System Identification, Exercises

Kristiaan Pelckmans (IT/UU, 2338)

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F, FRI Uppsala University, Information Technology

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Requirements Exam

- Exercises alike and working knowledge.
- Required (print assignments, add solution pages, put name/option and hand in at lecture)
  - Answers to computer exercises.
  - Homework assignments.
  - Answers to laboratory session.
- Handed out during respective labs.
- Deadline - end lectures (1 march)
- Written exam ± 18 march.
- Ph.D. - please contact me = `kp@it.uu.se`.

Book

- Introduction (Chap.2)
- Nonparametric Methods (Chap.3)
- Linear Regression (Chap.4)
- Input Signals (Chap.5)
- Model Parametrizations (Chap.6)
- Prediction Error Methods (Chap.7)
- Instrumental Variable Methods (Chap.8)
- Recursive Identification (Chap.9)
- Identification of Systems Operating in Closed Loop (Chap.10)
- Model Validation (Chap.11)

Key formulas

- Deterministic vs. stochastic.
- Expectation (for ergodic, stationary timeseries $(y(t))_t$)

$$E[f(y(t))] = \frac{1}{N}\sum_{t=1}^{N} f(y(t))$$

- $(y(t))_t$ and $(e(t))_t$ independent timeseries iff $\forall f, g$

$$E[f(y(t))g(e(t))] = E[f(y(t))]E[g(e(t))]$$

- Bias $\theta_0 - E[\hat{\theta}_N]$, consistent if $\lim_{N\to\infty} \hat{\theta}_N = \theta_0$
- System, model, parametrisation, estimator.
- Covariance of an estimate

$$\mathrm{cov}(\hat{\theta}_N) = E\left[(\hat{\theta}_N - \theta_0)^T(\hat{\theta}_N - \theta_0)\right]$$

- Least Squares estimator
- PE, SI, PI

Preparation Exercise 1

Given a system

$$H_1(z) = \frac{b}{z+a}, \quad H_1(z) = \frac{b_0 z + b_1}{z^2 + a_1 z + a_2}$$

1. If this filters white noise, zero mean, unit variance and

   $$\phi_y(\omega) = \frac{1}{2\pi} \frac{0.75}{1.25 - \cos(\omega)}.$$

   What is the variance of the filtered signal?

2. What happens to the output of the second system when you move the poles of $H_2(z)$ towards the unit circle?

3. Where to place the poles to get a 'low-pass' filter?

4. Where to put the poles in order to have a resonance top at $\omega = 1$?

5. How does a resonant system appear on the different plots?

6. What happens if $H_2(z)$ got a zero close to the unit circle?

Preparation Exercise 2

Determine the covariance function for an $AR(1)$ process

$$y(t) + a(y(t-1)) = e(t)$$

where $e(t)$ white, zero mean and unit variance.

Preparation Exercise 3

Determine the covariance function for a $MA(1)$ process

$$y(t) = e(t) + ce(t-1)$$

where $e(t)$ white, zero mean and unit variance.

Consider a general $MA(n)$. For which values τ is it in general true that $r(\tau) = 0$?

Preparation Exercise 4

Given an input $u(t)$ shaped by an ARMA filter,

$$A(q^{-1})x(t) = C(q^{-1})v(t)$$

where $v(t)$ white, zero mean and variance λ_v^2. Given noisy observations of this signal, or

$$y(t) = x(t) + e(t)$$

where $e(t)$ white, zero mean and variance λ_e^2 and uncorrelated to $v(t)$. Rewrite this as a ARMA process, what would be the corresponding variance of the 'noise'? How would the spectrum of $y(t)$ look like?

Exercise 2.2

Convergence rates for consistent estimators.

For most consistent estimators of the parameters of stationary processes, the estimation error $\hat{\theta} - \theta_0$ tends to zero as $1/N$ when $N \to \infty$. For nonstationary processes, faster convergence rates may be expected. To see this, derive the variance of the least squares estimate in the model

$$y(t) = \alpha t + e(t),\ t = 1, \ldots, N$$

with $e(t)$ white noise, zero mean and variance λ^2.

Exercise 2.3

Illustration of unbiasedness and consistency properties. Let $\{x_i\}_i$ be a sequence of i.i.d. Gaussian random variables with mean μ and variance σ. Both are unknown. Consider the following estimate of μ:

$$\hat{\mu} = \frac{1}{N}\sum_{i=1}^{N} x_i$$

and the following two estimates of σ:

$$\begin{cases} \hat{\sigma}_1 = \frac{1}{N}\sum_{i=1}^{N}(x_i - \hat{\mu})^2 \\ \hat{\sigma}_2 = \frac{1}{N-1}\sum_{i=1}^{N}(x_i - \hat{\mu})^2 \end{cases}$$

determine the mean and the variance of the estimates $\hat{\mu}, \hat{\sigma}_1$ and $\hat{\sigma}_2$. Discuss their bias and consistency properties. Compare $\hat{\sigma}_1$ and $\hat{\sigma}_2$ in terms of their Mean Square Error (mse).

Exercise 2.4

Least square estimates with white noise as inputs.

