mathbf{P}_{PEM}$ (recall that PEM is efficient for Gaussian disturbances). An "appropriate" choice of parameters leads to the optimal IVM. For example, (single output)

$$\begin{aligned}\boldsymbol{z}(t) &= H^{-1}(q^{-1}) \tilde{\boldsymbol{\varphi}}(t) \\ F(q^{-1}) &= H^{-1}(q^{-1}) \\ \boldsymbol{Q} &= \mathbf{I}\end{aligned}$$

where $\tilde{\boldsymbol{\varphi}}(t)$ is the noise-free part of $\boldsymbol{\varphi}(t)$. Then,

$$\mathbf{P}_{IV}^{opt} = \lambda^2 \left\{E\left[H(q^{-1}) \tilde{\boldsymbol{\varphi}}(t) H(q^{-1}) \tilde{\boldsymbol{\varphi}}^T(t)\right]\right\}^{-1}$$

and $\mathbf{P}_{IV} \geq \mathbf{P}_{IV}^{opt} \geq \mathbf{P}_{PEM}$.



## Approximative implementation of the optimal IVM

Note that the optimal instruments can not be implemented as it requires knowledge of the undisturbed output, the noise variance ($\lambda^2$), and the shaping filter $H(q^{-1})$. Fortunately, it is possible to find an approximate (iterative) implementations.

One way is the following four-step IV estimator:

(i) Use the least-squares estimate of

$$y(t) = \boldsymbol{\varphi}^T(t) \boldsymbol{\theta} \quad \Rightarrow \quad \hat{\boldsymbol{\theta}}_N^{(1)}$$

(ii) Use the IV estimator with the instruments

$$\boldsymbol{z}^{(1)}(t) = \Big[ -x^{(1)}(t-1) \quad \ldots \quad -x^{(1)}(t-n_a) \quad u(t-1) \quad \ldots u(t-n_b) \Big]$$

where $x_t^{(1)} = \frac{\hat{B}_N^{(1)}(q^{-1})}{\hat{A}_N^{(1)}(q^{-1})} u_t \quad \Rightarrow \quad \hat{\boldsymbol{\theta}}_N^{(2)}$.

(iii) Estimate $H(q^{-1})$. Postulate an AR model, and use the least-squares method

$$L(q^{-1})\hat{w}_N^{(2)}(t) = e(t), \quad \Rightarrow \quad \hat{L}_N(q^{-1})$$

where $\hat{w}_n^{(2)}(t) = \hat{A}_N^{(2)}(q^{-1})y(t) - \hat{B}_N^{(2)}(q^{-1})u(t)$

(iv) Use the IV estimator with $F(q^{-1}) = \hat{L}(q^{-1})$, and

$$\boldsymbol{z}^{(2)}(t) = \hat{L}_N(q^{-1})[-x^{(2)}(t-1) \ldots - x^{(2)}(t-n_a)\, u(t-1) \ldots u(t-n_b)]$$



## Summary IVM

- The implementation of the PEM is computationally complex for many model structures.
- The computationally convenient LS method is normally biased for such model structures (i.e. for correlated disturbances).
- The IV method uses **instruments** that are uncorrelated with the disturbances to make the "LS-like" solution consistent.
- The parameters obtained by the IV method are thus consistent (if the instruments are chosen with care) but has a (slightly) higher variance than the PEM estimates.
- Approximately optimal IV methods can be implemented in an iterative manner to achieve the lowest possible variance of the IV estimates.