Chapter 11

11.1 Dynamic Models

11.1.1 Exercises + Solutions

Exercise 1.0: Hello world

1. Given a discrete system $G(z) = \frac{K}{\tau - z}$ with $K = 1$ and $\tau = 2$. Is it BIBO stable? Why/not?

   Solution : No. The system has a pole outside the unit circle and therefore is unstable.

2. Given a system which outputs positive values for any input. Is it LTI? Why/not?

   Solution: No. For LTI system if the input $u(t)$ produces the output $y(t)$ which is strictly positive then $-u(t)$ produces the output $-y(t)$ which is negative.

3. Can you solve a least squares estimate for θ for a system satisfying $x_i\theta = y_i$ for any $\{(x_i, y_i)\}_i$? Why/not?

   Solution: No. The least square solution for the system $x_i\theta = y_i$ given $\{(x_i, y_i)\}_{i=1}^n$ is

   $$\hat{\theta} = \frac{\sum_{i=1}^n x_i y_i}{\sum_{i=1}^n x_i^2}$$

   in case $\sum_{i=1}^n x_i^2 \neq 0$.

4. Is the *median* estimate optimal in a least squares sense? Why/not?

   Solution: No. The *mean* value (sample average) is optimal in a least square sense. The *median* estimate is optimal for the absolute objective value.

5. If we are to model a certain behavior and we know some of the physics behind it - should we go for a black box model? Why/not?

   Solution: No. But we don't have to. We better go for the white box model. The white box model is formalized in terms of physical laws, chemical relations, or other theoretical considerations.

6. If we have a very fast system (time constants smaller than $O(10^{-2})s$). Can we get away with slow sampling? Why/not?

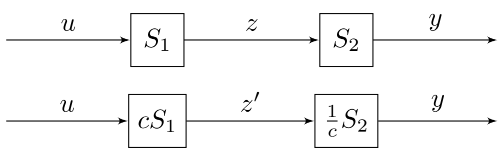
   Solution: No. A fast system does need a fast sampling (In general).

7. Does a non-causal model allow an impulse representation? Why/not?

   Solution: No. From the impluse response of a system $y_t = \sum_{k=0}^{n} h_k u_{t-k}$ one can see that the output does not depend on future inputs.

8. Is a sequence of two nontrivial LTIs identifiable from input-output observations? Why/not?

   Solution: No. Consider the following sequence of two LTIs



   where c is a constant parameter. The two LTIs have the same input-output behavior but different systems. Thus it is not identifiable from the input u and output y observations.

9. Is an ARMAX system linear in the parameters of the polynomials? Why/not?

   Solution: No. For example the ARMAX model

$$y(t) + ay(t-1) = bu(t-1) + e(t) + ce(t-1)$$

   is not LIP, since c and $e(t-1)$ are unknown.

10. Is an OE model LIP? Why/not?

    Solution: No. The OE model

$$y(t) = \frac{B(q^{-1})}{F(q^{-1})} u(t) + e(t)$$

    which can be written as $F(q^{-1})e(t) = F(q^{-1})y(t) - B(q^{-1})u(t)$. A simple example is

$$y(t) + f_1 y(t-1) = u(t-1) + e(t) + f_1 e(t-1)$$

    since f_1 and $e(t-1)$ are unknown, $f_1 e(t-1)$ is not LIP.

Exercise 1.2: Least Squares with Feedback

Consider the second-order AR model

$$y_t + a_0 y_{t-1} = b_0 u_{t-1} + e_t$$

where u_t is given by feedback as

$$u_t = -K y_t.$$

Show that given realizations of this signal we cannot estimate a_0, b_0 separately, but we can estimate $a_0 + b_0 K$.

Solution: (Book p. 26)

Exercise 1.3: Determine the time constant T from a step response.

A first order system $Y(s) = G(s)U(s)$ with

$$G(s) = \frac{K}{1+sT}e^{-s\tau}$$

or in time domain as a differential equation

$$T\frac{dy(t)}{dt} + y(t) = Ku(t-\tau)$$

derive a formula of the step response of an input $u_t = I(t > 0)$.

Solution: The step response of the system $T\frac{dy(t)}{dt} + y(t) = Ku(t-\tau)$ is

$$y(t) = \begin{cases} o & t < \tau \\ K(1 - exp(-(t-\tau)/T)) & \end{cases}$$

the tangent at $t = \tau$ is given as

$$y'(t) = \frac{K}{T}(t-\tau)$$

The tangent reaches the steady state value K at time $t = \tau + T$.

Exercise 1.4: Step response as a special case of spectral analysis.

Let $(y_t)_t$ be the step response of an LTI $H(q^{-1})$ to an input $u_t = aI(t \geq 0)$. Assume $y_t = 0$ for $t < 0$ and $y_t \approx c$ for $t > N$. Justify the following rough estimate of H

$$\hat{h}_k = \frac{y_k - y_{k-1}}{a}, \ \forall k = 0, \ldots, N$$

and show that it is approximatively equal to the estimate provided by the spectral analysis.

Solution:

The response y_t of the LTI system to the input u_t is

$$y_t = \sum_{k=0}^{t} h_k u_{t-k} = a\sum_{k=0}^{t} h_k$$

and since y_t remains constant for values $t > N$ it follows that

$$h_t = \frac{y_t - y_{t-1}}{a}$$

for $t = 0, 1, 2, \ldots, n$. Since $h_t \approx 0$ for large n, thus the possible estimate of the transfer function is:

$$\begin{aligned}
\hat{H}(e^{i\omega}) &= \sum_{k=0}^{n} h_k \exp(-i\omega k) \\
&= \frac{1}{a}\sum_{k=0}^{n}(y_k - y_{k-1}) \exp(-i\omega k) \\
&\approx \frac{1}{a}\sum_{k=0}^{n} y_k \exp(-i\omega k) - \frac{1}{a} y_k \exp(-i\omega k) exp(-i\omega) = \frac{1}{a} Y_n(\omega)(1 - \exp(-i\omega))
\end{aligned}$$

Now

$$U(\omega) = \sum_{k=0}^{\infty} u_k \exp(-i\omega k) = a \sum_{k=0}^{\infty} \exp(-i\omega k) = \frac{a}{1 - \exp(-i\omega)}$$