Given a system

$$y(t) + a_0 y(t-1) = b_0 u(t-1) + e(t) + c_0 e(t-1)$$

with $e(t)$ white, zero mean and variance λ^2, and zero mean white noise input $(u(t))_t$ with variance σ^2, uncorrelated with noise $e(s), s \leq t$, then

$$\begin{cases} E[y(t)u(t)] = 0 \\ E[y(t)y(t-1)] = \frac{-a_0 b_0^2 + (c_0 - a_0)(1 - a_0 c_0)\lambda^2}{(1 - a_0^2)} \\ E[y(t)u(t-1)] = b_0 \sigma^2 \end{cases}$$

rewrite as a model LIP:

$$y(t) = (y(t-1), u(t-1))^T (\alpha, \beta)$$

Application of LS yields estimates for $N \to \infty$

$$\begin{cases} \hat{\alpha} = a_0 + \frac{-c_0(1-a_0^2)\lambda^2}{b_0^2\sigma^2+(1+c_0^2-2a_0c_0)\lambda^2} \\ \hat{\beta} = b_0 \end{cases}$$

Exercise 2.5

Least square estimates of the same system with a step function $u(t) = \sigma I(t > 0)$ as inputs.

Let $S = \frac{b_0}{1+a_0}$. The covariance matrices become

$$\begin{cases} E[y^2(t)] = S^2\sigma^2 + \frac{(1+c_0^2-2a_0c_0)\lambda^2}{1-a_0^2} \\ E[u^2(t)] = \sigma^2 \\ E[y(t)u(t)] = S\sigma^2 \\ E[y(t)u(t-1)] = S\sigma^2 \\ E[y(t)y(t-1)] = S^2\sigma^2 + \frac{(c_0-a_0)(1-a_0c_0)\lambda^2}{1-a_0^2} \end{cases}$$

Application of LS yields estimates for $N \to \infty$

$$\begin{cases} \hat{\alpha} = a_0 - \frac{c_0(1-a_0^2)}{(1+c_0^2-2a_0c_0)} \\ \hat{\beta} = b_0 - b_0c_0\left(\frac{1-a_0}{1+c_0^2-2a_0c_0}\right) \end{cases}$$

Exercise 2.6

Least square estimates of the same system with a step function $u(t) = \sigma I(t > 0)$ as inputs (Ct'd).

Verify that the gain S is estimated correctly when $N \to \infty$ by

$$\hat{S} = \frac{\hat{\beta}}{1 + \hat{\alpha}} = \frac{b_0}{1 + a_0}$$

In the noise-free case where $\lambda = 0$, we run into troubles, the covariance matrix is noninvertible:

$$\begin{bmatrix} E[y^2(t)] & E[y(t)u(t)] \\ E[u(t)y(t)] & E[u^2(t)] \end{bmatrix}$$

How are all possible solutions characterized?

Exercise 3.1

Determine the time constant T from a step response.

A first order system $Y(s) = G(s)U(s)$ with

$$G(s) = \frac{K}{1 + sT} e^{-s\tau}$$

or in time domain as a differential equation

$$T \frac{dy(t)}{dt} + y(t) = Ku(t - \tau)$$

derive a formula of the step response of an input $u(t) = I(t > 0)$.

Exercise 3.7

Correlation analysis with truncated weighting function.

$$\begin{bmatrix} r_{uy}(0) \\ \vdots \\ r_{uy}(M-1) \end{bmatrix} \begin{bmatrix} r_u(0) & & r_u(M-1) \\ & \ddots & \\ r_u(M-1) & & r_{uy}(0) \end{bmatrix} \begin{bmatrix} \hat{h}(0) \\ \vdots \\ \hat{h}(M-1) \end{bmatrix}$$

1. Let an input $u(t)$ be white noise, note that regardless of M the solution $\hat{h}(k) = h_0(k)$ for $k = 0, \ldots, M-1$.

2. Consider the input ($|\alpha| < 1$, and $v(t)$ zero mean white noise with variance σ^2).

$$u(t) - \alpha u(t-1) = v(t)$$

and assume a first order system

$$y(t) + ay(t-1) = bu(t-1), \; |a| < 1$$

Then

$$\begin{cases} h(0) = 0 \\ h(k) = b(-a)^{k-1}, \ \forall k \geq 1 \\ \hat{h}(k) = h_0(k), \ k = 0, \ldots, M - 2\hat{h}(M-1) = \frac{h(M-1)}{1+a\alpha} \end{cases}$$

Hint:

$$\begin{bmatrix} 1 & \alpha & \ldots & \alpha^{M-1} \\ \alpha & 1 & & \\ & & \ddots & \\ \alpha^{M-1} & & & 1 \end{bmatrix} = \frac{1}{1-\alpha^2} \begin{bmatrix} 1 & -\alpha & & \\ -\alpha & 1+\alpha^2 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}$$

Exercise 3.10

Step response as a special case of spectral analysis.

Let $(y(t))_t$ be the step response of an LTI $H(q^{-1})$ to an input $u(t) = aI(t \geq 1)$. Assume $y(t) = 0$ for $t < 1$ and $y(t) \approx c$ for $t > N$. Justify the following rough estimate of H

$$\hat{h}(k) = \frac{y(k) - y(k-1)}{a}, \ \forall k = 0, \ldots, N$$

and show that it is approximatively equal to the estimate provided by the spectral analysis.